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Jonathan Rosenberg

Algebraic K-Theory and Its Applications

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Jonathan Rosenberg
Department of Mathematics
University of Maryland
College park, MD 20742 USA

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Preface

Algebraic K-theory is the branch of algebra dealing with linear algebra (especially in the limiting case of large matrices) over a general ring R instead of over a field. It associates to any ring R a sequence of abelian groups $K_i(R)$. The first two of these, K_0 and K_1, are easy to describe in concrete terms; the others are rather mysterious. For instance, a finitely generated projective R-module defines an element of $K_0(R)$, and an invertible matrix over R has a "determinant" in $K_1(R)$. The entire sequence of groups $K_i(R)$ behaves something like a homology theory for rings.

Algebraic K-theory plays an important role in many areas, especially number theory, algebraic topology, and algebraic geometry. For instance, the class group of a number field is essentially $K_0(R)$, where R is the ring of integers, and "Whitehead torsion" in topology is essentially an element of $K_1(\mathbb{Z}\pi)$, where π is the fundamental group of the space being studied. K-theory in algebraic geometry is basic to Grothendieck's approach to the Riemann-Roch problem. Some formulas in operator theory, involving determinants and determinant pairings, are best understood in terms of algebraic K-theory. There is also substantial evidence that the higher K-groups of fields and of rings of integers are related to special values of L-functions and encode deep arithmetic information.

This book is based on a one-semester course I gave at the University of Maryland in the fall of 1990. Most of those attending were second- or third-year graduate students interested in algebra or topology, though there were also a number of analysis students and faculty colleagues from other areas. I tried to make the course (and this book) fairly self-contained, and to assume as a prerequisite only the standard one-year graduate algebra course, based on a text such as [Hungerford], [Jacobson], or [Lang], and the standard introductory graduate course on algebraic and geometric topology, covering the fundamental group, homology, the notions of simplicial and CW-complex, and the definition and basic properties of manifolds. As taught at Maryland, the graduate algebra course includes the most basic definitions and concepts of category theory; a student who hasn't yet seen these ideas could consult any of the above algebra texts or an introduction to category theory such as [Mac Lane]. Since many graduate algebra courses do not include much in the way of algebraic number theory, I have

included many topics such as the basic theory of Dedekind rings and the Dirichlet unit theorem, which may be familiar to some readers but not to all. I've tried in this book to presuppose as little topology as possible beyond a typical introductory course, and to develop what is needed as I go along, but to give the reader a flavor of some of the important applications of the subject. A reader with almost no topology background should still be able to follow most of the book except for parts of Sections 1.6, 1.7, 2.4, 4.4, and 6.3, and most of Chapter 5 (though I would hope this book might encourage him or her to take a more systematic course in topology). A problem one always has in writing a book such as this is to decide what to do about spectral sequences. They are usually not mentioned in first-year graduate courses, and yet they are indispensable for serious work in homological algebra and K-theory. To avoid having to give an introduction to spectral sequences which might scare off many readers, I have avoided using spectral sequences directly anywhere in the text. On the other hand, I have made indirect reference to them in many places, so that the reader who has heard of them will often see why they are relevant to the subject and how they could be used to simplify some of the proofs.

For the most part, this book tends to follow the notes of the original course, with a few additions here and there. The major exceptions are that Chapters 3 and 5 have been greatly expanded, and Chapter 6 on cyclic homology has been added even though there was no time for it in the original course. Cyclic homology is a homology theory for rings which may be viewed as the "linearized version" of algebraic K-theory, and it's becoming increasingly clear that it is both a useful computational tool and a subject of independent interest with its own applications.

Each chapter of this book is divided into sections, and I have used a single numbering system for all theorems, lemmas, exercises, definitions, and formulas, to make them easier to locate. Thus a reference such as 1.4.6 means the 6th numbered item in Section 4 of Chapter 1, whether that item is a theorem, a corollary, an exercise, or a displayed formula. The exercises are an integral part of the book, and I have tried to put at least one interesting exercise at the end of every section. The reader should not be discouraged if he finds some of the exercises too difficult, since the exercises vary from the routine to the very challenging.

I have used a number of more-or-less standard notations without special reference, but the reader who is puzzled by them will be able to find most of them listed in the Notational Index in the back of the book.

Why This Book?

The reader might logically ask how this book differs from its "competition." [Bass] remains an important reference, but it is too comprehensive to use as a text for an elementary course, and also it predates the definition of K_2, let alone of higher K-theory or of cyclic homology. My original course was based on the notes by Milnor [Milnor], which are highly recommended. However, I found that [Milnor] is hard to use as a textbook, for

the following three reasons:

(1) Milnor writes for a working mathematician, and sometimes leaves out details that graduate students might not be able to provide for themselves.
(2) There are no exercises, at least in the formal sense.
(3) The subject has changed quite a bit since Milnor's book was written.

For the working algebraist already familiar with the contents of [Milnor] who wants to learn about Quillen K-theory and its applications in algebraic geometry, [Srinivas] is an excellent text, but it would have been far beyond the reach of my audience. The notes of Berrick [Berrick] give a more elementary introduction to Quillen K-theory than [Srinivas], but are rather sketchy and do not say much about applications, and thus again are not too suitable for a graduate text. And [LluisP] is very good for an up-to-date survey, but is, as the title says, an overview rather than a textbook. For cyclic homology, the recent book by Loday [LodayCH] is excellent, but to be most useful requires the reader already to know something about K-theory. Also, I do not believe that there is any book available that discusses the applications of algebraic K-theory in functional analysis (which are discussed here in 2.2.10–2.2.11, 4.4.19–4.4.24, 4.4.30, 6.3.8–6.3.17, and 6.3.29-30). Thus for all these reasons it seemed to me that another book on K-theory is needed. I hope this book helps at least in part to fulfill that need.

Acknowledgements

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Added August, 1995. I am quite grateful for the opportunity to correct a few mistakes in the second printing. I thank several colleagues for their comments on this book, and especially thank Bruce Blackadar, Andreas Rosenschon, and Dan Voiculescu for their corrections.

Contents

1

K_0 of Rings

1. Defining K_0

K-theory as an independent discipline is a fairly new subject, only about 35 years old. (See [Bak] for a brief history, including an explanation of the choice of the letter K to stand for the German word *Klasse*.) However, special cases of K-groups occur in almost all areas of mathematics, and particular examples of what we now call K_0 were among the earliest studied examples of abelian groups. More sophisticated examples of the idea of the definition of K_0 underlie the Euler-Poincaré characteristic in topology and the Riemann-Roch theorem in algebraic geometry. (The latter, which motivated Grothendieck's first work on K-theory, will be briefly described below in §3.1.) The Euler characteristic of a space X is the alternating sum of the Betti numbers; in other words, the alternating sum of the dimensions of certain vector spaces or free R-modules $H_i(X; R)$ (the homology groups with coefficients in a ring R). Similarly, when expressed in modern language, the Riemann-Roch theorem gives a formula for the difference of the dimensions of two vector spaces (cohomology spaces) attached to an algebraic line bundle over a non-singular projective curve. Thus both involve a **formal difference** of two free modules (over a ring R which can be taken to be $\mathbb{C}$). The group $K_0(R)$ makes it possible to define a similar formal difference of two finitely generated **projective modules** over any ring R.

We begin by recalling the definition and a few basic properties of projective modules. **Unless we say otherwise, we shall assume all rings have a unit, we shall require all ring homomorphisms to be unit-preserving, and we shall always use the word module to mean "left module."**

1.1.1. Definition. Let R be a ring. A **projective module** over R means an R-module P with the property that any surjective R-module homomorphism $\alpha : M \to P$ has a right inverse $\beta : P \to M$. An equivalent way of phrasing this is that whenever one has a diagram of R-modules and

R-module maps

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{\psi} & N \end{array}$$

with $M \xrightarrow{\psi} N$ surjective, one can fill this in to a commutative diagram

$$\begin{array}{ccc} & & P \\ & {\scriptstyle \theta}\swarrow & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{\psi} & N. \end{array}$$

Indeed, given the diagram-completion property and a surjective R-module homomorphism $\alpha : M \to P$, one can take $N = P$, $\varphi = id_P$, and $\psi = \alpha$, and the resulting $\theta : P \to M$ is a right inverse for α, *i.e.*, satisfies $\alpha \circ \theta = id_P$.

In the other direction, suppose any surjective R-module homomorphism $\alpha : M \to P$ has a right inverse $\beta : P \to M$, and suppose one is given a diagram of R-modules and R-module maps

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{\psi} & N \end{array}$$

with $M \xrightarrow{\psi} N$ surjective. Replacing $M \xrightarrow{\psi} N$ by $M \oplus P \xrightarrow{\psi \oplus id_P} N \oplus P$ and $\varphi : P \to N$ by $(\varphi, id_P) : P \to N \oplus P$, we may suppose φ is one-to-one, and then replacing N by the image of φ and M by $\psi^{-1}(\operatorname{im} \varphi)$, we may assume it's an isomorphism. Then take $\alpha = \varphi^{-1} \circ \psi$ and the right inverse $\beta : P \to M$ enables us to complete the diagram.

When $\alpha : M \to P$ is surjective and $\beta : P \to M$ is a right inverse for α, then $p = \beta \circ \alpha$ is an idempotent endomorphism of M, since

$$\begin{aligned} (\beta \circ \alpha)^2 &= (\beta \circ \alpha) \circ (\beta \circ \alpha) \\ &= \beta \circ (\alpha \circ \beta) \circ \alpha \\ &= \beta \circ id_P \circ \alpha = \beta \circ \alpha, \end{aligned}$$

and then $x \mapsto (\alpha(x), (1-p)(x))$ gives an isomorphism $M \cong P \oplus (1-p)(M)$.

Using this observation, we can now prove the fundamental characterization of projective modules.

1.1.2. Theorem. *Let R be a ring. An R-module is projective if and only if it is isomorphic to a direct summand in a free R-module. It is finitely generated and projective if and only if it is isomorphic to a direct summand in R^n for some n.*

Proof. If P is projective, choose a free module F and a surjective R-module homomorphism $\alpha : F \to P$ by taking F to be the free module on some

generating set for P, and α to be the obvious map sending a generator of F to the corresponding generator of P. We are using the universal property of a free module: To define an R-module homomorphism out of a free module, it is necessary and sufficient to specify where the generators should go. If P is finitely generated, then F will be isomorphic to R^n for some n. The observation above then shows P is isomorphic to a direct summand in a free R-module, which we can take to be R^n for some n if P is finitely generated.

For the converse, observe first that free modules F are projective, since given a surjective R-module homomorphism $\alpha : M \to F$ with F free, one can for each generator x_i of F choose some $y_i \in M$ with $\alpha(y_i) = x_i$, and then one can define a right inverse to α by using the universal property of a free module to define an R-module homomorphism $\beta : F \to M$ with $\beta(x_i) = y_i$. Next, suppose $F = P \oplus Q$ and F is a free module. Given a surjective R-module homomorphism $\alpha : M \to P$, $\alpha \oplus id_Q$ is a surjective R-module homomorphism $(M \oplus Q) \to (P \oplus Q) = F$, so it has a right inverse. Now restrict this right inverse to P and project into M to get a right inverse for α. Finally, if $F = R^n$ with standard generators $x_1, \ldots, x_n$, then P is generated by $p(x_i)$, where p is the identity on P and 0 on Q. Thus a direct summand in R^n is finitely generated and projective. □

We're now almost ready to define K_0 of a ring R. First of all, note that the isomorphism classes of finitely generated projective modules over R form an abelian semigroup $\operatorname{Proj} R$, in fact a monoid, with $\oplus$ as the addition operation and with the 0-module as the identity element. To see that this makes sense, there are a few easy things to check. First of all, $\operatorname{Proj} R$ is a set! (This wouldn't be true if we didn't take isomorphism classes, but in fact we have a very concrete model for $\operatorname{Proj} R$ as the set of split submodules of the R^n, $n \in \mathbb{N}$, divided out by the equivalence relation of isomorphism.) Secondly, direct sum is well defined on isomorphism classes, *i.e.*, if $P \cong P'$ and $Q \cong Q'$, then $P \oplus Q \cong P' \oplus Q'$. And thirdly, direct sum is commutative ($P \oplus Q \cong Q \oplus P$) and associative ($(P \oplus Q) \oplus V \cong P \oplus (Q \oplus V)$) once we pass to isomorphism classes.

In general, though, $\operatorname{Proj} R$ is not a group, and may not even have the cancellation property

$$a + b = c + b \Rightarrow a = c.$$

It's therefore convenient to **force it** into being a group, even though this may result in the loss of some information. The idea of how to do this is very simple and depends on the following, which is just a generalization of the way $\mathbb{Z}$ is constructed from the additive semigroup of positive integers, or $\mathbb{Q}^\times$ is constructed from the multiplicative semigroup of non-zero integers, or a ring is "localized" by the introduction of formal inverses for certain elements.

1.1.3. Theorem. *Let S be a commutative semigroup (not necessarily having a unit). There is an abelian group G (called the* **Grothendieck group** *or* **group completion** *of S), together with a semigroup homo-*

morphism $\varphi : S \to G$, such that for **any** *group H and homomorphism $\psi : S \to H$, there is a unique homomorphism $\theta : G \to H$ with $\psi = \theta \circ \varphi$.*

Uniqueness holds in the following strong sense: if $\varphi' : S \to G'$ is any other pair with the same property, then there is an isomorphism $\alpha : G \to G'$ with $\varphi' = \alpha \circ \varphi$.

Proof. We will outline two constructions. The simplest is to define G to be the set of equivalence classes of pairs (x, y) with $x, y \in S$, where $(x, y) \sim (u, v)$ if and only if there is some $t \in S$ such that

$$x + v + t = u + y + t \quad \text{in } S. \tag{1.1.4}$$

Denote by $[(x, y)]$ the equivalence class of (x, y). Then addition is defined by the rule

$$[(x, y)] + [(x', y')] = [(x + x', y + y')] .$$

(It is easy to see that this is consistent with the equivalence relation, and that the associative rule holds.)

Note that for any x and y in S,

$$[(x, x)] = [(y, y)]$$

since $x + y = y + x$. Let 0 be this distinguished element $[(x, x)]$. This is an identity element for G, *i.e.*, G is a monoid, since for any x, y, and t in S,

$$(x + t, y + t) \sim (x, y).$$

Also, G is a group since

$$[(x, y)] + [(y, x)] = [(x + y, x + y)] = 0.$$

We define $\varphi : S \to G$ by

$$\varphi(x) = [(x + x, x)] ,$$

and it is easy to see that this is a homomorphism. Note that the image of φ generates G as a group, since

$$[(x, y)] = \varphi(x) - \varphi(y)$$

in G. Given a group H and homomorphism $\psi : S \to H$, the homomorphism $\theta : G \to H$ with $\psi = \theta \circ \varphi$ is defined by

$$\theta ([(x, y)]) = \psi(x) - \psi(y).$$

Alternatively, one may define G to be the free abelian group on generators $[x]$, $x \in S$, divided out by the relations that if $x + y = z$ in S, then the elements $[x] + [y] = [z]$ in G. Note that $[(x, y)]$ in the previous construction corresponds to $[x] - [y]$ in this second construction. The map

φ is $x \mapsto [x]$, and of course any homomorphism from S into a group H must factor through G by construction.

To prove the uniqueness, suppose $\varphi' : S \to G'$ has the same universal property. First of all, $\varphi'(S)$ must generate G', since otherwise, if G'' is the subgroup generated by the image of φ', then there are two homomorphisms $\theta : G' \to G' \oplus G'/G''$ with

$$(\varphi', 0) = \theta \circ \varphi',$$

namely, $\theta = (id, 0)$ and $\theta = (id, q)$, q the quotient map. By the universal properties for G and G', there must be maps $\alpha : G \to G'$ with $\varphi' = \alpha \circ \varphi$ and $\beta : G' \to G$ with $\varphi = \beta \circ \varphi'$. But then $\alpha \circ \beta = id$ on the image of φ', hence on all of G', so α is a left inverse to β. Similarly $\beta \circ \alpha = id$ on the image of φ; hence α is also a right inverse to β, as required. □

Remarks. The assignment $S \rightsquigarrow G = G(S)$ is in fact a functor from the category of abelian semigroups to the category of abelian groups, since if $\gamma : S \to S'$ is a homomorphism of semigroups, it induces a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\gamma} & S' \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi'}\downarrow \\ G(S) & \longrightarrow & G(S'), \end{array}$$

where the arrow at the bottom is uniquely determined by the universal property of $G(S)$.

In fancier language, Theorem 1.1.3 just asserts that the forgetful functor F from the category of abelian groups to the category of abelian semigroups has a left adjoint, since

$$\mathrm{Hom}_{\mathrm{Semigroups}}(S, FH) \cong \mathrm{Hom}_{\mathrm{Groups}}(G, H).$$

This could also have been deduced from the adjoint functor theorem (see [Freyd] or [Mac Lane]).

It is convenient that we do not have to assume that cancellation ($x+z = y+z \Rightarrow x = y$) holds in S. Indeed, the map $\varphi : S \to G$ is injective if and only if cancellation holds in S. One of the reasons for introducing Grothendieck groups is that semigroups without cancellation are usually very hard to handle; yet in many cases their Grothendieck groups are fairly tractable.

1.1.5. Definition. Let R be a ring (with unit). Then $K_0(R)$ is the Grothendieck group (in the sense of Theorem 1.1.3) of the semigroup $\mathrm{Proj}\, R$ of isomorphism classes of finitely generated projective modules over R.

Note that K_0 is a **functor**; in other words, if $\varphi : R \to R'$ is an R-module homomorphism, there is an induced homomorphism $K_0(\varphi) = \varphi_* : K_0(R) \to K_0(R')$ satisfying the usual conditions $id_* = id$, $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$. To see this, observe first that φ induces a homomorphism $\mathrm{Proj}\, R \to \mathrm{Proj}\, R'$

via $[P] \mapsto [R' \otimes_\varphi P]$, for P a finitely generated projective module over R. As required, $R' \otimes_\varphi P$ is finitely generated and projective over R', since if $P \oplus Q \cong R^n$, then

$$(R' \otimes_\varphi P) \oplus (R' \otimes_\varphi Q) \cong R' \otimes_\varphi (P \oplus Q) \cong (R' \otimes_\varphi R^n) = R'^n.$$

And of course, the tensor product commutes with direct sums so we get a homomorphism. Functoriality of K_0 now follows from functoriality of the Grothendieck group construction.

1.1.6. Example. If R is a field, or more generally a division ring (*i.e.*, a skew-field), then any finitely generated R-module is a finitely generated R-vector space and so has a basis and a well-defined dimension. This dimension is the only isomorphism invariant of the module, so we see that $\operatorname{Proj} R \cong \mathbb{N}$, the additive monoid of natural numbers. Since the group completion of $\mathbb{N}$ is $\mathbb{Z}$, $K_0(R) \cong \mathbb{Z}$, with the isomorphism induced by the dimension isomorphism $\operatorname{Proj} R \to \mathbb{N}$. The inclusion of a field F into an extension field F' induces the identity map from $\mathbb{Z}$ to itself, since $\dim_{F'}(F' \otimes_F P) = \dim_F P$ for any F-vector space P.

This same example also shows why we only use **finitely generated** projective modules in defining K_0. If R is a field, the same arguments show that the monoid of isomorphism classes of **countably generated** modules is isomorphic to the extended natural numbers $\mathbb{N} \cup \{\infty\}$, with the usual rule of transfinite arithmetic, $n + \infty = \infty$ for any n. This is no longer a monoid with cancellation; in fact, any two elements become isomorphic after adding ∞ to each one. Thus the Grothendieck group of this monoid is trivial. A similar phenomenon happens with infinitely generated modules over an arbitrary ring; see Exercise 1.1.8.

1.1.7. Exercise. Let S be the abelian monoid with elements $a_{n,m}$, where $n \in \mathbb{N}$, and

$$\begin{cases} m = 0 \text{ if } n = 0 \text{ or } 1, \\ m \in \mathbb{Z} \text{ if } n = 2, \\ m \in \mathbb{Z}/2 \text{ if } n \geq 3. \end{cases}$$

The semigroup operation is given by the formula

$$a_{n,m} + a_{n',m'} = a_{n+n',m+m'},$$

where $m + m'$ is to be computed in $\mathbb{Z}$ if $n + n' \leq 2$ and in $\mathbb{Z}/2$ if $n + n' \geq 3$. (If for instance $n = 2$ and $n' \geq 1$, then m is to be interpreted mod 2.) We shall see in §1.6 that S is isomorphic to $\operatorname{Proj} R$ with $R = C^{\mathbb{R}}(S^2)$, the continuous real-valued functions on the 2-sphere. Compute $G(S)$ and the map $\varphi : S \to G(S)$. Determine the image of S in G, and show that while $\varphi^{-1}(0) = 0$, φ is not injective.

1.1.8. Exercise (the "Eilenberg swindle"). Show that for any ring R, the Grothendieck group of the semigroup of isomorphism classes of **countably** generated projective R-modules vanishes.

1.1.9. Exercise. Recall that if a ring R is commutative, then every left R-module is automatically a right R-module as well, so that the tensor product of two left R-modules makes sense.

(1) Show that the tensor product of two finitely generated projective modules is again finitely generated and projective.
(2) Show that the tensor product makes $K_0(R)$ into a commutative ring with unit. (The class of the free R-module R is the unit element.)

2. K_0 from idempotents

There is another approach to K_0 which is a little more concrete and therefore often convenient. If P is a finitely generated projective R-module, we may assume (replacing P by an isomorphic module) that $P \oplus Q = R^n$ for some n, and we can consider the R-module homomorphism p from R^n to itself which is the identity on P and 0 on Q. Clearly p is idempotent, *i.e.*, $p^2 = p$. Since any R-module homomorphism $R^n \to R^n$ is determined by the n coordinates of the images of each of the standard basis vectors, it corresponds to multiplication on the **right** (since R is acting on the left) by an $n \times n$ matrix. In other words, P is given by an idempotent $n \times n$ matrix p which determines P up to isomorphism.

On the other hand, different idempotent matrices can give rise to the same isomorphism class of projective modules. (When R is a field, the only invariant of a projective module P is its dimension, which corresponds to the rank of the matrix p. When the characteristic of the field is zero, the rank of an idempotent matrix is just its trace.) So to compute $K_0(R)$ from idempotent matrices, we need to describe the equivalence relation on the idempotent matrices that corresponds to isomorphism of the corresponding modules.

1.2.1. Lemma. *If p and q are idempotent matrices over a ring R (of possibly different sizes), the corresponding finitely generated projective R-modules are isomorphic if and only if it is possible to enlarge the sizes of p and q (by adding zeroes in the lower right-hand corner) so that they have the same size $N \times N$ and are conjugate under the group of invertible $N \times N$ matrices over R, $GL(N, R)$.*

Proof. The condition is sufficient since if $u \in GL(N, R)$ and $upu^{-1} = q$, then right multiplication by u induces an isomorphism from $R^N q$ to $R^N p$. So the problem is to prove necessity of the condition. Suppose p is $n \times n$ and q is $m \times m$, and $R^n p \cong R^m q$. We can extend an isomorphism $\alpha : R^n p \to R^m q$ to an R-module homomorphism $R^n \to R^m$ by taking $\alpha = 0$ on the complementary module $R^n(1-p)$, and by viewing the image $R^m q$

as embedded in R^m. Similarly extend α^{-1} to an R-module homomorphism $\beta : R^m \to R^n$ which is 0 on $R^m(1-q)$. Once we've done this, α is given by right multiplication by an $n \times m$ matrix a, and β is given by right multiplication by an $m \times n$ matrix b. We also have the relations $ab = p$, $ba = q$, $a = pa = aq$, $b = qb = bp$. The trick is now to take $N = n + m$ and to observe that

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(with usual block matrix notation) and that

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}$$
$$= \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

Thus $\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}$ is invertible and conjugates $p \oplus 0$ to $0 \oplus q$. The latter is of course conjugate to $q \oplus 0$ by a permutation matrix. □

Now we can give a simple description of Proj R.

1.2.2. Definition. Let R be a ring. Denote by $M(n, R)$ the collection of $n \times n$ matrices over R and by $GL(n, R)$ the group of $n \times n$ matrices over R. We embed $M(n, R)$ in $M(n+1, R)$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ (this is a **non-unital** ring homomorphism) and $GL(n, R)$ in $GL(n+1, R)$ by the group homomorphism $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Denote by $M(R)$ and $GL(R)$ the infinite unions of the the $M(n, R)$, resp. $GL(n, R)$. Note that $M(R)$ is a ring **without unit** and $GL(R)$ is a group. It is important to remember that each matrix in $M(R)$ has finite size. Let Idem(R) be the set of idempotent matrices in $M(R)$, and note that $GL(R)$ acts on Idem(R) by conjugation.

Now we can restate Lemma 1.2.1.

1.2.3. Theorem. *For any ring R,* Proj R *may be identified with the set of conjugation orbits of $GL(R)$ on* Idem(R). *The semigroup operation is induced by $(p, q) \mapsto \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$. (One only has commutativity and associativity after passage to conjugacy classes.) $K_0(R)$ is the Grothendieck group of this semigroup.*

Using this fact we can now show that K_0 is invariant under passage from R to $M_n(R)$ and commutes with direct limits. We will also construct an example of a ring for which K_0 vanishes.

1.2.4. Theorem ("Morita invariance"). *For any ring R and any positive integer n, there is a natural isomorphism $K_0(R) \xrightarrow{\cong} K_0(M_n(R))$.*

Proof. Via the usual identification of $M_k(M_n(R))$ with $M_{kn}(R)$,

$$\text{Idem}(M_n(R)) = \text{Idem}(R) \quad \text{and} \quad GL(M_n(R)) = GL(R).$$

The result therefore follows immediately from Theorem 1.2.3. □

Next we show that K_0 is a *continuous* functor, *i.e.*, that it commutes with (direct) limits. A **direct system** or **directed system** in a category is a collection $(A_\alpha)_{\alpha\in I}$ of objects, indexed by a partially ordered set I with the property that if α, $\beta \in I$, there is some $\gamma \in I$ with $\gamma \geq \alpha$, $\gamma \geq \beta$. In addition, one supposes there are morphisms $\varphi_{\alpha\beta} : A_\alpha \to A_\beta$ defined whenever $\alpha \leq \beta$, with the compatibility condition

$$\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}, \qquad \alpha \leq \beta \leq \gamma.$$

A (direct) **limit** for such a system is an object $A = \varinjlim A_\alpha$, together with morphisms $\psi_\alpha : A_\alpha \to A$ satisfying the compatibility condition $\psi_\alpha = \psi_\beta \circ \varphi_{\alpha\beta}$ whenever $\alpha \leq \beta$, with the universal property that compatible morphisms

$$\xi_\alpha : A_\alpha \to B, \qquad \xi_\alpha = \xi_\beta \circ \varphi_{\alpha\beta},$$

must factor as $\xi\circ\psi_\alpha$ for some $\xi : A \to B$. For example, if G is the increasing union of an increasing sequence

$$G_1 \subseteq G_2 \subseteq \cdots$$

of subgroups, it is their categorical direct limit in the category of groups (with respect to the obvious inclusion maps), and similarly if one replaces groups by rings or other algebraic objects.

1.2.5. Theorem. *Let $(R_\alpha)_{\alpha\in I}$, $(\theta_{\alpha\beta} : R_\alpha \to R_\beta)_{\alpha<\beta}$ be a direct system of rings and let $R = \varinjlim R_\alpha$ be the direct limit of the system. Then $K_0(R) \cong \varinjlim K_0(R_\alpha)$.*

Proof. Applying K_0, we obtain a directed system of abelian groups $(K_0(R_\alpha))_{\alpha\in I}$, $(\theta_{\alpha\beta,*} : K_0(R_\alpha) \to K_0(R_\beta))_{\alpha<\beta}$ and thus a limit group $\varinjlim K_0(R_\alpha)$. By the universal property of the direct limit, there is a natural map $\varinjlim K_0(R_\alpha) \to K_0(R)$. We want to show this is an isomorphism. To prove surjectivity, first observe that each $p \in \mathrm{Idem}(R)$ is a matrix with finitely many entries, each one of which must come from some R_α. If we choose γ greater than or equal to all of these indices α, then p is the image of a matrix in $\mathrm{Idem}(R_\gamma)$, hence the class $[p]$ of p in $K_0(R)$ is in the image of the natural map $K_0(R_\gamma) \to K_0(R)$, hence in the image of $\varinjlim K_0(R_\alpha) \to K_0(R)$. Since the $[p]$, $p \in \mathrm{Idem}(R)$, generate $K_0(R)$, this proves surjectivity.

Now we prove injectivity. Suppose $x \in \varinjlim K_0(R_\alpha)$ and $x \mapsto 0$ in $K_0(R)$. We may suppose x comes from $K_0(R_\alpha)$ for some α and is of the form $[p]-[q]$, p, $q \in \mathrm{Idem}(R_\alpha)$. The fact that $x \mapsto 0$ means that the images of p and of q in $\mathrm{Idem}(R)$ are stably isomorphic in the sense of (1.1.4). Without loss of generality, we may first add on zeroes in the lower right corners of p and q, then replace p and q by $p \oplus 1_r$ and $q \oplus 1_r$, so that when mapped into $\mathrm{Idem}(R)$, p and q represent the same element of $\mathrm{Proj}\, R$, hence are conjugate under $GL(R)$. (This is by Theorem 1.2.3.) Once again, the matrix that does the conjugating must come from some $GL(R_\gamma)$, $\gamma \geq \alpha$, and then $[p] - [q] \mapsto 0$ in $K_0(R_\gamma)$, hence $x = 0$ in the direct limit. □

1.2.6. Example: a ring with vanishing K_0. We shall also use Theorem 1.2.3 to construct a ring R for which all projective modules are stably isomorphic to one another (in the sense of (1.1.4)), hence for which $K_0(R) = 0$. Let k be a field and let V be an infinite-dimensional vector space over k. Let $R = \mathrm{End}_k(V)$. If p, $q \in \mathrm{Idem}(R)$, then p and q are idempotents in some $M_n(R)$. Consider $p \oplus 1 \oplus 0$ and $q \oplus 1 \oplus 0$ in

$$\begin{aligned} M_{n+2}(R) &\cong \mathrm{End}_k(k^{n+2}) \otimes_k R \cong \mathrm{End}_k(k^{n+2}) \otimes_k \mathrm{End}_k(V) \\ &\cong \mathrm{End}_k(V^{n+2}) \cong \mathrm{End}_k(V) \cong R, \end{aligned}$$

since V^{n+2} and V have the same dimension over k when V is infinite-dimensional. Now $0 \le \operatorname{rank} p \le \dim_k(V^n) = n \dim V = \dim V$, and similarly $0 \le \operatorname{rank} q \le \dim V$, whereas $\operatorname{rank} 1_R = \dim V$. So

$$\dim V \le \operatorname{rank}(p \oplus 1 \oplus 0) \le \dim V + \dim V = \dim V$$

and $\operatorname{rank}(p \oplus 1 \oplus 0) = \dim V$. Similarly, $\operatorname{rank}(q \oplus 1 \oplus 0) = \dim V$ and

$$\begin{aligned} \operatorname{rank}\left((1 \oplus 1 \oplus 1)(p \oplus 1 \oplus 0)\right) &= \operatorname{rank}\left((1-p) \oplus 0 \oplus 1\right) = \dim V, \\ \operatorname{rank}\left((1 \oplus 1 \oplus 1)(q \oplus 1 \oplus 0)\right) &= \operatorname{rank}\left((1-q) \oplus 0 \oplus 1\right) = \dim V. \end{aligned}$$

Since $p \oplus 1 \oplus 0$ and $q \oplus 1 \oplus 0$ are idempotent endomorphisms of a vector space and have the same rank and corank, they are conjugate. Hence $p \oplus 1 \oplus 0 \cong q \oplus 1 \oplus 0$ and hence $[p] = [q]$ in $K_0(R)$.

1.2.7. Exercise: construction of a simple ring for which K_0 is not finitely generated. Let k be a (commutative) field and define a map of rings $\phi_n : M_{2^n}(k) \to M_{2^{n+1}}(k)$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Show that the induced map on K_0 is multiplication by 2 (when we use the isomorphisms $K_0(M_{2^n}(k)) \cong K_0(k) \cong \mathbb{Z}$, $K_0(M_{2^{n+1}}(k)) \cong K_0(k) \cong \mathbb{Z}$ defined by Theorem 1.2.4). Deduce that if $A = \varinjlim(M_{2^n}(k), \phi_n)$, then

$$K_0(A) = \varinjlim \left(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \cdots \right) \cong \mathbb{Z}[\tfrac{1}{2}].$$

Note that since matrix rings over fields are simple, A is a limit of simple rings and so is simple. (One needs to show that if $x \in A$, then the 2-sided ideal generated by x is everything, or that there exist elements a_j, b_j in A with $1 = \sum_j a_j x b_j$. However, x must lie in (the canonical image of) one of the approximating rings M_{2^n}, and one can construct the elements there by simplicity of the matrix ring.)

1.2.8. Exercise: behavior of K_0 under Cartesian products. Let $R = R_1 \times R_2$, a Cartesian product of rings. By using the obvious decompositions $\mathrm{Idem}(R) = \mathrm{Idem}(R_1) \times \mathrm{Idem}(R_2)$ and $GL(R) = GL(R_1) \times GL(R_2)$, show that $\mathrm{Proj}\, R \cong \mathrm{Proj}\, R_1 \times \mathrm{Proj}\, R_2$ and hence that $K_0(R) \cong K_0(R_1) \oplus K_0(R_2)$. Generalize to arbitrary finite products.

1.2.9. Exercise: construction of rings with quite general countable torsion-free K_0.

(1) Use Theorem 1.2.4 and Exercise 1.2.8 to show that if k is a field and R is a finite product of r matrix rings over k, then $K_0(R) \cong \mathbb{Z}^r$.

(2) Show that a homomorphism $\mathbb{Z}^r \to \mathbb{Z}^{r'}$ given by right multiplication by a matrix $A \in M_{r,r'}(\mathbb{Z})$ can be implemented by a unital homomorphism of rings as in (1) if and only if all the entries of the matrix A are non-negative and no row or column of A is identically 0.

(3) Generalizing Exercise 1.2.7, show that any countable torsion-free abelian group can be realized as $K_0(R)$ of a ring. (Write the group as an inductive limit of a sequence of finitely generated free abelian groups, with maps given by matrices as in (2).)

3. K_0 of PIDs and local rings

We're now ready to begin computing K_0 for more rings of practical interest. Recall that a PID (*principal ideal domain*) is a commutative integral domain (ring without zero-divisors) in which every ideal can be generated by a single element. Standard examples are $\mathbb{Z}$ and a polynomial ring in one variable over a field. More general polynomial rings will be discussed in Chapter 3.

1.3.1. Theorem. *If R is a PID, every finitely generated projective module over R is isomorphic to R^n for some unique n, called the* **rank** *of the module. The rank induces an isomorphism $K_0(R) \to \mathbb{Z}$.*

Proof. Needless to say, this follows from the general structure theorem for finitely generated modules over a PID, which we presume most readers have seen in an algebra course. However, since there's an easier proof that will motivate what we'll do for Dedekind rings, we give it here. Let M be a finitely generated projective module over R. We may assume that M is embedded in some R^n. We argue by induction on n that M is isomorphic to R^k for some $k \leq n$. If $n = 0$, there is nothing to prove. So assume the result for smaller values of n and let $\pi : R^n \to R$ be projection on the last coordinate. Note that π maps M onto an R-submodule of R, *i.e.*, an ideal. If $\pi(M) = 0$, then we may view M as embedded in $\ker \pi \cong R^{n-1}$ and use the inductive hypothesis. Otherwise, $\pi(M)$ is a non-zero ideal and so is isomorphic to R as an R-module (by the PID property). So $\pi(M)$ is projective and hence M splits as $\ker \pi|_M \oplus R$ (recall the remarks in 1.1.1). Since we may view $\ker \pi|_M$ as embedded in R^{n-1}, we may apply the inductive hypothesis to conclude that it's isomorphic to $R^{k'}$, $k' \leq n-1$. So $M \cong R^k$ with $k = k' + 1 \leq (n-1) + 1 = n$.

Finally, we need to know that the rank k of M is well defined. This follows from the fact that we may also characterize it as the dimension of $F \otimes_R M$ over F, where F is the field of fractions of R. The calculation of K_0 is as in 1.1.6. $\square$

Remark. The proof actually showed a little more, namely that every submodule M of a finitely generated free R-module is free. We never explicitly used the fact that M is projective.

For any ring R with unit, there is a unique ring homomorphism $\iota : \mathbb{Z} \to R$ sending 1 to the identity element of R. By Theorem 1.3.1, $K_0(\mathbb{Z}) \cong \mathbb{Z}$, so we obtain a map $\iota_* : \mathbb{Z} \to K_0(R)$. The image of this map is the subgroup of $K_0(R)$ generated by the finitely generated free R-modules. In general, the map ι_* need not be injective; in Example 1.2.6, it is even 0.

1.3.2. Definition. The **reduced K_0-group** of R is the quotient

$$\tilde{K}_0(R) = K_0(R)/\iota_*(\mathbb{Z}).$$

Note that we have seen that $\tilde{K}_0(R)$ vanishes if R is a division ring or a PID. In general, $\tilde{K}_0(R)$ measures the *non-obvious* part of $K_0(R)$. We will see in the next section that it recaptures a famous classical invariant of Dedekind rings.

Next we compute K_0 for local rings (which are not necessarily commutative). We begin with a review of some useful general ring theory.

1.3.3. Definition. A ring R (not necessarily commutative) is **local** if the non-invertible elements of R constitute a proper 2-sided ideal M of R. Examples of commutative local rings include $k[[t]]$, the ring of formal power series over a field k, and $\mathbb{Z}_{(p)}$, the ring of rational numbers of the form $\frac{a}{b}$, where p is a prime, $b \neq 0$, and $p \nmid b$. For an example of a non-commutative local ring, let S be any non-commutative unital k-algebra, where k is a field, and let

$$R = \left\{a_0 + a_1 t + a_2 t^2 + \cdots \in S[[t]] : a_0 \in k\right\}.$$

Since any power series in R with $a_0 \neq 0$ is invertible (by the usual algorithm for inverting power series), and since the elements in R with $a_0 = 0$ constitute an ideal, R is a local ring.

1.3.4. Proposition. *For a ring R (not necessarily commutative), the following are equivalent:*

(*a*) *R has a unique maximal left ideal, and a unique maximal right ideal, and these coincide.*
(*b*) *R is local.*

Proof. (b) $\Rightarrow$ (a). If R is local with ideal M of non-invertible elements, no element of $R \smallsetminus M$ can lie in a proper left ideal or proper right ideal, hence M is both the unique maximal left ideal and the unique maximal right ideal.

Now let's show (a) $\Rightarrow$ (b). Assume (a) and let $x \in R$. If x does not have a left inverse, then Rx is a proper left ideal, which by Zorn's Lemma lies in a maximal left ideal, which by (a) is unique. Similarly, if x does not have a right inverse, then x lies in the unique maximal right ideal. Thus all non-invertible elements lie a proper 2-sided ideal M. □

1.3.5. Corollary. *In a local ring, an element with a one-sided inverse is invertible.*

Remark. Note that replacing (a) by the condition that R has a unique maximal 2-sided ideal gives a very different class of rings in the non-commutative case. A simple ring R (one with no 2-sided ideals other than 0 and R) need not be local; a matrix ring over a field is a counterexample, since a sum of singular matrices need not be singular.

1.3.6. Definition. If R is any ring, the **radical** (or **Jacobson radical**) of R is the intersection of the maximal left ideals. By Proposition 1.3.4, in a local ring, the radical coincides with the maximal ideal.

1.3.7. Proposition. *For any ring R, the radical of R is a 2-sided ideal.*

Proof. If I is a maximal left ideal, the annihilator of R/I in R certainly is contained in I. Hence

$$\bigcap_{I \text{ a max. left ideal}} \operatorname{Ann}_R(R/I) \subseteq \bigcap_I I = \operatorname{rad} R.$$

On the other hand,

$$\operatorname{Ann}_R(R/I) = \bigcap_{\dot{x} \in R/I,\, \dot{x} \neq 0} \operatorname{Ann}_R(\dot{x}),$$

an intersection of maximal left ideals. So $\operatorname{rad} R$ is exactly the intersection of the $\operatorname{Ann}_R(R/I)$, and so is 2-sided. □

Remark. The proof showed that the radical of R is the set of elements that annihilate all simple left R-modules. One observation we will need later is that since every simple module for $M_n(R)$ is isomorphic to one of the form $R^n \otimes_R M$ with M a simple R-module, any matrix all of whose entries lie in $\operatorname{rad} R$ must annihilate all such modules, hence must be in the radical of $M_n(R)$.

1.3.8. Proposition. *For any ring R, the radical coincides with*

$$\{x \in R : \forall a \in R, 1 - ax \text{ has a left inverse}\}$$

and with the intersection of the maximal right ideals.

Proof. First we show that $\operatorname{rad} R$ is contained in the indicated set. If x lies in every maximal left ideal, then Rx lies in every maximal left ideal. Suppose $a \in R$ and $1 - ax$ does not have a left inverse. Then $1 - ax$ lies in a proper left ideal, hence in a maximal left ideal M. Since $ax \in M$, we have $1 \in M$, a contradiction.

Conversely, suppose that for all $a \in R$, $1 - ax$ has a left inverse. Let M be a maximal left ideal. If $x \notin M$, then $Rx + M = R$. Thus for some $a \in R$, $1 - ax \in M$, a contradiction. So $\operatorname{rad} R$ coincides with

$$\{x \in R : \forall a \in R, 1 - ax \text{ has a left inverse}\}.$$

Similarly, we can define the right radical

$$\begin{aligned}\text{r-rad}\, R &= \bigcap \text{max. right ideals}\\ &= \{x \in R : \forall a \in R,\ 1 - xa \text{ has a right inverse}\}.\end{aligned}$$

Since $\operatorname{rad} R$ is a right ideal by 1.3.7, if $x \in \operatorname{rad} R$ and $a \in R$, there is a $c \in R$ with $(1-c)(1-xa) = 1$. This gives $(1-xa)(1-c) = 1 + xac - cxa$, and since $x \in \operatorname{rad} R$, $xac - cxa \in \operatorname{rad} R$. Thus $1 + xac - cxa$ has a left inverse, which shows $1-c$ has a left inverse. Since it also has a right inverse, namely $1 - xa$, they coincide, and $1 - xa$ is invertible with inverse $1 - c$. Hence $\operatorname{rad} R \subseteq \text{r-rad}\, R$. By symmetry, $\text{r-rad}\, R \subseteq \operatorname{rad} R$ and the two coincide. □

1.3.9. Theorem (Nakayama's Lemma). *Suppose R is a ring and M is a finitely generated R-module such that $(\operatorname{rad} R)M = M$. Then $M = 0$.*

Proof. Suppose $M \neq 0$. Pick a set of generators $x_1, \dots, x_m$ for M with m as small as possible. (This implies in particular that each $x_j \neq 0$.) By the assumption that $(\operatorname{rad} R)M = M$, there are elements $r_1, \dots, r_m$ in $\operatorname{rad} R$ such that

$$x_m = r_1 x_1 + \cdots + r_m x_m.$$

Hence

$$(1 - r_m)x_m = r_1 x_1 + \cdots + r_{m-1} x_{m-1}.$$

By Proposition 1.3.8, $1 - r_m$ is invertible; hence x_m can be expressed as a linear combination of $x_1, \dots, x_{m-1}$. This contradicts the assumption that m was as small as possible. □

1.3.10. Corollary. *If R is a ring, M is a finitely generated R-module, and $x_1, \dots, x_m \in M$, then $x_1, \dots, x_m$ generate M if and only if their images $\dot{x}_1, \dots, \dot{x}_m$ generate $M/(\operatorname{rad} R)M$ as an $R/\operatorname{rad} R$-module.*

Proof. The "only if" statement is trivial. Suppose $\dot{x}_1, \dots, \dot{x}_m$ generate $M/(\operatorname{rad} R)M$. Let $N = Rx_1 + \cdots + Rx_m \subseteq M$ and consider M/N. This satisfies the hypotheses of Nakayama's Lemma, so $M/N = 0$ and $M = N$. □

1.3.11. Theorem. *If R is a local ring, not necessarily commutative, then every projective finitely generated R-module is free with a uniquely defined rank. In particular, $K_0(R) \cong \mathbb{Z}$ with generator the isomorphism class of a free module of rank 1.*

Proof. Note $R/\operatorname{rad} R$ is a division ring D. If M is a finitely generated projective R-module, we may assume $M \oplus N = R^k$ for some k. Then $M/(\operatorname{rad} R)M$ and $N/(\operatorname{rad} R)N$ are D-modules, hence are free, say of ranks m and n, respectively, with $m+n = k$. Choose basis elements and pull them back to elements $x_1, \dots, x_m \in M$, $x_{m+1}, \dots, x_k \in N$. By Corollary 1.3.10, these generate R^k. We want to show that $x_1, \dots, x_k$ are a free basis for R^k. This will show in particular that $x_1, \dots, x_m$ are a linearly independent generating set for M, so that M is free with the uniquely determined rank

$$\operatorname{rank} M = \dim_D M/(\operatorname{rad} R)M.$$

Let $e_1, \dots, e_k$ be the standard free basis for R^k. Since we now have two generating sets for R^k, each can be expressed in terms of the other, and there are elements a_{ij}, $b_{ij} \in R$ with

$$e_i = \sum_{j=1}^{k} a_{ij} x_j, \quad x_i = \sum_{j=1}^{k} b_{ij} e_j.$$

Thus we get

$$e_i = \sum_{j=1}^{k} a_{ij} \sum_{l=1}^{k} b_{jl} e_l,$$

so

$$\sum_{j=1}^{k} \sum_{l=1}^{k} (a_{ij} b_{jl} - \delta_{il}) e_l = 0,$$

and if $A = (a_{ij})$, $B = (b_{ij})$, this means (since the e_l are linearly independent) that $AB = I$. Substituting the other way, we get

$$\sum_{j=1}^{k} \sum_{l=1}^{k} (b_{ij} a_{jl} - \delta_{il}) x_l = 0,$$

and since the x_l are linearly independent modulo the radical of R, this shows $BA - I \in M_n(\operatorname{rad} R) \subsetneq \operatorname{rad} M_n(R)$ (using the remark following 1.3.7). By Proposition 1.3.8, BA is invertible, hence B is invertible. Since A was a left inverse for B, this shows it is also a right inverse, *i.e.*, $BA = I$. This proves the $x_1, \dots, x_m$ are a free basis for R^k. □

Part of the interest in local rings stems from the importance of localization as a technique for studying more general commutative rings. Recall that if R is a commutative ring, the set $\operatorname{Spec} R$ of prime ideals in R becomes a topological space, called the **spectrum** of R, when equipped with the so-called **Zariski topology**. The closed sets E_I in this topology are parameterized by the ideals I of R, where for $I \trianglelefteq R$,

$$E_I = \{P \in \operatorname{Spec} R : P \supseteq I\}.$$

1.3.12. Proposition. *Let R be a commutative ring and let* $\operatorname{Spec} R$ *be its prime ideal spectrum. If P is a finitely generated projective R-module, then P has a well-defined* **rank function** $\operatorname{rank} P : \operatorname{Spec} R \to \mathbb{N}$, *and this function is continuous. In particular, if R is an integral domain, it is constant. Furthermore, for any commutative ring R, there is a splitting $K_0(R) \cong \mathbb{Z} \oplus \tilde{K}_0(R)$.*

Proof. Given $\mathfrak{p} \in \operatorname{Spec} R$, $P_{\mathfrak{p}} \cong R_{\mathfrak{p}} \otimes_R P$ is a finitely generated projective module over $R_{\mathfrak{p}}$, which is a local ring. So by Proposition 1.3.11, it is free with a well-defined rank, which is the dimension of the associated module

over the field $R_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, where $\mathfrak{m}_{\mathfrak{p}}$ is the unique maximal ideal of $R_{\mathfrak{p}}$. Since $\mathfrak{m}_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R \mathfrak{p}$, the rank at $\mathfrak{p}$ may also be computed by first taking $P/\mathfrak{p}P$, which is a module over the integral domain $R/\mathfrak{p}$, then taking the dimension of the associated vector space over the field of fractions of $R/\mathfrak{p}$.

Next we prove continuity of the rank function. One way of seeing this is via the idempotent picture. Suppose P is defined by an idempotent matrix $p \in M_n(R)$. Then $\operatorname{rank}_{\mathfrak{p}} P = k$ if and only if the image of p in $M_n(R/\mathfrak{p})$ has rank k. Thus $\operatorname{rank}_{\mathfrak{p}} P \le k$ if and only if every $(k+1) \times (k+1)$ submatrix of p has a determinant in $\mathfrak{p}$. This is clearly a closed condition, since it's equivalent to saying $\mathfrak{p}$ contains certain specific elements of R, and the most general closed set in $\operatorname{Spec} R$ is of the form $\{\mathfrak{p} : \mathfrak{p} \supseteq I\}$ for some ideal I. But it's also an open condition since

$$\operatorname{rank} p \le k \Longleftrightarrow \operatorname{rank}(1-p) \ge n-k.$$

To prove the final remarks, note that if R is an integral domain, then (0) is an open point in $\operatorname{Spec} R$, hence $\operatorname{Spec} R$ is connected and $\operatorname{rank} P$ must be constant. The splitting map $K_0(R) \to \mathbb{Z}$ for a general commutative ring is obtained simply by fixing a point $\mathfrak{p} \in \operatorname{Spec} R$ and computing the rank there. □

1.3.13. Exercise (The finite generation hypothesis in Nakayama's Lemma is necessary). Show from Nakayama's Lemma that if R is a left Noetherian ring and $(\operatorname{rad} R)^2 = \operatorname{rad} R$, then $\operatorname{rad} R = 0$.

Let R be the ring of germs at 0 of continuous functions $\mathbb{R} \to \mathbb{R}$. Show that R is a local ring, with radical the germs of functions f with $f(0) = 0$, and that $(\operatorname{rad} R)^2 = \operatorname{rad} R$. ($R$ is not Noetherian, which is why this is possible.)

1.3.14. Exercise. Compute $K_0(\mathbb{Z}/(m))$ in terms of m, for any integer $m > 0$. Hint: write m as a product of prime powers and use the Chinese Remainder Theorem to get a corresponding splitting of $\mathbb{Z}/(m)$ as a product of local rings. Then use Theorem 1.3.11 and Exercise 1.2.8.

4. K_0 of Dedekind domains

A particularly rich family of rings for which K_0 is interesting are the Dedekind domains. We begin with the definition and basic properties of these domains, and then proceed to the most important examples, namely, the rings of integers in number fields. **In this section R will always denote a commutative integral domain embedded in its field of fractions F.**

1.4.1. Definition. A non-zero R-submodule I of F is called a **fractional ideal** of R if there exists some $a \in R$ with $aI \subseteq R$. Clearly a non-zero ideal of R may be viewed as a fractional ideal; for emphasis, such an ideal is called an **integral ideal**. Also, if $\frac{a}{b} \in F$ (a, $b \in R$; a, $b \neq 0$), then $R(\frac{a}{b})$ is a fractional ideal since $bR(\frac{a}{b}) \subseteq R$. Such a fractional ideal is called

principal. One can multiply fractional ideals, and under multiplication they form an abelian monoid with identity element R.

1.4.2. Definition. R is called a **Dedekind domain** or **Dedekind ring** if the fractional ideals under multiplication are a group, *i.e.*, if given a fractional ideal I, there is a fractional ideal I^{-1} with $I^{-1}I = R$. Observe that necessarily $I^{-1} = \{a \in F : aI \subseteq R\}$. For if $J = \{a \in F : aI \subseteq R\}$, then $I^{-1}I \subseteq R$ so $I^{-1} \subseteq J$, but then

$$R = II^{-1} \subseteq IJ \subseteq R,$$

so $II^{-1} = IJ$ and $I^{-1} = I^{-1}IJ = J$.

1.4.3. Definition. Note that the principal fractional ideals are a subgroup of the fractional ideals isomorphic to $F^\times/R^\times$. The **class group** of the Dedekind domain R is defined to be

$$C(R) = \{\text{group of fractional ideals}\}/\{\text{group of principal fractional ideals}\}.$$

1.4.4. Proposition. *The class group of a Dedekind domain may also be identified with the set of R-module isomorphism classes of integral fractional ideals.*

Proof. Clearly any fractional ideal is isomorphic to an integral one I (via multiplication by some element of $R \smallsetminus \{0\}$). And if $I = (J)(Ra)$, then multiplication by a implements an R-module isomorphism $J \to I$. Conversely, if $\varphi : I \to J$ is an R-module isomorphism and $a_0 \in I \smallsetminus \{0\}$, then for any $a \in I$,

$$\varphi(a_0 a) = a_0\varphi(a) = a\varphi(a_0),$$

so $\varphi(a_0)I = a_0 J$ and $[I] = [J]$ in $C(R)$. $\square$

1.4.5. Theorem. *If R is Dedekind, then every fractional ideal is finitely generated and projective. In particular, R is Noetherian.*

Proof. Let I be a fractional ideal. Since $I^{-1}I = R$, there are elements $x_1, \dots, x_n \in I^{-1}$ and $y_1, \dots, y_n \in I$ such that $\sum_{i=1}^n x_i y_i = 1$. If $b \in I$, then $b = \sum (bx_i)y_i$ with $bx_i \in I^{-1}I = R$, so $y_1, \dots, y_n$ generate I. Thus I is finitely generated. Since every ideal of R is finitely generated, R is Noetherian.

But in addition, the homomorphism $R^n \to I$ defined by $(a_1, \dots, a_n) \mapsto \sum a_i y_i$ splits, with right inverse $b \mapsto (bx_1, \dots, bx_n)$, by the same calculation. So I is isomorphic to a direct summand in R^n and so is projective by Theorem 1.1.2. $\square$

1.4.6. Corollary. *If R is Dedekind, then every finitely generated projective R-module is isomorphic to a direct sum of ideals. In particular, the isomorphism classes of the ideals generate $K_0(R)$.*

Proof. We use the same argument as in the proof of Theorem 1.3.1. Let M be a finitely generated projective module over R. We may assume

that M is embedded in some R^n. We argue by induction on n that M is isomorphic to a direct sum of k ideals for some $k \le n$. If $n = 0$, there is nothing to prove. So assume the result for smaller values of n and let $\pi : R^n \to R$ be projection on the last coordinate. Note that π maps M onto an R-submodule of R, *i.e.*, an ideal. If $\pi(M) = 0$, then we may view M as embedded in $\ker \pi \cong R^{n-1}$ and use the inductive hypothesis. Otherwise, $\pi(M)$ is a non-zero ideal I and so is projective by Theorem 1.4.5. Hence M splits as $\ker \pi|_M \oplus I$ (recall the remarks in 1.1.1). Since we may view $\ker \pi|_M$ as embedded in R^{n-1}, we may apply the inductive hypothesis to conclude that it's isomorphic to a direct sum of k' ideals, $k' \le n-1$. So M is a direct sum of k ideals with $k = k' + 1 \le (n-1) + 1 = n$. □

Our next goal is to relate $K_0(R)$ to $C(R)$, but first we need to develop more of the theory of Dedekind domains. This will also enable us to prove a useful characterization of Dedekind domains that will show that the ring of algebraic integers in a number field is a Dedekind domain. The next theorem generalizes the "fundamental theorem of arithmetic" (unique factorization of an integer into primes).

1.4.7. Theorem. *In a Dedekind domain R, every prime integral ideal is maximal. And every proper integral ideal can be factored uniquely (up to renumbering of the factors) into prime (or maximal) ideals. The group of fractional ideals is the free (multiplicative) abelian group on the (non-zero) prime ideals.*

Proof. (a) Suppose $0 \subsetneqq I \subsetneqq R$ and I is prime but not maximal. Then there exists an integral ideal J with $I \subsetneqq J \subsetneqq R$. Let $K = J^{-1}I$; since $I \subsetneqq J$, $K \subsetneqq J^{-1}J = R$. Since $JK = I$ and I is prime but $J \not\subseteq I$, $K \subseteq I$. But then $I = JK \subseteq JI \subsetneqq RI = I$, a contradiction. So I is maximal.

(b) **Existence of factorizations.** Let

$$\mathcal{C} = \{\text{proper integral ideals that are not products of prime ideals}\}.$$

If this is empty, we're done. Otherwise, since every ascending chain of ideals in R has a maximal element (R is Noetherian by Theorem 1.4.5), $\mathcal{C}$ has a maximal element I by Zorn's Lemma. I can't be a maximal ideal (otherwise it would be prime itself and would have a trivial factorization $I = I$) so $I \subsetneqq I_1 \subsetneqq R$ for some ideal I_1. Let $I_2 = I_1^{-1}I$. This is also an ideal in R since $I \subseteq I_1$, and since $I \subsetneqq I_1$, it is a proper ideal containing I properly. Since I_1 and I_2 are both strictly bigger than I and I was maximal in $\mathcal{C}$, both have factorizations into primes. But since $I = I_1 I_2$, multiplying gives a factorization of I, a contradiction.

(c) **Uniqueness of factorizations.** Suppose $P_1 \cdots P_m = Q_1 \cdots Q_n$ with P_i, Q_j prime and $m \le n$. Then $P_1 \supseteq P_1 \cdots P_m = Q_1 \cdots Q_n$ so some Q_j lies in P_1. After renumbering if necessary, we may assume $Q_1 \subseteq P_1$. Write $Q_1 = S_1 P_1$ by the Dedekind property (where $S_1 = P_1^{-1} Q_1$). Multiplying through by P_1^{-1} gives $P_2 \cdots P_m = S_1 Q_2 \cdots Q_n$. Continuing by induction, we get down to the case where $m = 1$, in which case it is clear

that we must have $n = 1$ and $Q_1 = P_1$. So factorizations into primes are unique.

(d) Clearly there's a map from the free abelian group on the prime ideals into the multiplicative group of the fractional ideals. By (b) above, it's surjective. If there is something non-trivial in the kernel, we have $P_1^{n_1} \cdots P_r^{n_r} = R$ for some distinct prime ideals P_j and some $n_j \in \mathbb{Z}$. If for some j, $n_j < 0$, multiply through by $P_j^{|n_j|}$. Then we end up with some ideal in R having two distinct factorizations, contradicting (c). □

1.4.8. Lemma. *Let R be any commutative ring and let I_1, I_2 be ideals in R. If $I_1 + I_2 = R$, then $I_1 I_2 = I_1 \cap I_2$.*

Proof. Clearly $I_1 I_2 \subseteq I_1 \cap I_2$. On the other hand, if $a_1 \in I_1$, $a_2 \in I_2$, and $a_1 + a_2 = 1$, then for $x \in I_1 \cap I_2$, $x = a_1 x + a_2 x \in I_1 I_2 + I_2 I_1 = I_1 I_2$. □

1.4.9. Lemma. *Let R be a Dedekind domain and let I be a fractional ideal, J an integral ideal. There exists $a \in I$ such that $I^{-1}a + J = R$.*

Proof. Let $P_1, \ldots, P_r$ be the distinct prime ideals that occur in the factorization of J given by Theorem 1.4.7. Choose $a_i \in IP_1 \cdots \hat{P_i} \cdots P_r$ with $a_i \notin IP_1 \cdots P_r$. Let $a = \sum a_i$. Note $a_i I^{-1} \subseteq P_j$ if $j \neq i$, but $a_i I^{-1} \not\subseteq P_i$, sice otherwise we'd have

$$a_i I^{-1} \subseteq \bigcap P_j = P_1 \cdots P_r \quad \text{by iterated use of (1.4.8),}$$

hence

$$a_i \in IP_1 \cdots P_r,$$

a contradiction. Now note that $I^{-1}a \not\subseteq P_j$ for any j. It's an integral ideal and this says $I^{-1}a + J$ can't be divisible by any P_j. But it can't be divisible by any other prime ideal, either, by the choice of a, so it can't be a proper ideal and must be all of R. □

This implies that a Dedekind domain doesn't miss being a PID by very much. If R is a PID, any fractional ideal is singly generated. In a Dedekind domain, the best one can say along these lines is the following.

1.4.10. Corollary. *If R is a Dedekind domain, any fractional ideal of R can be generated by at most two elements.*

Proof. Let I be a fractional ideal, $0 \neq b \in I$. Let $J = bI^{-1}$, which is an integral ideal. By Lemma 1.4.9, there is some $a \in I$ with $aI^{-1} + bI^{-1} = R$. Then $I = Ra + Rb$. □

1.4.11. Lemma. *Suppose R is a Dedekind domain and I_1, I_2 are fractional ideals for R. Then $I_1 \oplus I_2 \cong R \oplus I_1 I_2$ as R-modules.*

Proof. Choose $a_1 \neq 0$ in I_1 and let $J = a_1 I_1^{-1}$, which is an integral ideal. Apply Lemma 1.4.9 with $I = I_2$. We get $a_2 \in I_2$ such that $I_2^{-1}a_2 + a_1 I_1^{-1} = R$. Choose $b_1 \in I_1^{-1}$, $b_2 \in I_2^{-1}$ with $a_1 b_1 + a_2 b_2 = 1$. Then

$$\begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

showing that $\begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}$, and

$$(x_1, x_2) \mapsto (x_1, x_2) \begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix}$$

gives the desired isomorphism (with inverse given by multiplication by the inverse matrix). □

1.4.12. Theorem. *Let R be a Dedekind domain. Then any projective R-module of rank k can be written as $R^{k-1} \oplus I$, with I an ideal, and the isomorphism class of I is uniquely determined. If P and Q are finitely generated projective modules of the same rank k, say $P \cong R^{k-1} \oplus I_1$ and $Q \cong R^{k-1} \oplus I_2$ for ideals I_1 and I_2, the map $[P] - [Q] \mapsto I_1 I_2^{-1}$ sets up an isomorphism from $\tilde{K}_0(R)$ to $C(R)$. In fact,*

$$\left[R^{k-1} \oplus I\right] \mapsto (k, [I])$$

sets up an isomorphism of abelian groups

$$K_0(R) \to \mathbb{Z} \oplus C(R).$$

As a commutative ring (see 1.1.9),

$$K_0(R) \cong \{(k, [I]) : k \in \mathbb{Z}, \quad [I] \in C(R)\},$$

with the operations

$$(1.4.13) \qquad \begin{cases} (k, [I]) + (k', [I']) = (k + k', [I][I']), \\ (k, [I]) \cdot (k', [I']) = (kk', [I]^{k'}[I']^{k}), \\ \text{rank} : (k, [I]) \mapsto k \in \mathbb{Z}. \end{cases}$$

Proof. By Corollary 1.4.6, every finitely generated projective module P over R is isomorphic to a direct sum $I_1 \oplus \cdots \oplus I_k$ of ideals, and by Proposition 1.3.12, P also has a well-defined rank. If I is an ideal, then $\operatorname{rank} I = \dim_F(F \otimes_R I) = \dim_F F = 1$, so the rank of P is just the number k of ideals in a direct sum decomposition. Using Lemma 1.4.11 iteratively, we can rework the decomposition into the form $R^{k-1} \oplus I$ with a single ideal I. The only problem is to show that if

$$(R^{k-1} \oplus I_1) \cong (R^{k-1} \oplus I_2),$$

then $I_1 \cong I_2$ as R-modules, or (equivalently, by Proposition 1.4.4) $[I_1] = [I_2]$ in $C(R)$. Once this is done, the formulae 1.4.13, and the identification of $\tilde{K}_0(R)$ with $C(R)$, then follow upon taking the direct sum or tensor product of $R^k \oplus I$ and of $R^{k'} \oplus I'$ and applying Lemma 1.4.11 iteratively.

So suppose we have an isomorphism

$$\alpha : (R^{k-1} \oplus I_1) \xrightarrow{\cong} (R^{k-1} \oplus I_2)$$

with inverse β. Since any R-module map from one ideal to another is given by multiplication by an element of F (compare the proof of Proposition 1.4.4), α and β are induced by right multiplication by $k \times k$ matrices A and B (with entries in F) which are inverses of each other. Now if X is the diagonal matrix with diagonal entries $(1, 1, \dots, 1, x)$, where $x \in I_1$, then right multiplication by X maps R^k into $R^{k-1} \oplus I_1$, hence right multiplication by XA maps R^k into $R^{k-1} \oplus I_2$. The rows of XA are the images of the standard basis vectors for R^k under this map, so they have their first $k-1$ entries in R and last entry in I_2. Thus expansion of the determinant along the last column shows that $\det(XA) \in I_2$. Since $\det X = x$, we obtain the condition $x \det A \in I_2$ for all $x \in I_1$. Similarly $y \det B = y(\det A)^{-1} \in I_1$ for all $y \in I_2$. So multiplication by $\det A$ implements an isomorphism from I_1 to I_2. □

We proceed now to the characterization of Dedekind domains. This will eventually make it possible to show that the rings of integers in number fields are Dedekind domains. Recall that a subring R of another ring S is called **integrally closed** in S if any element of S which is a root of a monic polynomial with coefficients in R actually lies in R.

1.4.14. Lemma. *Let R be a Noetherian integral domain which is integrally closed in its field of fractions F. Suppose I is a fractional ideal of R. Then $\{s \in F : sI \subseteq I\} = R$.*

Proof. Since R is Noetherian, I is finitely generated. Let $S = \{s \in F : sI \subseteq I\}$. Clearly $R \subseteq S$. But if $s \in S$, s is integral over R, by the following argument. Choose generators a_j for I. Then there are elements $b_{jk} \in R$ such that $sa_j = \sum b_{jk} a_k$. Thus if $B = (b_{jk})$, s is an eigenvalue of B and so is a root of its characteristic polynomial, which is a monic polynomial with coefficients in R. Hence $s \in R$ since R is integrally closed. Thus $S \subseteq R$. □

1.4.15. Lemma. *Let R be a Noetherian commutative ring and let I be a non-zero proper ideal of R. Then I contains a product of non-zero prime ideals.*

Proof. Suppose the result is false, and let $\mathcal{C}$ be the family of non-zero proper ideals of R which do **not** contain a product of non-zero prime ideals. Since R is Noetherian, $\mathcal{C}$ must contain a maximal element (under inclusion), say I. Clearly I is not prime, so there must be a, $b \in R$ with $ab \in I$, a, $b \notin I$. We have $I \subsetneqq I + Ra$, $I \subsetneqq I + Rb$. If $I + Ra = R$, then $(I + Ra)(I + Rb) = I + Rb \supsetneqq I$, while on the other hand $(I + Ra)(I + Rb) \subseteq I + Rab \subseteq I$, a contradiction. So $I \subsetneqq I + Ra \subsetneqq R$. Similarly $I \subsetneqq I + Rb \subsetneqq R$. Since I was maximal in $\mathcal{C}$, $I + Ra$ and $I + Rb$ do not lie in $\mathcal{C}$. Thus $I + Ra \supseteq P_1 \cdots P_r$, $I + Rb \supseteq Q_1 \cdots Q_s$, for some prime ideals P_j and Q_k. Then $I = (I + Ra)(I + Rb) \supseteq P_1 \cdots P_r Q_1 \cdots Q_s$, a contradiction. □

1.4.16. Lemma. *Let R be a Noetherian integral domain in which every prime ideal is maximal. Let I be a non-zero proper ideal of R. Then there exists $c \in F$ with $c \notin R$ such that $cI \subseteq R$.*

Proof. Let $a \neq 0$ in I. Then Ra contains by Lemma 1.4.15 a product of non-zero prime ideals, say $P_1 \cdots P_m$, and we may assume m is chosen to be minimal with this property. Let P be a maximal ideal containing I. Then

$$P_1 \cdots P_m \subseteq Ra \subseteq I \subseteq P,$$

so some $P_j \subseteq P$, say $P_1 \subseteq P$. Since all prime ideals are maximal, we have $P_1 = P$. If $m = 1$, then $I = Ra = P$ is maximal and $a^{-1} \notin R$, $a^{-1}I \subseteq R$. If $m \geq 2$, then by minimality of m, $Ra \not\supseteq P_2 \cdots P_m$. Choose $b \in P_2 \cdots P_m$ with $b \notin Ra$, and let $c = \frac{b}{a}$. Then $c \notin R$ but

$$cI \subseteq cP_1 = a^{-1}bP_1 \subseteq a^{-1}P_1 \cdots P_m \subseteq a^{-1}Ra = R. \quad \square$$

1.4.17. Theorem. *A commutative integral domain R is Dedekind if and only if it has the following three properties:*

(*a*) *Every non-zero prime ideal is maximal.*
(*b*) *R is integrally closed in its field of fractions F.*
(*c*) *R is Noetherian.*

Proof. If R is a Dedekind domain, it satisfies (c) by Theorem 1.4.5 and (a) by Theorem 1.4.7. Suppose $a \in F$, $a \neq 0$, and a is integral over R. Then a is a root of some monic polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_0, \ldots, a_{n-1} \in R$. Consider $M = R + Ra + Ra^2 + \cdots + Ra^{n-1}$. This is an R-submodule of F, and since $a^n = -a_{n-1}a^{n-1} - \cdots - a_0$, it is stable under multiplication by a. If we write $a = \frac{p}{q}$, p, $q \in R$ and $q \neq 0$, then $q^{n-1}M \subseteq R$, so M is a fractional ideal. Multiplying $aM \subseteq M$ by M^{-1} gives $aR \subseteq R$, so $a \in R$. This shows R is integrally closed.

Now we show the conditions (a)–(c) imply R is Dedekind. Suppose R satisfies (a)–(c) and I is a fractional ideal. Let $J = \{a \in F : aI \in R\}$. We want to show $IJ = R$, so that J is an inverse for I. Now IJ is an integral ideal. Let $K = \{a \in F : aIJ \in R\}$. By definition, $K(IJ) = (KJ)I \subseteq R$, so $KJ \subseteq J$. By Lemma 1.4.14, $K \subseteq R$. On the other hand, if $IJ \subsetneq R$, then $K \supsetneq R$ by Lemma 1.4.16, a contradiction. So $IJ = R$ and I is invertible. $\square$

1.4.18. Theorem. *Let F be a number field, i.e., a finite algebraic extension of $\mathbb{Q}$, and let R be the ring of algebraic integers in F, that is, the integral closure of $\mathbb{Z}$ in F. Then R is a Dedekind domain.*

Proof. We need to check the conditions of Theorem 1.4.17. Condition (b) is the easiest. $R \subseteq F$, and if $a \in F$ is integral over R, then it is integral over $\mathbb{Z}$ by "transitivity of integrality," hence already contained in R. So R is integrally closed.

To check (a), let $\mathfrak{p}$ be a non-zero prime ideal in R. Then $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$. We claim it can't be zero. Indeed, if $b \neq 0$ is in $\mathfrak{p}$,

the product $N_{\mathbb{Q}(b)/\mathbb{Q}}(b)$ of the conjugates of b (in some Galois extension $K \supseteq F$) is $\pm$ the constant term of the minimal polynomial of b, which by the assumption that $b \in R$ has coefficients in $\mathbb{Z}$. Now this product of the conjugates of b is a product of b with a product c of other algebraic integers, and since $bc \in \mathbb{Z} \subseteq F$, $c \in F$ and is integral over $\mathbb{Z}$. Hence $c \in R$ and $0 \neq bc \in Rb \cap \mathbb{Z} \subseteq \mathfrak{p} \cap \mathbb{Z}$. Thus $\mathfrak{p} \cap \mathbb{Z}$ is a non-zero prime ideal in $\mathbb{Z}$, *i.e.*, $\mathfrak{p} \cap \mathbb{Z} = (p)$ for some prime number p. Since F is a finite algebraic extension of $\mathbb{Q}$, $R/\mathfrak{p}$ must be contained in a finite algebraic extension of $\mathbb{Z}/(\mathfrak{p} \cap \mathbb{Z}) = \mathbb{Z}/(p)$, in other words in a finite field of characteristic p. Since a finite integral domain is a field, $R/\mathfrak{p}$ is a field, *i.e.*, $\mathfrak{p}$ is a maximal ideal.

It remains to check (c), *i.e.*, that R is Noetherian. One way of seeing this is by using the trace. Recall that if $x \in F$, $\mathrm{Tr}_{F/\mathbb{Q}}(x)$ is the trace of the linear operator of multiplication by x on F, when we regard F as an n-dimensional vector space over $\mathbb{Q}$, where $n = [F : \mathbb{Q}]$. The trace pairing $(x, y) \mapsto \mathrm{Tr}_{F/\mathbb{Q}}(xy)$ is a non-degenerate symmetric $\mathbb{Q}$-bilinear pairing on F (since for $x \neq 0$ in F, $\mathrm{Tr}_{F/\mathbb{Q}}(xx^{-1}) = n \neq 0$). Choose elements $\lambda_1, \dots, \lambda_n \in R$ which span F over $\mathbb{Q}$. (One may obtain such elements by taking any basis elements for F over $\mathbb{Q}$ and then multiplying them by suitably large (ordinary) integers to kill off any denominators in the coefficients of their minimal polynomials.) Then

$$x \mapsto \left(\mathrm{Tr}_{F/\mathbb{Q}}(x\lambda_1), \dots, \mathrm{Tr}_{F/\mathbb{Q}}(x\lambda_n)\right)$$

is an embedding of R into $\mathbb{Z}^n$. In particular, R is a finitely generated $\mathbb{Z}$-module, so any ascending chain of ideals in R is an ascending chain of submodules in a finitely generated $\mathbb{Z}$-module, and so terminates (since $\mathbb{Z}$ is Noetherian). Thus R is Noetherian. □

Finally, we show that the Dedekind domains given by Theorem 1.4.18, which are the main subject of study in algebraic number theory, have **finite** class groups. The computation of these groups is not easy and is a problem of major interest.

1.4.19. Theorem. *Let F be a number field, i.e., a finite algebraic extension of $\mathbb{Q}$, and let R be the ring of algebraic integers in F, that is, the integral closure of $\mathbb{Z}$ in F. Then the class group $\tilde{K}_0(R)$ is finite.*

Proof. The proof requires the notion of the **norm** of an ideal. If I is an integral ideal of R, with prime factorization $P_1^{n_1} \cdots P_r^{n_r}$, then by the Chinese Remainder Theorem,

$$R/I \cong R/P_1^{n_1} \times \cdots \times R/P_r^{n_r}.$$

Since R/P_j is a finite field for each j (by the proof of Theorem 1.4.18) and $R/P_j^{n_j}$ clearly has a composition series with n_j composition factors, each isomorphic to R/P_j, $R/P_j^{n_j}$ is finite with $|R/P_j|^{n_j}$ elements, and R/I is finite. Thus we can define

$$\|I\| = |R/I| = |R/P_1|^{n_1} \cdots |R/P_r|^{n_r} = \prod_{j=1}^{r} \|P_j\|^{n_j}.$$

It is clear that this norm is multiplicative:

$$\|I_1 I_2\| = \|I_1\| \cdot \|I_2\|.$$

If I happens to be a principal ideal (a), note that since $\mathrm{N}_{F/\mathbb{Q}}(a)$ is the determinant of the $\mathbb{Z}$-linear operator of multiplication by a on R (which is isomorphic to $\mathbb{Z}^n$ as a $\mathbb{Z}$-module), Ra has index $|\mathrm{N}_{F/\mathbb{Q}}(a)|$ in R and thus

$$\|(a)\| = |\mathrm{N}_{F/\mathbb{Q}}(a)|.$$

Recall from the proof of Theorem 1.4.18 that if P is a prime ideal with $P \cap \mathbb{Z} = (p)$, then R/P is a finite extension of $\mathbb{Z}/(p)$ of degree $\leq n = [F : \mathbb{Q}]$, so that $\|P\| = p^j$ for some j with $1 \leq j \leq n$. Thus for any $C > 0$, $\|P\| \leq C$ implies $p \leq C$ for the corresponding p. On the other hand, for a fixed prime number p, there are only finitely many prime ideals $P \subset R$ with $P \cap \mathbb{Z} = (p)$ (namely, those prime ideals occurring in the prime factorization of Rp). So putting all of this together, we see there are only finitely many ideals I satisfying $\|I\| \leq C$.

To prove the theorem, it therefore suffices to show that there is a constant $C > 0$ such that every element of $C(R)$ has a representative I with $\|I\| \leq C$. Choose a basis $\lambda_1, \ldots, \lambda_n$ for R as a $\mathbb{Z}$-module. (That such a basis exists was shown in the proof of Theorem 1.4.18.) Let Λ be the maximal absolute value of a conjugate of one of the λ_j in $\mathbb{C}$ and let $C = n^n \Lambda^n$. Choose any element of $C(R)$ and represent it by a fractional ideal of the form $K = J^{-1}$, with J an integral ideal. We will show there is another representative I for the same ideal class with $\|I\| \leq C$. Consider the set

$$S = \{a_1\lambda_1 + \cdots + a_n\lambda_n : a_j \in \mathbb{Z},\ 0 \leq a_j \leq [\|J\|^{\frac{1}{n}}].\}$$

(The square brackets denote the "greatest integer" function.) This set has

$$([\|J\|^{\frac{1}{n}}] + 1)^n > \|J\| = |R/J|$$

elements, so there must by the pigeonhole principle be two elements η and ζ of S with the same image in R/J. Let $\xi = \eta - \zeta \in J$ and let $I = (\xi)J^{-1} = (\xi)K \simeq K$. This is an integral ideal and $(\xi) = IJ$, so that

$$\|I\|\|J\| = \|(\xi)\| = |\mathrm{N}_{F/\mathbb{Q}}(\xi)|.$$

On the other hand, since ξ is the difference of two elements of S, we may write

$$\xi = a_1\lambda_1 + \cdots + a_n\lambda_n \quad \text{with } |a_j| \leq [\|J\|^{\frac{1}{n}}],$$

so

$$\begin{aligned}
\|I\|\|J\| &= |\mathrm{N}_{F/\mathbb{Q}}(\xi)| \\
&= \prod_{\sigma : F \hookrightarrow \mathbb{C}} |a_1\sigma(\lambda_1) + \cdots + a_n\sigma(\lambda_n)| \\
&\leq \prod_{\sigma : F \hookrightarrow \mathbb{C}} (|a_1||\sigma(\lambda_1)| + \cdots + |a_n||\sigma(\lambda_n)|) \\
&\leq \prod_{\sigma : F \hookrightarrow \mathbb{C}} \left(n[\|J\|^{\frac{1}{n}}]\Lambda\right) \\
&\leq n^n \|J\| \Lambda^n = C\|J\|,
\end{aligned}$$

proving the desired estimate. □

1.4.20. Exercise (Construction of a non-trivial torsion element in a class group). Let $R = \mathbb{Z}[\sqrt{-5}]$.

(1) Show that R is the algebraic closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-5})$, so that R is a Dedekind domain by Theorem 1.4.18.

(2) Show that $\mathfrak{p} = (3,\ 2+\sqrt{-5})$ is a prime ideal in R. Hint: it's easy to see that $R/\mathfrak{p}$ is a field of 3 elements, so that $\mathfrak{p}$ is a maximal ideal.

(3) Show that $\mathfrak{p}$ is not principal. Hint: show that neither of the two generators divides the other, and that if there were a single generator $a + b\sqrt{-5}$, then

$$(a + b\sqrt{-5})(c + d\sqrt{-5}) = 3 \text{ for some } a, b, c, d \in \mathbb{Z},$$

and (multiplying by complex conjugates)

$$(a^2 + 5b^2)(c^2 + 5d^2) = 9.$$

If the factorization is non-trivial, $a^2 + 5b^2 = 3$, which is impossible.

(4) Show that $\mathfrak{p}$ is an element of order 2 in the class group $C(R)$. Hint: by (2), it is not of order 1. Show that $\mathfrak{p}^2 = (2 + \sqrt{-5})$.

(5) In fact, $C(R)$ is the cyclic group of order 2 generated by $\mathfrak{p}$, though it is hard to prove this by such elementary methods. Can you supply a proof?

(6) Suppose we replace R by the integral closure R' of $\mathbb{Z}_{(3)}$ in $\mathbb{Q}(\sqrt{-5})$. This is a localization of R that will have the property that if $\mathfrak{p}$ is a maximal ideal in R', then $\mathfrak{p} \cap \mathbb{Z}_{(3)} = (3)$. Show that R' is also Dedekind and compute its class group.

1.4.21. Exercise (A ring of algebraic integers that is almost, but not quite, Dedekind). Let $R = \mathbb{Z}[\sqrt{-3}]$, with field of fractions $F = \mathbb{Q}(\sqrt{-3})$.

(1) Show that R is not integrally closed in F, so that R is **not** a Dedekind domain, by Theorem 1.4.17.

(2) Exhibit a fractional ideal in R that does not have an inverse. Is this fractional ideal a projective module?

1.4.22. Exercise. Show that a Dedekind domain R with only finitely many prime ideals is a PID, using the following (slightly non-standard) sketch:

(1) Let $P_1, \ldots, P_n$ be a complete list of the distinct maximal ideals. Show using the Chinese Remainder Theorem that

$$R/\operatorname{rad} R \cong R/P_1 \times \cdots \times R/P_n,$$

a finite product of fields.

(2) Let P be an integral ideal of R. Show using (1) and the fact that P has rank 1 at each prime ideal that $P/(\operatorname{rad} R)P$ is free of rank 1, hence principal.

(3) Lift a generator of $P/(\operatorname{rad} R)P$ to a generator of P using Corollary 1.3.10.

1.4.23. Exercise (Complete calculation of a non-trivial class group). In this exercise, let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, the ring of (real-valued) polynomial functions on the circle.

(1) Show that R is a Dedekind domain.
(2) If $\mathfrak{p}$ is a prime (and thus maximal) ideal in R, show that $R/\mathfrak{p}$ is an algebraic extension of $\mathbb{R}$, and thus isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. Show that both possibilities can occur, and that in the first case, $\mathfrak{p}$ is of the form $(x - \alpha, y - \beta)$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 1$, and that in the second case, $\mathfrak{p}$ is a principal ideal generated by some linear polynomial $y + b$, where $b \in \mathbb{R}$, $|b| > 1$, or $x + ay + b$, where $a, b \in \mathbb{R}$, $b^2 - a^2 > 1$. Deduce that the class group $\tilde{K}_0(R)$ is generated by the classes of the ideals $(x-\alpha, y-\beta)$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 1$.
(3) Show that if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals of the form $(x-\alpha_j, y-\beta_j)$, respectively, where $\alpha_j, \beta_j \in \mathbb{R}$ and $\alpha_j^2+\beta_j^2 = 1$, $j = 1, 2$, then $\mathfrak{p}_1\mathfrak{p}_2$ is a principal ideal, with generator a linear polynomial vanishing at both (α_1, β_1) and (α_2, β_2), if these points are distinct, or else the linear polynomial $\alpha_1 x + \beta_1 y - 1$, if $\mathfrak{p}_2 = \mathfrak{p}_1$. Conclude that all non-principal prime ideals of R define the same element of the class group, and that this element is of order 2, hence that $\tilde{K}_0(R) \cong \mathbb{Z}/2$.

1.4.24. Exercise (More on class groups of quadratic number fields). Let d be a square-free integer and let $F = \mathbb{Q}(\sqrt{d})$, which is the most general quadratic extension of $\mathbb{Q}$.

(1) Show that the ring R of algebraic integers in F is $\mathbb{Z}[\sqrt{d}]$, provided that $d \equiv 2$ or $3 \mod 4$, and is $\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ if $d \equiv 1 \mod 4$. (This explains Exercises 1.4.20(1) and 1.4.21(1).)
(2) Let $p \in \mathbb{N}$ be a (rational) prime. Show that $R/(p)$ is a two-dimensional algebra over the field $\mathbb{F}_p$ of p elements, and that there are exactly three possibilities for $R/(p)$:
   (a) $R/(p) \cong \mathbb{F}_p[x]/(x^2)$ contains a nilpotent element. In this case we say p is **ramified**. Show that this case happens exactly when p divides d or, if $d \equiv 2$ or $3 \mod 4$, when $p = 2$.
   (b) $R/(p) \cong \mathbb{F}_{p^2}$ is a field, so the principal ideal (p) in R is maximal. In this case we say p is **inert**.
   (c) $R/(p) \cong \mathbb{F}_p \times \mathbb{F}_p$. In this case we say the prime p **splits** in F.

   (Hint: suppose $R = \mathbb{Z}[\xi]$ with $\xi^2 = d$, which is the case if $d \equiv 2$ or $3 \mod 4$. Then $R/(p) \cong \mathbb{F}_p[x]/(x^2 - d)$, so you just have to analyze whether the polynomial $x^2 - d$ has 0, 1, or 2 roots in $\mathbb{Z}/(p)$. The case $d \equiv 1 \mod 4$ is similar; it's just that the polynomial is different.)
(3) Show that in case (a), the ramified case, $(p) = \mathfrak{p}^2$ for some prime ideal $\mathfrak{p}$ of R, and that in case (c), the split case, $(p) = \mathfrak{p}_1\mathfrak{p}_2$ for some distinct prime ideals of R. In either case, if R has no elements of norm p, then the prime ideals occurring cannot be principal and

are thus non-trivial in $C(R)$. Thus show that in the ramified case, one gets an element of $C(R)$ of order 2.

(4) Show how Exercise 1.4.20 fits into this general framework.

5. Relative K_0 and excision

One of the things that makes K-theory so computable and useful is the fact that it behaves like a "homology theory" for rings. (The precise connection with a cohomology theory for topological spaces will be made in the next section.) In particular, when R is a ring containing a two-sided ideal I, there is an exact sequence relating $K_0(R)$, $K_0(R/I)$, and a certain "relative K-group." This exact sequence looks something like the exact sequence in cohomology for a pair of topological spaces (X, A):

$$H^j(X, A) \to H^j(X) \to H^j(A).$$

The first aim of this section is to define the relative group $K_0(R, I)$ and the exact sequence relating it to $K_0(R)$ and $K_0(R/I)$. Then we prove an algebraic analogue of the excision axiom for homology and develop some applications.

1.5.1. Definition. Let R be a ring and $I \subseteq R$ an ideal (in this section, always two-sided). The **double of R along I** is the subring of the Cartesian product $R \times R$ given by

$$D(R, I) = \{(x, y) \in R \times R : x - y \in I\}.$$

Note that if p_1 denotes projection onto the first coordinate, then there is a split exact sequence

$$0 \to I \to D(R, I) \xrightarrow{p_1} R \to 0, \tag{1.5.2}$$

in the sense that p_1 is split surjective (with splitting map given by the diagonal embedding of R in $D(R, I)$) and that $\ker p_1$ may be identified with I.

1.5.3. Definition. The **relative K_0-group** of a ring R and an ideal I is defined by

$$K_0(R, I) = \ker\left((p_1)_* : K_0(D(R, I)) \to K_0(R)\right).$$

Relative K-theory is closely linked to the phenomenon that while any matrix over R/I can be lifted to a matrix over R, an invertible matrix cannot always be lifted to an invertible matrix. The following lemma will also be used in the next chapter.

1.5.4. Lemma. *Let R be a ring and $I \subseteq R$ an ideal. Then if $A \in GL(n, R/I)$, the $2n\times 2n$ matrix $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ lifts to a matrix in $GL(2n, R)$.*

Proof. Note that

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ lifts "as is" to an invertible matrix over R. If B and C are any (not necessarily invertible) matrices in $M_n(R)$ lifting A and A^{-1}, respectively, then

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -C & 1 \end{pmatrix}$$

are invertible and lift

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix}.$$

Now just multiply. □

1.5.5. Theorem. *Let R be a ring and $I \subseteq R$ an ideal. Then there is a natural short exact sequence*

$$K_0(R, I) \to K_0(R) \xrightarrow{q_*} K_0(R/I),$$

where q_* is induced by the quotient map $q : R \to R/I$ and the map $K_0(R, I) \to K_0(R)$ is induced by $p_2 : D(R, I) \to R$.

Proof. For simplicity of notation in the proof, if A is an element of R or a matrix with entries in R, we will often denote $q(A)$, the corresponding matrix over R/I, by $\dot{A}$. First consider an element $[e] - [f] \in K_0(R, I)$, where $e = (e_1, e_2)$, $f = (f_1, f_2) \in \mathrm{Idem}(D(R, I))$. The image of $[e] - [f]$ in $K_0(R \times R) \cong K_0(R) \times K_0(R)$ (using (1.2.8)) is $([e_1] - [f_1], [e_2] - [f_2])$. So

$$q_* \circ (p_2)_*([e] - [f]) = q_*([e_2] - [f_2]) = [\dot{e}_2] - [\dot{f}_2],$$

whereas $[e_1] - [f_1] = 0$ since by assumption $[e] - [f] \in \ker(p_1)_*$. But since $e, f \in D(R, I)$, $\dot{e}_1 = \dot{e}_2$ and $\dot{f}_1 = \dot{f}_2$. Thus

$$[\dot{e}_2] - [\dot{f}_2] = [\dot{e}_1] - [\dot{f}_1] = q_*([e_1] - [f_1]) = 0.$$

Hence the image of the first map is contained in the kernel of the second.

Now suppose $e, f \in \mathrm{Idem}(R)$ and $q_*([e] - [f]) = [\dot{e}] - [\dot{f}] = 0$. Then $\dot{e}$ and $\dot{f}$ are stably equivalent, so for suitably large r,

$$\dot{e} \oplus \dot{1}_r = q(e \oplus 1_r) \sim \dot{f} \oplus \dot{1}_r = q(f \oplus 1_r)$$

under $GL(R/I)$. Replacing e by $e \oplus 1_r$ and f by $f \oplus 1_r$, we may assume $\dot{f} = \dot{g}\dot{e}(\dot{g})^{-1}$ for some matrix $\dot{g} \in GL(R/I)$. In general, $\dot{g}$ will not lift to a matrix in $GL(R)$. However, $\dot{g} \oplus (\dot{g})^{-1}$ does conjugate $\dot{e} \oplus \dot{0}$ to $\dot{f} \oplus \dot{0}$, and lifts to a matrix h in $GL(R)$ by Lemma 1.5.4. Thus we may replace f by $f \oplus 0$ and e by $h(e \oplus 0)h^{-1}$ without changing $[e]$ and $[f]$, and reduce to the case where $\dot{e} = \dot{f}$. This means $(e, f) \in \mathrm{Idem}(D(R, I))$. Then $[(e, e)] - [(e, f)]$ is a class in $K_0(D(R, I))$ which maps to 0 under $(p_1)_*$ and to $[e] - [f]$ under $(p_2)_*$. This completes the proof of exactness. The naturality of the sequence (under homomorphisms $R \to R'$ sending $I \to I'$) is obvious from the definition of the maps and from functoriality of K_0. □

Remark. In general, the map $K_0(R) \to K_0(R/I)$ is not surjective, and the map $K_0(R, I) \to K_0(R)$ is not injective. The one exception will be the case where the ring homomorphism $R \to R/I$ splits. In this case it is obvious that the map $K_0(R) \to K_0(R/I)$ is split surjective, and it will also turn out (see 1.5.11 below) that $K_0(R, I)$ is the kernel of this map.

Next we want to prove the analogue of the excision theorem for topological homology. Recall that this says that under suitable hypotheses, the relative homology $H_\bullet(X, A)$ is unchanged when a large subset U of A is removed from both A and X. Under optimal circumstances (for instance, for CW-pairs), $H_\bullet(X, A) \cong \tilde{H}_\bullet(X/A)$ and thus only depends on the "difference" between X and A. The analogous statement for K_0 turns out to be true, and says that the relative group $K_0(R, I)$ only depends on the "difference" between R and R/I, which is measured by I (with its structure as a ring without unit). In fact, it turns out that K_0 makes sense and is functorial even for rings **without** unit and for non-unital ring homomorphisms. With this language, we show that $K_0(R, I) \cong K_0(I)$.

1.5.6. Definition. Let I be a ring that doesn't necessarily have a unit element. The **ring obtained by adjoining a unit element to** I, denoted I_+, is **as an abelian group** just $I \oplus \mathbb{Z}$, with multiplication defined by the rule

$$(x, n) \cdot (y, m) = (xy + ny + mx, mn),$$

$x, y \in I$; $m, n \in \mathbb{Z}$. It is an easy exercise to check that this is indeed a ring with unit, the unit element being $(0, 1)$. The notation I_+ is suggested by topology, where X_+ is standard notation for a space X with a disjoint basepoint added.

It is useful to note that if $\alpha : I \to I'$ is a homomorphism in the category of rings without unit, it automatically extends uniquely to a unital homomorphism $I_+ \xrightarrow{\alpha_+} I'_+$.

Remark. The reader might wonder what happens if I already has a unit element, say e. In this case, there is a unital isomorphism $\alpha : I_+ \to I \times \mathbb{Z}$ (the Cartesian product of rings) defined by

$$\alpha(x, n) = (x + ne, n),$$

since

$$\begin{aligned}\alpha((x, n)\cdot(y, m)) &= \alpha(xy+ny+mx, mn)\\ &= (xy+ny+mx+mne, mn)\\ &= (x+ne, n)\cdot(y+me, m)\\ &= \alpha(x, n)\cdot\alpha(y, m).\end{aligned}$$

1.5.7. Definition. Let I be a ring that doesn't necessarily have a unit. Note that one has a split exact sequence

$$0 \to I \to I_+ \xrightarrow{\rho} \mathbb{Z} \to 0. \tag{1.5.8}$$

Define

$$K_0(I) = \ker\left(\rho_* : K_0(I_+) \to K_0(\mathbb{Z}) \cong \mathbb{Z}\right).$$

At first sight, there might appear to be some ambiguity here, since if I has a unit, we have given two different definitions of $K_0(I)$. However, by the remark above, in this case $I_+ \cong I \times \mathbb{Z}$, so $K_0(I_+) \cong K_0(I) \oplus K_0(\mathbb{Z})$, and $\ker \rho_*$ just picks out the first summand. So the new definition agrees with the old one in this case.

Also, this new definition makes K_0 into a functor from the category of non-unital rings to abelian groups. This observation is occasionally useful even when one wants to deal only with rings with unit. For instance, if R is a ring with unit, there is a **non-unital** homomorphism $R \to M_n(R)$ defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. The reader can check that the homomorphism induced by this non-unital homomorphism is the Morita invariance isomorphism of Theorem 1.2.4.

1.5.9. Theorem (Excision). *If I is a two-sided ideal in a ring R, then $K_0(R, I) \cong K_0(I)$ (and thus does not depend on R, only on the structure of I as a ring without unit).*

Proof. Define a unital homomorphism $\gamma : I_+ \to D(R, I)$ by

$$(x, n) \mapsto (n\cdot 1, n\cdot 1 + x), \quad x \in I, \quad n \in \mathbb{Z},$$

and note that the diagram

$$\begin{array}{ccc} I_+ & \xrightarrow{\gamma} & D(R, I) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathbb{Z} & \xrightarrow{\iota} & R \end{array}$$

commutes. Hence $\gamma_* : K_0(I_+) \to K_0(D(R, I))$ sends $\ker \rho_*$ to $\ker(p_1)_*$, *i.e.*, maps $K_0(I)$ to $K_0(R, I)$.

Next we show that this map is surjective. Consider a class $[e]-[f] \in K_0(R, I)$, where $e = (e_1, e_2)$, $f = (f_1, f_2) \in \mathrm{Idem}(D(R, I))$ and $[e_1] = [f_1]$ in $K_0(R)$. After replacing e and f by $e \oplus 1_r$ and $f \oplus 1_r$ for a suitably large r, we may assume that e_1 and f_1 are conjugate under $GL(R)$, say $e_1 = gf_1g^{-1}$ for some invertible matrix g. Replacing (f_1, f_2) by (gf_1g^{-1}, gf_2g^{-1}), we may assume that in fact $f_1 = e_1$. Next, if e is an $s \times s$ matrix, we may replace e and f by $e \oplus (1_s - e_1, 1_s - e_1)$ and by $f \oplus (1_s - e_1, 1_s - e_1)$. Note that there is an invertible $2s \times 2s$ matrix h with entries in R conjugating $e_1 \oplus (1_s - e_1)$ to $1_s \oplus 0_s$. Conjugating everything by h finally reduces us to the case where $e = (1_s \oplus 0_s, e_2)$, $f = (1_s \oplus 0_s, f_2)$. Since e and f are matrices over $D(R, I)$, $e_2 - (1_s \oplus 0_s)$ and $f_2 - (1_s \oplus 0_s)$ have entries in I. Now $[e] - [f]$ is clearly in the image of $K_0(I)$.

Finally, we have to show γ_* is injective on $K_0(I)$. We may represent a general element of $K_0(I)$ by $[e]-[f]$, where e, $f \in \mathrm{Idem}(I_+)$ and $\mathrm{rank}\,\rho(e) = \mathrm{rank}\,\rho(f)$. As above, if f is an $r \times r$ matrix, we may stabilize by taking direct sums with $1_r - f$ and conjugating, and thus assume $f = 1_r$, $\mathrm{rank}\,\rho(e) = r$. We may also assume $g\rho(e)g^{-1} = 1_r$ for some $g \in GL(\mathbb{Z})$. Viewing g as an element of $GL(I_+)$ via the split exact sequence 1.5.8, we may replace e by geg^{-1} and assume that $\rho(e) = 1_r$. Now if $\gamma_*([e] - [1_r]) = 0$, this means

$$[(1_r, e)] = [(1_r, 1_r)] \quad \text{in } K_0(D(R, I)).$$

We may stabilize if necessary by increasing r and assume that there is a matrix $(g_1, g_2) \in GL(D(R, I))$ with

$$g_1 1_r g_1^{-1} = 1_r, \quad g_2 e g_2^{-1} = 1_r.$$

Then $(1, g_1^{-1}g_2) \in GL(D(R, I))$ and

$$(g_1^{-1}g_2)e(g_1^{-1}g_2)^{-1} = g_1^{-1}(g_2eg_2^{-1})g_1 = g_1^{-1}1_r g_1 = 1_r.$$

Since $g_1^{-1}g_2 \equiv 1 \mod I$, $g_1^{-1}g_2$ lies in $GL(I_+)$ and this says $[e] - [1_r] = 0$ in $K_0(I)$, proving that the kernel of γ_* is trivial. □

1.5.10. Examples.

(a) Suppose $R = \mathbb{Z}$ and $I = (m)$, where $m > 0$. Thus $R/I = \mathbb{Z}/(m)$. $K_0(R/I)$ was computed in Exercise 1.3.14; the map $K_0(R) \to K_0(R/I)$ is always injective but in general has a free abelian cokernel of rank $k-1$, where k is the number of distinct prime factors of m. As a ring without unit, I is the free abelian group on a generator t satisfying $t^2 = mt$. Hence $I_+ \cong \mathbb{Z}[t]/(t^2 - mt)$, a fairly complicated ring. $K_0(I)$ is not so easy to compute directly, though we will find a way to compute it in the next chapter. It turns out to be a finite abelian group.

(b) For applications to topology (see Section 1.7 below), rings of the form $R = \mathbb{Z}G$, the integral group ring of a group G, are of particular importance. It is a long-standing conjecture that when G is torsion-free, $\tilde{K}_0(R) = 0$. This is known in some cases, for instance

when G is free abelian; this case will be treated in Chapter 3. For finite groups, $\tilde{K}_0(\mathbb{Z}G)$ is often non-trivial and contains interesting arithmetic information. Consider the simplest example, when G is cyclic of prime order p, say with generator t. Then $R = \mathbb{Z}G$ may be identified with $\mathbb{Z}[t]/(t^p - 1)$. If $\xi = e^{2\pi i/p}$, a primitive p-th root of unity, and if $S = \mathbb{Z}[\xi]$, then S is the ring of integers in the cyclotomic field $\mathbb{Q}(\xi)$, hence is a Dedekind domain by Theorem 1.4.18. There is a surjective homomorphism $R \to S$ defined by sending $t \mapsto \xi$. Since the cyclotomic polynomial $f_p(t) = t^{p-1} + \cdots + t + 1$ is irreducible, any polynomial $g(t) \in \mathbb{Z}[t]$ with $g(\xi) = 0$ must be divisible by f_p. In particular, anything in the kernel I of the map $R \to S$ must be a multiple of f_p. Note that as an element of R, $f_p^2 = pf_p$. Thus I in this example is, as a ring without unit, the same as in the last example if we specialize to the case $m = p$. In particular, $K_0(R, I) = K_0(\mathbb{Z}, (p))$. It is a result of Rim, which we will discuss later on, that the map $R \to S$ induces an isomorphism on K_0. In particular, $\tilde{K}_0(R) \cong C(S)$, the class group of the cyclotomic field. This is known to be non-zero for primes $p \geq 23$. (See Example 3.3.5(b) below.) The smallest group G for which $\tilde{K}_0(\mathbb{Z}G)$ is non-trivial is the quaternion group of order 8—in this case, $\tilde{K}_0(\mathbb{Z}G)$ is of order 2 and an explicit generator is exhibited in Exercise 1.7.20(3) below.

1.5.11. Exercise. The excision theorem may be interpreted as saying that the split exact sequence 1.5.2 gives rise to a split exact sequence of K_0-groups, the first group of which is $K_0(I)$. The same holds by definition in the case of the split exact sequence 1.5.8. Using ideas from the proof of the excision theorem, show that if

$$0 \to I \to R \to R/I \to 0$$

is split exact (*i.e.*, I is an ideal in a ring R, and there is a splitting homomorphism $R/I \to R$), then

$$0 \to K_0(I) \to K_0(R) \to K_0(R/I) \to 0$$

is split exact.

6. An application: Swan's Theorem and topological K-theory

To many mathematicians, the term K-theory suggests not algebraic K-theory but topological K-theory, an exceptional cohomology theory on compact Hausdorff spaces defined using vector bundles. The connection between the two comes from specializing what we have done to the case where R is a ring of continuous functions. In this context, the Excision

Theorem (1.5.9) gives the excision property for this cohomology theory. We do not attempt here to cover any of the deep properties of topological K-theory, the most fundamental of which is the Bott Periodicity Theorem, but we at least give a quick introduction to the fundamentals. This provides an interesting application of what we have done so far, as well as a useful motivation for a number of results and constructions in future chapters. The reader who wants to see more details can consult any of the texts [Atiyah], [Husemoller], or [Karoubi].

1.6.1. Definition. Let X be a topological space (in most of what we will do, assumed to be compact Hausdorff) and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A **$\mathbb{F}$-vector bundle** (in the weakest sense) consists of a topological space E and a continuous open surjective map $p : E \to X$, with extra structure defined by the following:

a) each fiber $p^{-1}(x)$ of p, $x \in X$, is a finite-dimensional vector space over $\mathbb{F}$;
b) there are continuous maps

$$E \times_p E \to E \quad \text{and} \quad \mathbb{F} \times E \to E$$

which restrict to vector addition and scalar multiplication on each fiber.

Such bundles $E \xrightarrow{p} X$ make up a category, in which the morphisms are commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & = & X \end{array}$$

for which the map $E \xrightarrow{f} E'$ is linear on each fiber.

For any X and any $n \in \mathbb{N}$, the category always includes the **trivial $\mathbb{F}$-vector bundle of rank n**, which is $X \times \mathbb{F}^n \xrightarrow{\pi_1} X$, where π_1 is projection on the first factor and the vector bundle structure is the obvious one coming from the vector space structure on the second factor.

The category has a binary operation called the **Whitney sum**, denoted $\oplus$. By definition, if $E \xrightarrow{p} X$ and $E' \xrightarrow{p'} X$ are $\mathbb{F}$-vector bundles over X, their Whitney sum is defined by

$$E \oplus E' = \{(x, x') : x \in E,\ x' \in E',\ p(x) = p'(x')\},$$

with the obvious map to X.

For most purposes we want a more restrictive definition. A (locally trivial) **$\mathbb{F}$-vector bundle** is a $\mathbb{F}$-vector bundle in the above sense with the additional property that for each $x \in X$, there is a neighborhood U of x in X and an isomorphism (in the category of $\mathbb{F}$-vector bundles) from $p^{-1}(U) \xrightarrow{p|_{p^{-1}(U)}} U$ to a trivial bundle of some rank over U. The **rank** of such a bundle is then a continuous function $X \to \mathbb{N}$ defined by $\text{rank}_x(E) = \dim p^{-1}(x)$. If X is connected, the rank must be constant.

1.6.2. Definition. If X is a compact Hausdorff space, let $\text{Vect}_{\mathbb{F}}(X)$ denote the monoid of isomorphism classes (in the category of $\mathbb{F}$-vector bundles) of locally trivial $\mathbb{F}$-vector bundles over X, with an addition operation induced by the Whitney sum. The 0-element of this monoid is the trivial bundle of rank 0. The **topological K-theory** of X is defined by $K^0_{\mathbb{F}}(X) = G(\text{Vect}_{\mathbb{F}}(X))$. Sometimes this is denoted simply $K(X)$ or $KU(X)$ if $\mathbb{F} = \mathbb{C}$, $KO(X)$ if $\mathbb{F} = \mathbb{R}$. (The "U" and "O" stand respectively for "unitary" and "orthogonal" after the names of isometric linear transformations.) We will often suppress mention of $\mathbb{F}$ when it is understood from context. If X is connected, the **reduced** topological K-theory is $\tilde{K}^0_{\mathbb{F}}(X) = \ker\left(\text{rank} : K^0_{\mathbb{F}}(X) \to \mathbb{Z}\right)$.

$K^0(X)$ is actually a contravariant functor from the category of compact Hausdorff spaces (and continuous maps) to the category of abelian groups. This follows from the fact that vector bundles **pull back** under continuous maps. If $X \xrightarrow{f} Y$ is continuous and $E \xrightarrow{p} Y$ is a vector bundle over Y, we define $f^*(Y)$ to be the fiber product

$$\{(x, e) : x \in X,\ e \in E,\ f(x) = p(e)\},$$

with the obvious map to X. The pull-back clearly induces a monoid homomorphism $f^* : \text{Vect}_{\mathbb{F}}(Y) \to \text{Vect}_{\mathbb{F}}(X)$ and thus a map $K^0(Y) \to K^0(X)$.

We're now ready for the connection between vector bundles and projective modules that explains the connection between topological and algebraic K-theory.

1.6.3. Theorem [Swan2]. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let X be a compact Hausdorff space, and let $R = C^{\mathbb{F}}(X)$ be the ring of continuous $\mathbb{F}$-valued continuous functions on X (with pointwise addition and multiplication). If $E \xrightarrow{p} X$ is a (locally trivial) $\mathbb{F}$-vector bundle over X, let*

$$\Gamma(X, E) = \{s : X \to E \text{ continuous } \mid p \circ s = id_X\}$$

be the set of continuous sections of p. Observe that this is naturally an R-module. Then $\Gamma(X, E)$ is finitely generated and projective over R, and every finitely generated projective module over R arises (up to isomorphism) from this construction. The map $E \rightsquigarrow \Gamma(X, E)$ induces an isomorphism of categories from the category of (locally trivial) vector bundles over X to the category of finitely generated projective R-modules. It also induces an isomorphism $K^0(X) \to K_0(R)$.

Proof. Let $E \xrightarrow{p} X$ be a (locally trivial) $\mathbb{F}$-vector bundle over X and let $\Gamma(X, E)$ be its R-module of sections. For each $x \in X$, there is an open neighborhood U over which E looks like a trivial bundle $U \times \mathbb{F}^n$ for some n. The n constant functions $e_j : U \to \mathbb{F}^n$ determined by the standard basis vectors of $\mathbb{F}^n$ clearly generate the sections of this trivial bundle as a module over the continuous functions. Since X is compact, we can cover X by finitely many such open sets U_i and choose a partition of unity (f_i)

subordinate to the covering. (Thus $0 \le f_i \le 1$, f_i is supported in U_i, and $\sum f_i \equiv 1$.) Multiplying the e_j corresponding to U_i by f_i, we get sections e_{ij} supported in U_i which clearly extend to all of X by taking them $= 0$ off U_i, and by construction, the e_{ij} generate $\Gamma(X, E)$ as an R-module. Hence $\Gamma(X, E)$ is finitely generated.

Next we show that $\Gamma(X, E)$ is projective. Choose generators s_j, $1 \le j \le k$, for $\Gamma(X, E)$ as an R-module. (These may or may not be the ones we just constructed above.) Consider the trivial bundle $X \times \mathbb{F}^k \xrightarrow{\pi_1} X$ and construct a morphism $\varphi : X \times \mathbb{F}^k \to E$ by

$$(x, v_1, \ldots, v_k) \mapsto \sum_{j=1}^{k} v_j s_j(x).$$

Since the $s_j(x)$ span $p^{-1}(x)$ for each x, this vector bundle morphism is surjective on each fiber. Define a subbundle of the trivial bundle by $E' = \ker \varphi$, *i.e.*, by $E'_x = \ker \varphi_x$. This is also locally trivial since one can check that it is trivial over any open set where E is trivial. We claim now that $E \oplus E' \cong X \times \mathbb{F}^k$, which will show that

$$\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, X \times \mathbb{F}^k) \cong R^k,$$

hence that $\Gamma(X, E)$ is a projective module over R.

The easiest way to do this is by introducing hermitian metrics, *i.e.*, inner products. A **hermitian metric** on E is a continuous map

$$\langle\ ,\ \rangle : E \times_X E \to \mathbb{F}$$

which restricts to a positive-definite inner product on each fiber of E (bilinear if $\mathbb{F} = \mathbb{R}$, sesquilinear if $\mathbb{F} = \mathbb{C}$). Such metrics clearly exist since they exist on trivial bundles (use the standard inner product on $\mathbb{F}^n$) and can be patched together using a partition of unity. Therefore we may choose such a metric on E and the metric on $X \times \mathbb{F}^k$ coming from the standard inner product on $\mathbb{F}^k$. With respect to these metrics, φ has an adjoint φ^* satisfying the usual relation

$$\langle \varphi v, w\rangle = \langle v, \varphi^* w\rangle.$$

Since φ is surjective on each fiber, φ^* will be injective on each fiber, with image the orthogonal complement of $E = \ker \varphi$. So φ^* gives an isomorphism of vector bundles from E to $E'^{\perp}$, showing that $E \oplus E' \cong X \times \mathbb{F}^k$, as desired.

Now we have to show that every finitely generated projective module over R corresponds to a vector bundle. Suppose P is such a module and $P \oplus Q = R^n \cong C(X, \mathbb{F}^n)$. Then we may view P as a collection of functions $X \to \mathbb{F}^n$ and let

$$E = \{(x, v_1, \ldots, v_n) \in X \times \mathbb{F}^n : \exists s \in P \text{ with } s(x) = (v_1, \ldots, v_n)\}.$$

Define $p : E \to X$ using projection onto the first factor. It is now quite easy to see that $E \xrightarrow{p} X$ is a vector bundle. Vector addition and scalar multiplication just come from vector addition and scalar multiplication in $\mathbb{F}^n$. (These operations map E into itself since P is an R-module.) We need only check the local triviality. Given $x \in X$, choose elements $e^1, \dots, e^r \in P$ such that $e^1(x), \dots, e^r(x)$ are a basis for the subspace $E_x = p^{-1}(x)$ of $\mathbb{F}^n$. Recall these are vector-valued functions; write $e^i = (e^i_1, \dots, e^i_n)$. Then since $e^1(x), \dots, e^r(x)$ are linearly independent, we can choose $1 \le j_1 < \dots < j_r \le n$ such that

$$e = \det \begin{pmatrix} e^1_{j_1} & e^1_{j_2} & \cdots & e^1_{j_r} \\ \vdots & & & \vdots \\ e^r_{j_1} & e^r_{j_2} & \cdots & e^r_{j_r} \end{pmatrix} \tag{1.6.4}$$

is non-zero at x. We may choose similar elements $f^1, \dots, f^{n-r} \in Q$ such that $f^1(x), \dots, f^{n-r}(x)$ are a basis for the image of Q in $\mathbb{F}^n$ at x. (The dimensions are complementary since $P \oplus Q = R^n \cong C(X, \mathbb{F}^n)$.) From the f^k we may construct an $(n-r) \times (n-r)$ determinant f, similar to (1.6.4), which is non-zero at x. Since e and f are continuous, there is some neighborhood U of x in which both $e \neq 0$ and $f \neq 0$. For $y \in U$, $e^1(y), \dots, e^r(y)$ are linearly independent and generate a rank-r free submodule of P. Similarly, $f^1(y), \dots, f^{n-r}(y)$ are linearly independent and generate a rank-$(n-r)$ free submodule of Q. By dimension counting, these must exhaust P and Q, so both P and Q are trivial over U. The statement about an equivalence of categories is now easy to check. □

Theorem 1.6.3 suggests that we should extend the definition of K^0 to the category of **locally compact** spaces and **proper** maps (maps that extend continuously to the one-point compactification) by letting $K^0(Y) = K_0(C_0^{\mathbb{F}}(Y))$, where $C_0^{\mathbb{F}}(Y)$ is the ring of functions vanishing at infinity on Y and we are using K-theory for rings without unit, as in Definition 1.5.7. The resulting theory is called **K-theory with compact supports**. See Exercise 1.6.14 below for a more geometric definition.

1.6.5. Proposition. *If X is a compact Hausdorff space and A is a closed subspace, there is (for $\mathbb{F}$ = either $\mathbb{C}$ or $\mathbb{R}$) an exact sequence induced by the inclusion $A \hookrightarrow X$:*

$$K^0(X \smallsetminus A) \to K^0(X) \to K^0(A).$$

Proof. Let $R = C^{\mathbb{F}}(X)$, and let I be the closed ideal of functions vanishing on A, which as a ring without unit is isomorphic to $C_0^{\mathbb{F}}(X \smallsetminus A)$, the functions vanishing at infinity on the locally compact space $X \smallsetminus A$. By the Tietze Extension Theorem, every continuous function on A is the restriction of a continuous function on X, hence R/I may be identified with $C^{\mathbb{F}}(A)$, with the quotient map $R \to R/I$ identified with restriction of functions. The result now follows immediately from Theorem 1.6.3 and Theorem 1.5.9. □

Proposition 1.6.5 shows in effect that K^0 satisfies two of the Eilenberg-Steenrod axioms for a cohomology theory: exact sequences and excision. It also satisfies the other key axiom, homotopy invariance, and we prove this next by using special properties of Banach algebras. Recall that a **Banach algebra** A is an algebra over $\mathbb{R}$ or $\mathbb{C}$ which also has the structure of a Banach space, such that for any $a, b \in A$, $\|ab\| \leq \|a\|\|b\|$. The principal examples for our purposes are $M_n(C^{\mathbb{F}}(X))$, X a compact Hausdorff space, or $M_n(C_0^{\mathbb{F}}(Y))$, Y a locally compact Hausdorff space. The latter does not have a unit.

1.6.6. Lemma. *Let A be a (real or complex) Banach algebra with unit and let $x \in A$ with $\|1 - x\| < 1$. Then for each $\alpha \in \mathbb{R}$ there is an element x^α in A with the usual properties ($x^1 = x$, $x^0 = 1$, $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$). In particular, x is invertible in A.*

Proof. Define x^α by the usual binomial power series for $(1 + (x - 1))^\alpha$. The norm of the n-th term in the series is bounded by the corresponding term in the series for $(1+\|x-1\|)^\alpha$, which converges absolutely. Since A is a Banach space, the series for x^α therefore converges absolutely. The relation $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$ follows as usual from multiplication of the series. □

1.6.7. Lemma. *Let A be a Banach algebra and let p, q be two idempotents in A with $\|p - q\| < \min\left(\|p\|^{-2}, \|q\|^{-2}\right)$. Then the projective A-modules Ap and Aq are isomorphic.*

Proof. Observe that pAp and qAq are Banach algebras with unit elements p and q, respectively. Since $\|p - q\| < \|p\|^{-2}$, multiplying by p on both sides gives $\|p-pqp\| < 1$, and similarly $\|q-qpq\| < 1$. So $x = (pqp)^{-\frac{1}{2}}$ makes sense in pAp and qpq is invertible in qAq, both by Lemma 1.6.6. Thus there is an $x \in pAp$ commuting with pqp with $x^2(pqp) = p$ and of course with $x = xp = px$. Observe then that

$$(xq)(qx) = xpqpx = x^2(pqp) = p, \tag{1.6.8}$$

that

$$p(xq) = xq = (xq)q, \quad q(qx) = qx = (qx)p, \tag{1.6.9}$$

and that

$$(qx^2q)(qpq) = qx^2qpq = qx^2(pqp)q = qpq. \tag{1.6.10}$$

The equation (1.6.10) says $(qx)(xq)$ is a left unit for qpq in qAq. But since qpq is invertible in qAq, $(qx)(xq)$ must be equal to the unit element of qAq, which is q. The equations (1.6.8) and (1.6.9), together with this fact, imply that right multiplication by qx gives an isomorphism from Aq onto Ap, whose inverse is right multiplication by xq (compare the calculation in Lemma 1.2.1). □

1.6.11. Corollary (Homotopy invariance of topological K-theory). *Let A and B be Banach algebras and let $\varphi_t : A \to B$, $0 \le t \le 1$, be a homotopy of homomorphisms from A to B. (This means exactly that there is a homomorphism $\varphi : A \to C([0, 1], B)$ which when composed with evaluation at t gives φ_t.) Then φ_0 and φ_1 induce the same map on K-theory $K_0(A) \to K_0(B)$.*

Proof. If necessary, adjoin units to A and B and extend φ_t to a homotopy of unital homomorphisms of unital algebras $A_+ \to B_+$. Since $K_0(A) \hookrightarrow K_0(A_+)$ and $K_0(B) \hookrightarrow K_0(B_+)$, this reduces us to the unital case. For simplicity, we therefore assume without loss of generality that A, B, and the homomorphisms are unital. For any $p \in \mathrm{Idem}(A)$, p lies in $M_n(A)$ for some n, so we may replace A and B by $M_n(A)$ and $M_n(B)$, respectively. (These are still unital Banach algebras, and φ_t extends naturally to $M_n(A)$ just by application of the homomorphism to each matrix entry separately.)

Then $\varphi_t(p)$ is a continuous path of idempotents in B. Choose C so that $\|\varphi_t(p)\| \le C$ for all t. We may partition the interval [0, 1] into subintervals such that $\|\varphi_t(p) - \varphi_s(p)\| < C^{-2}$ for t, s in the same subinterval. By Lemma 1.6.7, the class of $\varphi_t(p)$ remains constant in each subinterval, hence remains constant in the whole interval. So φ_0 and φ_1 induce the same map $\mathrm{Idem}(A) \to \mathrm{Idem}(B)$ and hence the same map on K_0. □

1.6.12. Corollary. *The functors $X \rightsquigarrow \mathrm{Vect}_{\mathbb{F}}(X)$ and $X \rightsquigarrow K^0(X)$ are homotopy-invariant functors from the category of compact Hausdorff topological spaces to the category of abelian monoids and the category of abelian groups, respectively. In particular, if X is contractible, all vector bundles over X are trivial, and $\tilde{K}^0(X) = 0$.*

Proof. Specialize to the case of Banach algebras of the form $M_n(C^{\mathbb{F}}(X))$. Since homotopic idempotents are equivalent, we deduce that the map from X to isomorphism classes of direct summands in a trivial bundle of rank n over X is a homotopy functor. The rest of the statements follow from this. □

1.6.13. Example. Corollary 1.6.12 shows that the classification of vector bundles, and hence the calculation of $K^0(X)$, are homotopy-theoretic in nature. Consider for instance the case where $X = S^n$. This is a union of two contractible hemispheres joined along the equator S^{n-1}. (If $n = 0$, the hemispheres are single points and the "equator" is the empty set.) Thus any rank-r bundle over X is trivial over the hemispheres and determined by the homotopy class of the "gluing data" along $Y = S^{n-1}$, which gives an isomorphism between the two trivializations of the bundle coming from the two hemispheres. Now an isomorphism between two trivial bundles $Y \times \mathbb{F}^r \xrightarrow{\pi_1} Y$ is just given by a continuous map $Y \to GL(r, \mathbb{F})$. So isomorphism classes of rank-r $\mathbb{F}$-vector bundles over S^n are in one-to-one correspondence with homotopy classes of maps $S^{n-1} \to GL(r, \mathbb{F})$. Furthermore, by polar decomposition, any matrix in $GL(r, \mathbb{F})$ can be written uniquely in the form up, where u is unitary if $\mathbb{F} = \mathbb{C}$, orthogonal if $\mathbb{F} = \mathbb{R}$, and p is positive-definite self-adjoint. The positive-definite self-adjoint ma-

trices form a contractible space (since one can write any such matrix as e^h with h hermitian and use the contraction given by e^{th}, $0 \le t \le 1$), so $GL(r, \mathbb{C})$ has a deformation retraction to $U(r)$ and $GL(r, \mathbb{R})$ has a deformation retraction to $O(r)$. Thus the isomorphism classes of rank-r $\mathbb{F}$-vector bundles over S^n are given by $\pi_{n-1}(U(r))$ if $\mathbb{F} = \mathbb{C}$, $\pi_{n-1}(O(r))$ if $\mathbb{F} = \mathbb{R}$. The 0-element of the homotopy group corresponds to the trivial bundle.

Now we can make some computations. $O(r)$ always has two components with identity component the rotation group $SO(r)$, and $U(r)$ is connected. Thus $\pi_0(U(r)) = 0$ and $\pi_0(O(r)) \cong \mathbb{Z}/2$, so $\widetilde{KU}^0(S^1) = 0$, $\widetilde{KO}^0(S^1) \cong \mathbb{Z}/2$. In low dimensions, one can check that $O(1) = \{1, -1\}$, $SO(2) \cong S^1$, $SO(3) \cong \mathbb{RP}^3$, $SO(4)$ has $S^3 \times S^3$ as a double cover, $U(1) \cong S^1$, $SU(2) \cong S^3$. Thus, for instance,

$$\pi_1(O(r)) = \begin{cases} 0, & r = 1, \\ \mathbb{Z}, & r = 2, \\ \mathbb{Z}/2, & r \ge 3, \end{cases}$$

so that $\mathrm{Vect}_{\mathbb{R}}(S^2)$ is the monoid described in Exercise 1.1.7. One finds similarly that $\pi_1(U(r)) = \mathbb{Z}$ for all r, so that $\widetilde{KU}^0(S^2) \cong \mathbb{Z}$. The calculations of $\widetilde{KO}^0(S^n)$ and of $\widetilde{KU}^0(S^n)$ for all n follow from the Bott Periodicity Theorem, which says that the answer only depends on the value of $n \mod 8$ in the real case or the value of $n \mod 2$ in the complex case. One obtains

$$\widetilde{KO}^0(S^n) = \begin{cases} 0, & r \not\equiv 0, 1, 2, 4 \mod 8, \\ \mathbb{Z}, & r \equiv 0, 4 \mod 8, \\ \mathbb{Z}/2, & r \equiv 1, 2 \mod 8, \end{cases}$$

$$\widetilde{KU}^0(S^n) = \begin{cases} 0, & r \text{ odd}, \\ \mathbb{Z}, & r \text{ even}. \end{cases}$$

1.6.14. Exercise. Give another description of K-theory with compact supports for a locally compact Hausdorff space Y in which $K^0(Y)$ is a set of equivalence classes of triples (E_0, E_1, φ), where E_0 and E_1 are (locally trivial) vector bundles over Y and φ is a morphism of vector bundles $E_0 \to E_1$ which is an isomorphism outside of a compact set, and with relations

(a) $$[E_0, E_1, \varphi] + [F_0, F_1, \psi] = [E_0 \oplus F_0, E_1 \oplus F_1, \varphi \oplus \psi],$$

(b) $$[E_0, E_1, \varphi] = [E_0, E_1, \varphi'] \quad \text{if } \varphi = \varphi' \text{ outside of a compact set},$$

(c) $$[E_0, E_1, \varphi] = 0 \quad \text{if } \varphi \text{ is an isomorphism}.$$

Impose the necessary equivalence relation to get an isomorphism with our old description of K^0. Hint: when Y is actually compact, condition (b) says that one can forget the φ altogether. In this case, the isomorphism of this description of K^0 with the usual one is given by

$$[E_0, E_1, \varphi] \mapsto [E_0] - [E_1].$$

1.6.15. Exercise. Show that if one defines $K^{-j}(X) = K^0(X \times \mathbb{R}^j)$ (using K-theory with compact supports) that the short exact sequence of Proposition 1.6.5 can be extended to a long exact sequence

$$\cdots \to K^{-j}(X \setminus A) \to K^{-j}(X) \to K^{-j}(A) \to K^{-j+1}(X \setminus A) \to \cdots .$$

Hint: the problem is construct the boundary map $K^0(A \times \mathbb{R}) \to K^0(X \setminus A)$. This can be done by letting Y be the space $(A \times (0, 1]) \cup X$, with $(a, 1)$ identified with $a \in X$ for $a \in A$. (Y is the "open mapping cone" of the inclusion $A \hookrightarrow X$.) One gets from Proposition 1.6.5 exact sequences

$$K^0(A \times (0, 1)) \to K^0(Y) \to K^0(X)$$

and

$$K^0(X \setminus A) \to K^0(Y) \to K^0(A \times (0, 1]).$$

Show using homotopy-invariance and excision that $K^0(A \times (0, 1])$ vanishes and that $K^0(X \setminus A) \to K^0(Y)$ is an isomorphism. Then splice these exact sequences together with the sequence

$$K^0(X \setminus A) \to K^0(X) \to K^0(A).$$

1.6.16. Exercise (The Karoubi Density Theorem [Karoubi, II.6. 15]**).** Let $\mathcal{A}$ and A be (unital) Banach algebras over $\mathbb{C}$, and let $\iota : \mathcal{A} \to A$ be a continuous injection of $\mathcal{A}$ into A as a dense subalgebra. Extend ι to matrices in the usual way, by applying it to each entry of the matrix. Assume that for all n, if $x \in M_n(\mathcal{A})$ and $\iota(x)$ is invertible in $M_n(A)$, then x is invertible in $M_n(\mathcal{A})$.

(1) Show that ι induces an isomorphism $K_0(\mathcal{A}) \to K_0(A)$. Hint for the surjectivity: if e is an idempotent in $M_n(A)$, then e can be approximated in the topology of A by an element x of $M_n(\mathcal{A})$. Show that the spectrum of x in $M_n(\mathcal{A})$ coincides with its spectrum in $M_n(A)$, and thus that x has spectrum close to $\{0, 1\}$. Deduce that the Banach subalgebra of $M_n(\mathcal{A})$ generated by x contains an idempotent f with $\iota(f)$ close to e, by justifying the definition

$$f = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - x},$$

where Γ is a contour in the complex plane encircling the part of the spectrum of x close to 1, and excluding the part of the spectrum of x close to 0. Then use Lemma 1.6.7.

(2) Show that the two hypotheses are satisfied if A is the algebra of continuous complex-valued functions on a compact subset X of $\mathbb{R}^n$ (equipped with the sup norm $\| \ \|$), and if $\mathcal{A}$ is the algebra of continuously differentiable functions on X, equipped with the norm

$$\|f\|_{\mathcal{A}} = \|f\| + \|\nabla f\|.$$

Deduce that "every vector bundle over X has a differentiable structure."

7. Another application: Euler characteristics and the Wall finiteness obstruction

In this final section of Chapter 1, we discuss the algebraic background of most of those applications of K_0 to topology that do not involve topological K-theory. While what we will be doing here is pure algebra, it is worth saying a bit about the topological motivation to explain what is going on. If X is a path-connected, locally 1-connected topological space with fundamental group G and $R = \mathbb{Z}G$, we can manufacture from X its **singular chain complex with local coefficients** $S_\bullet(X)$. This is a chain complex of free R-modules which is the same thing as the usual singular chain complex of the universal cover $\tilde{X}$ of X, together with the R-module structure coming from the action of G on $\tilde{X}$ by covering transformations. Furthermore, the chain homotopy equivalence class of the chain complex $S_\bullet(X)$ only depends on the homotopy equivalence class of the space X. The chain complex $S_\bullet(X)$ is quite large in general; for most spaces of interest, the R-modules in it are not even countably generated. However, if X is a finite CW-complex, then $S_\bullet(X)$ is chain homotopy equivalent to the **cellular chain complex with local coefficients** $C_\bullet(X)$, a chain complex of free R-modules with only finitely many non-zero chain groups and with each of these chain groups finitely generated. Thus an obvious necessary condition for a space X to be homotopy-equivalent to a finite CW-complex is for $S_\bullet(X)$ to be chain-homotopy-equivalent to a finitely generated complex of free R-modules.

Under some circumstances, it is easy to check not this condition but something weaker, called **finite domination**. The space X is finitely dominated if up to homotopy it is a retract of a finite CW-complex; in other words, if there is a finite CW-complex Y and there are maps $f : X \to Y$, $g : Y \to X$ with $g \circ f \simeq id_X$. An important question is then whether this implies that X is homotopy-equivalent to some (other) finite CW-complex. (It is not hard to show that X is homotopy-equivalent to **some** CW-complex (see [Varadarajan, Theorem 3.9] or [Spanier, Ch. 7, Exercise G6]), but this complex is not necessarily finite.) This question was answered by C. T. C. Wall in an important series of papers. Wall showed that if X is finitely dominated, then $S_\bullet(X)$ is chain-homotopy-equivalent to a finitely generated complex of projective R-modules. The **Wall finiteness obstruction** of X is then the "Euler characteristic" of this complex in the group $\tilde{K}_0(R)$. Though we will not show here that vanishing of the obstruction is sufficient for finiteness (for this see [Wall] or [Varadarajan]), it will be clear that it is necessary. The Wall obstruction occurs in many problems in geometric topology, such as the question studied by Siebenmann of when a non-compact manifold is homeomorphic to the interior of a compact manifold with boundary. For this and other geometric problems related to the Wall obstruction, see [Weinberger, Ch. 1, §1 and §4].

We shall now provide an abstract treatment of the Wall finiteness obstruction for chain complexes of R-modules, as an outgrowth of the classical theory of the Euler-Poincaré characteristic for topological spaces. Since we

don't assume the reader is very familiar with homological algebra, we begin with a review of some classical notions and facts. The reader who has had a course in homological algebra or homology theory can probably skip ahead to 1.7.9 after reviewing the statements of Theorems 1.7.4 and 1.7.7.

1.7.1. Definition. Let R be a ring (with unit). A **chain complex** of R-modules is a pair $(C_\bullet, d)$, where $C_\bullet$ is a $\mathbb{Z}$-graded R-module and d is an R-module homomorphism $C \to C$ of degree -1 such that $d^2 = 0$. (In other words, d is defined by maps $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.) Recall that the **homology** of such a chain complex is $H(C) = \ker d / \operatorname{im} d$; more precisely, $H_n = \ker d_n / \operatorname{im} d_{n+1}$. Elements of $\ker d$ are called **cycles** and elements of $\operatorname{im} d$ are called **boundaries**. The chain complex is called **acyclic** if $H(C) = 0$, *i.e.*, if the sequence

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is exact.

1.7.2. Definition. If $(C_\bullet, d)$, $(C'_\bullet, d')$ are chain complexes of R-modules, a **chain map** between them is an R-module homomorphism $\varphi : C \to C'$ of degree 0 intertwining d and d', *i.e.*, is given by maps $\varphi_n : C_n \to C'_n$ such that $d'_n \circ \varphi_n = \varphi_{n-1} \circ d_n$. It is immediate that such a φ induces maps on homology $\varphi_* : H_n(C) \to H_n(C')$. If $\varphi : C \to C'$ and $\psi : C \to C'$ are chain maps, a **chain homotopy** between them is an R-module homomorphism $s : C \to C'$ of degree $+1$ such that

$$s \circ d + d' \circ s = \varphi - \psi. \tag{1.7.3}$$

Chain homotopy is an equivalence relation on chain maps. We write $\varphi \simeq \psi$ if there is a chain homotopy between them. A chain homotopy from id_C to 0 is called a **chain contraction**, and if such a homotopy exists, $C_\bullet$ is called **chain-contractible**.

Note that (1.7.3) implies that $\varphi_* = \psi_*$ on homology. Indeed if $dx = 0$, then

$$\varphi(x) - \psi(x) = s \circ d(x) + d' \circ s(x) = d'(s(x)),$$

so that $\varphi(x)$ and $\psi(x)$ lie in the same homology class. Thus if a chain complex is chain-contractible, it is acyclic. The converse is false without additional conditions.

If there exist chain maps $\varphi : C \to C'$ and $\psi : C' \to C$ such that $\psi \circ \varphi \simeq id_C$ and $\varphi \circ \psi \simeq id_{C'}$, then we say C and C' are **chain-homotopy-equivalent**. This of course implies by our previous remark that φ_* is an isomorphism on homology with inverse ψ_*.

1.7.4. Proposition. *If $(C_\bullet, d)$ is an acyclic chain complex of* **projective** *R-modules and $C_\bullet$ is bounded below, i.e., $C_j = 0$ for j sufficiently small, then $C_\bullet$ is chain-contractible.*

Proof. Without loss of generality assume $C_j = 0$ for $j < 0$. (Otherwise reindex.) We construct a contraction $s_n : C_n \to C_{n+1}$ by induction on n to satisfy the needed condition

$$s_{n-1} \circ d_n + d_{n+1} \circ s_n = id_{C_n}. \tag{$*_n$}$$

At the same time, we also show by induction that $\ker d_n$ is a direct summand in C_n. To begin the induction, set $s_j = 0$ for $j < 0$ and note that by the assumptions that $H_0(C) = 0$ and $C_{-1} = 0$, $d_1 : C_1 \to C_0$ must be surjective. Since C_0 is projective, d_1 must have a right inverse s_0, so $(*_0)$ holds. Furthermore, $\operatorname{im} d_1 = \ker d_0 = C_0$ is projective.

For the inductive step, assume we've constructed s_j for $j < n$ to satisfy $(*_j)$ and we know $\ker d_j = \operatorname{im} d_{j+1}$ is a direct summand in C_j for $j < n$, hence projective by Theorem 1.1.2. We shall construct s_n to satisfy $(*_n)$. By inductive assumption, $C_{n-1} = (\operatorname{im} d_n) \oplus Q_{n-1}$ for some projective Q_{n-1}. On $\operatorname{im} d_n = \ker d_{n-1}$, $s_{n-2} \circ d_{n-1} = 0$, so $d_n \circ s_{n-1}$ is the identity. Thus, by $(*_{n-1})$, s_{n-1} is a right inverse for

$$d_n : C_n \to \operatorname{im} d_n \subseteq C_{n-1}.$$

Therefore $s_{n-1} \circ d_n$ is an idempotent endomorphism of C_n with image Q_n complementary to $\ker d_n$, and $\ker d_n = \operatorname{im} d_{n+1}$ is R-projective. Since

$$d_{n+1} : C_{n+1} \to \operatorname{im} d_{n+1} = \ker d_n$$

is surjective, it has a right inverse s_n. Extend s_n to all of C_n by making it 0 on Q_n. Then $(*_n)$ is satisfied and we've completed the inductive step. The Proposition now follows by induction. ☐

1.7.5. Definition. Suppose $\varphi : (C_\bullet, d) \to (C'_\bullet, d')$ is a chain map between chain complexes of R-modules. Its **mapping cone** is $(C''_\bullet, d'')$, where $C''_j = C_{j-1} \oplus C'_j$ (note the degree shift in the first summand!) and

$$d''_j(c, c') = (-d_{j-1}c, \varphi(c) + d'_j(c')).$$

This is a chain complex since

$$\begin{aligned} d''_{j-1} \circ d''_j(c, c') &= \left(d_{j-2} \circ d_{j-1}c, \varphi(-d_{j-1}c) + d'_{j-1}\left(\varphi(c) + d'_j(c')\right)\right) \\ &= (0, -\varphi \circ d_{j-1}(c) + d'_{j-1} \circ \varphi(c) + 0) = (0, 0). \end{aligned}$$

1.7.6. Theorem (Fundamental Theorem of Homological Algebra). *Suppose*

$$0 \to (C'_\bullet, d') \xrightarrow{\alpha} (C_\bullet, d) \xrightarrow{\beta} (C''_\bullet, d'') \to 0$$

is a short exact sequence of chain complexes. (This means α and β are chain maps and the sequence of R-modules

$$0 \to C'_j \xrightarrow{\alpha} C_j \xrightarrow{\beta} C''_j \to 0$$

is exact for each j.) Then there is an induced long exact sequence of homology modules

$$\cdots \to H_j(C') \xrightarrow{\alpha_*} H_j(C) \xrightarrow{\beta_*} H_j(C'') \xrightarrow{\partial} H_{j-1}(C') \to \cdots.$$

Proof. This is the quintessential "diagram chase." First we go through the definition of $H_j(C'') \xrightarrow{\partial} H_{j-1}(C')$; then we go through the proof of exactness. Let $[x'']$ be a class in $H_j(C'')$ represented by $x'' \in C''_j$ with $d''x'' = 0$. Since β is surjective, $x'' = \beta(x)$ with $x \in C_j$. Since $d'' \circ \beta(x) = 0$ and β is a chain map, $\beta \circ d(x) = 0$, *i.e.*, $d(x) \in \ker \beta = \operatorname{im} \alpha$. Hence $d(x) = \alpha(x')$ for some $x' \in C_{j-1}$. We claim $d'(x') = 0$, so that x' is a "cycle," *i.e.*, represents a class in $H_{j-1}(C')$. Indeed, since α is a chain map, $\alpha \circ d'(x') = d \circ \alpha(x') = d^2(x) = 0$. But α was injective, so $d'(x') = 0$. Now let $\partial[x''] = [x']$. We leave to the reader the simple argument that shows this is independent of the choice of x'' within its homology class and independent of the choice of the lift x of x''.

We proceed now to the proof of exactness. The construction of $\partial[x'']$ above gives $\alpha_*(\partial[x'']) = [\alpha(x')] = [0]$, and also shows that if $[x''] = \beta_*[x]$ for some $[x] \in H_j(C)$, then $\partial[x''] = 0$ (since $d(x) = 0$). Also, $\beta_* \circ \alpha_* = 0$ since $\beta \circ \alpha = 0$. So the image of each map in our sequence is contained in the kernel of the next one.

For the reverse containments, suppose for instance that $x \in C_j$, $d(x) = 0$, and $\beta_*[x] = 0$ in $H_j(C'')$. Then $\beta(x) = d''(y'')$ for some $y'' \in C''_{j+1}$. Since β is surjective, we may choose $y \in C_{j+1}$ with $\beta(y) = y''$. Since $d'' \circ \beta(y) = \beta \circ d(y) = \beta(x)$, $x - d(y) \in \ker \beta = \operatorname{im} \alpha$, and $[x] \in \operatorname{im} \alpha_*$. Thus $\ker \beta_* \subseteq \operatorname{im} \alpha_*$.

Next, suppose $x \in C''_j$, $d''(x) = 0$, and $\partial[x''] = 0$ in $H_{j-1}(C')$. By the description of ∂ above, this means $x'' = \beta(x)$ with $d(x) = \alpha(x')$ and $x' = d'(y')$, $y' \in C'_j$. Then $d \circ \alpha(y') = \alpha \circ d'(y') = \alpha(x') = d(x)$, so $x - \alpha(y') \in \ker d$. Since also $\beta(x - \alpha(y')) = \beta(x) = x''$, this shows $[x''] = \beta_*[x - \alpha(y')]$. Hence $\ker \partial \subseteq \operatorname{im} \beta_*$.

Finally, suppose $x' \in C_{j-1}$, $d'(x') = 0$, and $\alpha_*[x'] = 0$ in $H_{j-1}(C)$. Then $\alpha(x') = d(x)$ for some $x \in C_j$. Let $x'' = \beta(x)$. Then $d''(x'') = \beta \circ d(x) = \beta \circ \alpha(x') = 0$, so x'' defines a class $[x'']$ in $H_j(C'')$. From the description of ∂, $\partial[x''] = [x']$, so $\ker \alpha_* \subseteq \operatorname{im} \partial$. This completes the proof of exactness. □

1.7.7. Theorem. *A chain map between chain complexes of R-modules is a chain homotopy equivalence if and only if its mapping cone is contractible. If the complexes are bounded below and consist of projective R-modules, then it is a homotopy equivalence if and only if the mapping cone is acyclic, or if and only if it induces an isomorphism on homology.*

Proof. Let $\varphi : (C_\bullet, d) \to (C'_\bullet, d')$ be a chain map and let $(C''_\bullet, d'')$ be its mapping cone. First observe that there is a short exact sequence of chain complexes

$$0 \to (C'_\bullet, d') \to (C''_\bullet, d'') \to (C_{\bullet-1}, -d) \to 0.$$

The maps here are the obvious ones: we map C'_j to $C''_j = C_{j-1} \oplus C'_j$ by $c' \mapsto (0, c')$, and we project C''_j onto the first summand C_{j-1}. The fact that these maps commute with the boundary maps is obvious from Definition 1.7.5. Since changing the sign of d doesn't change the homology of C, we obtain from Theorem 1.7.6 an exact sequence

$$(1.7.8) \qquad \cdots \to H_n(C'') \to H_{n-1}(C) \xrightarrow{\partial} H_{n-1}(C') \to H_{n-1}(C'') \to \cdots .$$

Here it is easy to check from the definition of ∂ that the map $H_{n-1}(C) \to H_{n-1}(C')$ is just φ_*. Thus φ_* being an isomorphism in all degrees is equivalent to the mapping cone C'' being acyclic. Furthermore, if C and C' are bounded below and consist of projective modules, then the same is true of C''. Hence, by Proposition 1.7.4, the mapping cone in this case is acyclic if and only if it is contractible.

It remains to show that φ is a homotopy equivalence if and only if C'' is contractible. Suppose $s'' : C'' \to C''$ is a chain contraction. Then we define $s : C \to C$, $s' : C' \to C'$, and $\psi : C' \to C$ by

$$\begin{aligned} s''(c,\, 0) &= (s(c),\, \cdots),\\ s''(0,\, c') &= (\psi(c'),\, -s'(c')). \end{aligned}$$

Since $d''s'' + s''d'' = id_{C''}$, we have

$$\begin{aligned} (c,\, 0) &= (-d\circ s(c),\, \cdots) + s''(-dc,\, \varphi(c))\\ &= (-d\circ s(c) + \psi\circ\varphi(c) - s\circ d(c),\, \cdots)\,,\\ (0,\, c') &= d''(\psi(c'),\, -s'(c')) + s''(0,\, d'(c'))\\ &= (-d\circ\psi(c'),\, \varphi\circ\psi(c') - d'\circ s'(c')) + (\psi\circ d'(c'),\, -s'\circ d'(c'))\,, \end{aligned}$$

which says

$$\begin{cases} -d\circ\psi(c') + \psi\circ d'(c') = 0\ \forall c' \quad (\psi \text{ is a chain map});\\ c = -d\circ s(c) + \psi\circ\varphi(c) - s\circ d(c)\ \forall c \quad (\psi\circ\varphi \underset{s}{\simeq} id_C);\\ c' = \varphi\circ\psi(c') - d'\circ s'(c') - s'\circ d'(c')\ \forall c' \quad (\varphi\circ\psi \underset{s'}{\simeq} id_{C'}). \end{cases}$$

In the other direction, suppose φ is a homotopy equivalence with homotopy inverse ψ, and suppose one has homotopies s from $\psi\circ\varphi$ to id_C and s' from $\varphi\circ\psi$ to $id_{C'}$. Let

$$\begin{aligned} s''(c,\, c') = (s(c) + \psi(c') + \psi\circ s'\circ\varphi(c)\psi\circ\varphi\circ s(c),&\\ -s'(c') + s'\circ\varphi\circ s(c) - (s')^2\circ\varphi(c))\,.& \end{aligned}$$

We will check that one obtains a chain contraction of C''. Note that

$$\begin{aligned} (d''s'' + s''d'')(c,\, c') &= d''\,(s(c) + \psi(c') + \psi\circ s'\circ\varphi(c) - \psi\circ\varphi\circ s(c),\\ &\quad -s'(c') + s'\circ\varphi\circ s(c) - (s')^2\circ\varphi(c))\\ &\qquad + s''\,(-d(c),\, \varphi(c) + d'(c'))\\ &= (-d\circ s(c) - d\circ\psi(c') - d\circ\psi\circ s'\circ\varphi(c)\\ &\quad + d\circ\psi\circ\varphi\circ s(c),\ \varphi\circ s(c) + \varphi\circ\psi(c')\\ &\qquad + \varphi\circ\psi\circ s'\circ\varphi(c) - \varphi\circ\psi\circ\varphi\circ s(c)\\ &\qquad -d'\circ s'(c') + d'\circ s'\circ\varphi\circ s(c) - d'\circ(s')^2\circ\varphi(c))\\ &\quad + (-s\circ d(c) + \psi\circ\varphi(c) + \psi\circ d'(c')\\ &\quad + \psi\circ s'\circ\varphi(-d(c)) + \psi\circ\varphi\circ s(d(c)),\\ &\quad - s'\circ\varphi(c) - s'\circ d'(c')\\ &\quad +s'\circ\varphi\circ s(-d(c)) - (s')^2\circ\varphi(-d(c)))\,. \end{aligned}$$

The first coordinate (after some regrouping) is

$$\begin{aligned}
&= [-d \circ s(c) - s \circ d(c)] + [-d \circ \psi(c') + \psi \circ d'(c')] \\
&\quad + [-d \circ \psi \circ s' \circ \varphi(c) - \psi \circ s' \circ \varphi \circ d(c)] \\
&\quad + [d \circ \psi \circ \varphi \circ s(c) + \psi \circ \varphi \circ s \circ d(c)] + \psi \circ \varphi(c) \\
&= -(d \circ s + s \circ d)(c) + (\psi \circ \varphi \circ s \circ d + \psi \circ \varphi \circ d \circ s)(c) \\
&\qquad - \psi \circ (d' \circ s' + s' \circ d') \circ \varphi(c) + \psi \circ \varphi(c) \\
&= c + \psi \circ \varphi \circ (\psi \circ \varphi - id_C)(c) \\
&\qquad - \psi \circ (\varphi \circ \psi - id_{C'}) \circ \varphi(c) \\
&= c.
\end{aligned}$$

The second coordinate (after some regrouping) is

$$\begin{aligned}
&= [\varphi \circ \psi(c') - d' \circ s'(c') - s' \circ d'(c')] \\
&\quad + [\varphi \circ s(c) + d' \circ s' \circ \varphi \circ s(c) - \varphi \circ \psi \circ \varphi \circ s(c)] \\
&\quad + [-s' \circ \varphi(c) - d' \circ (s')^2 \circ \varphi(c) + \varphi \circ \psi \circ s' \circ \varphi(c)] \\
&\quad - s' \circ \varphi \circ s \circ d(c) + (s')^2 \circ \varphi \circ d(c) \\
&= c' + (id_{C'} - \varphi \circ \psi + d' \circ s') \circ \varphi \circ s(c) \\
&\qquad + (\varphi \circ \psi - id_{C'} - d' \circ s') \circ s' \circ \varphi(c) \\
&\qquad - s' \circ \varphi \circ s \circ d(c) + (s')^2 \circ \varphi \circ d(c) \\
&= c' - s' \circ d' \circ \varphi \circ s(c) \\
&\qquad + s' \circ d' \circ s' \circ \varphi(c) \\
&\qquad - s' \circ \varphi \circ s \circ d(c) + (s')^2 \circ \varphi \circ d(c) \\
&= c' - s' \circ \varphi \circ (d \circ s + s \circ d)(c) \\
&\qquad + s' \circ (d' \circ s' + s' \circ d') \circ \varphi(c) \\
&= c' - s' \circ \varphi \circ (\psi \circ \varphi - id_C)(c) \\
&\qquad + s' \circ (\varphi \circ \psi - id_{C'}) \circ \varphi(c) \\
&= c'.
\end{aligned}$$

This confirms that s'' is a chain contraction of C''. □

Now we're ready to introduce the connection with $K_0(R)$.

1.7.9. Definition. A chain complex $(C_\bullet, d)$ of R-modules is called **bounded** if the modules C_j are non-zero for only finitely many j, and is called **of finite type** if it is bounded and all the C_j are finitely generated. (The connection with topology is that the cellular chain complex of a finite CW-complex is of finite type, and the cellular chain complex of a finite-dimensional CW-complex is bounded (with non-zero chain groups only in the dimensions of the cells of the complex).)

If $(C_\bullet, d)$ is a chain complex of finite type of projective R-modules, we define its **Euler characteristic** by

$$\chi(C) = \sum_{j=-\infty}^{\infty} (-1)^j [C_j] \quad (\text{in } K_0(R)).$$

Note that this is really a finite sum, and that d is not used in the definition of $\chi(C)$. Also define $\tilde{\chi}(C)$ to be the image of $\chi(C)$ in $\tilde{K}_0(R)$.

1.7.10. Proposition ("Euler-Poincaré Principle"). *The Euler characteristic is additive on short exact sequences of complexes of finite type. In other words, if*

$$0 \to C' \to C'' \to C \to 0$$

is a short exact sequence of chain complexes of finite type of projective R-modules, then $\chi(C'') = \chi(C') + \chi(C)$. Furthermore, if $(C_\bullet, d)$ is a chain complex of finite type of projective R-modules, **and if all its homology modules are projective,** *then*

$$\chi(C) = \sum_{j=-\infty}^{\infty} (-1)^j [H_j(C)].$$

Proof. Since any short exact sequence of projective modules splits, if

$$0 \to C' \to C'' \to C \to 0$$

is a short exact sequence of chain complexes of finite type of projective R-modules, then $C''_j \cong C'_j \oplus C_j$, hence $[C''_j] = [C'_j] + [C_j]$ and the formula $\chi(C'') = \chi(C') + \chi(C)$ follows upon taking the alternating sum over j.

Next, suppose $(C_\bullet, d)$ is a chain complex of finite type of projective R-modules and all the homology modules $H_j(C)$ are R-projective. Let $Z_j = \ker(d_j : C_j \to C_{j-1})$, $B_j = \operatorname{im}(d_{j+1} : C_{j+1} \to C_j)$. We have short exact sequences

$$(*) \qquad 0 \to Z_{j+1} \to C_{j+1} \xrightarrow{d_{j+1}} B_j \to 0,$$

$$(**) \qquad 0 \to B_j \to Z_j \to H_j \to 0.$$

Since H_j is assumed projective, $(**)$ splits, and $Z_j \cong B_j \oplus H_j$. Since the complex is assumed to be of finite type, we may assume (after reindexing) that $C_j = 0$ for $j < 0$, in which case $C_0 = Z_0$ is projective; hence, since $Z_0 \cong B_0 \oplus H_0$, B_0 is projective. Thus $C_1 \xrightarrow{d_1} B_0$ must split and so $C_1 \cong B_0 \oplus Z_1$. This implies Z_1 is projective, and since $Z_1 \cong B_1 \oplus H_1$,

B_1 is projective. Continuing by induction, all the B_j and Z_j are projective and all of the above short exact sequences split. Thus we obtain

$$[Z_{j+1}] + [B_j] = [C_{j+1}] \quad \text{(from (*))},$$

$$[B_j] + [H_j] = [Z_j] \quad \text{(from (**))}.$$

Substituting in the definition of $\chi(C)$, we obtain

$$\begin{aligned}
\chi(C) &= \sum_{j=-\infty}^{\infty} (-1)^j [C_j] \\
&= \sum_{j=-\infty}^{\infty} (-1)^j ([Z_j] + [B_{j-1}]) \\
&= \sum_{j=-\infty}^{\infty} (-1)^j ([H_j] + [B_j] + [B_{j-1}]) \\
&= \sum_{j=-\infty}^{\infty} (-1)^j [H_j] + \sum_{j=-\infty}^{\infty} (-1)^j [B_j] \sum_{j=-\infty}^{\infty} (-1)^j [B_j] \\
&= \sum_{j=-\infty}^{\infty} (-1)^j [H_j].
\end{aligned}$$ □

1.7.11. Corollary. *The Euler characteristic is well defined on chain complexes of projective R-modules which are homotopy-equivalent to complexes of finite type of projective R-modules, and is constant on homotopy equivalence classes. It is also additive on short exact sequences of such chain complexes.*

Proof. Suppose $(C_\bullet, d)$ is a chain complex of projective R-modules which is homotopy-equivalent to a chain complex of finite type $(C^1_\bullet, d^1)$ of projective R-modules. We define $\chi(C) = \chi(C^1)$. Of course, to know that this makes sense, we need to check that it is independent of the choice of C^1. If $(C^2_\bullet, d^2)$ is another possible choice, then C^1 and C^2 are each homotopy-equivalent to C, hence are homotopy-equivalent to each other. Let $\varphi : C^1 \to C^2$ be a homotopy equivalence between them and let C^3 be its mapping cone. Since C^1 and C^2 are of finite type and consist of projective R-modules, the same is true of C^3. Furthermore, from the short exact sequence

$$0 \to (C^2_\bullet) \to (C^3_\bullet) \to (C^1_{\bullet-1}) \to 0$$

and Proposition 1.7.10, we obtain that

$$\chi(C^3) = \chi(C^2) - \chi(C^1).$$

But C^3 is acyclic by Theorem 1.7.7, so its homology modules are 0 and hence $\chi(C^3) = 0$ by Proposition 1.7.10 again. Thus $\chi(C^1) = \chi(C^2)$,

as required. The same calculation shows that homotopy-equivalent chain complexes have the same Euler characteristic. Finally, additivity on short exact sequences also follows immediately from Proposition 1.7.10 and the fact that short exact sequences of projective modules must split. □

We're now finally ready for Wall's theorem.

1.7.12. Theorem [**Wall**]. *Let $(C_\bullet, d)$ be a chain complex of projective R-modules which is homotopy-equivalent to a chain complex of finite type of projective R-modules. Then $(C_\bullet, d)$ is homotopy-equivalent to a chain complex of finite type of* **free** *R-modules if and only if $\tilde{\chi}(C) = 0$ in $\tilde{K}_0(R)$.*

Proof. Suppose $(C_\bullet, d)$ is homotopy-equivalent to $(C'_\bullet, d')$ of finite type, with both complexes consisting of projective modules. By Corollary 1.7.11, $\chi(C) = \chi(C')$; hence $\tilde{\chi}(C) = \tilde{\chi}(C')$. If C' consists of finitely generated free modules, then clearly $\tilde{\chi}(C') = 0$ so $\tilde{\chi}(C) = 0$.

On the other hand, suppose $\tilde{\chi}(C') = 0$. It will be enough to show C' is homotopy-equivalent to a complex of finite type consisting of free R-modules. Suppose $C'_j = 0$ for j outside of an interval $\{k, k+1, \dots, k+n\}$. Choose projective modules $Q_n, \dots, Q_0$ such that $C'_{k+n} \oplus Q_n$ is free, $C'_{k+n-1} \oplus Q_n \oplus Q_{n-1}$ is free, and in general such that $C'_{k+j} \oplus Q_{j+1} \oplus Q_j$ is free for $0 \le j < n$. If $(T_\bullet, d_T)$ is chain-contractible, then replacing $(C'_\bullet, d')$ by $(C'_\bullet, d') \oplus (T_\bullet, d^T)$ doesn't change its homotopy class. So let $(T^j_\bullet, d^{T^j})$ be defined by

$$T^j_i = \begin{cases} 0, & i \neq k+j,\ k+j-1, \\ Q_j, & i = k+j,\ k+j-1, \end{cases}$$

with $d^{T^j}_{k+j} : Q_j \to Q_j$ the identity map. This is clearly contractible, so now

$$(C''_\bullet, d'') = (C'_\bullet, d') \oplus \bigoplus_{j=0}^{n} (T^j_\bullet, d^{T^j})$$

is homotopy-equivalent to $(C'_\bullet, d')$ and has free modules in all degrees except $k-1$. Thus

$$0 = \tilde{\chi}(C') = \tilde{\chi}(C'') = (-1)^k [C''_{k-1}] \quad (\text{in } \tilde{K}_0(R)),$$

so C''_{k-1} is stably free. Choose a finitely generated free R-module F such that $C''_{k-1} \oplus F \cong F$. Let $(T_\bullet, d^T)$ be defined by

$$T_j = \begin{cases} 0, & j \neq k-1,\ k-2, \\ F, & j = k-1,\ k-2, \end{cases}$$

with $d^T_{k-1} : F \to F$ the identity map. This is clearly contractible, and $(C''_\bullet, d'') \oplus (T_\bullet, d^T)$ now has free modules in all degrees. So $(C_\bullet, d)$ is homotopy-equivalent to a chain complex of finite type of free R-modules. □

When R is Noetherian, we can also relate finite generation of a chain complex C to finite generation of its homology, as shown in the following theorem.

1.7.13. Theorem. *Let R be a (left) Noetherian ring. If $(C_\bullet, d)$ is a bounded chain complex of projective R-modules, then $H_j(C)$ is finitely generated over R for all j if and only if C is homotopy-equivalent to a complex of finite type of projective R-modules. In particular, if the homology modules of C are finitely generated, its Euler characteristic is well defined.*

Proof. One direction is easy. If C is homotopy-equivalent to a complex of finite type, then its homology is the same as that of a complex of finite type, so we might as well assume C is already of finite type. If R is Noetherian and C_j is finitely generated, then its submodule $Z_j = \ker d_j$ must also be finitely generated, hence $H_j(C)$, which is a quotient of Z_j, is finitely generated. Thus all homology modules are finitely generated.

For the converse, without loss of generality, assume $C_j = 0$ for $j < 0$ and for $j > n$. We first construct by induction on m, starting at 0 and continuing up to $m = n$, a complex of finite type $(C'_j, d')_{j \le m}$ of free R-modules and a chain map

$$\varphi : (C'_\bullet, d') \to (C_\bullet, d)$$

which induces isomorphisms on homology through degree $m-1$. Of course we take $C'_j = 0$ for $j < 0$ and for $j > n$. To begin the induction, note that since $C_{j-1} = 0$, $H_0(C) = C_0/\operatorname{im} d_1$. Choose a finite set of generators $[x_1], \dots, [x_r]$ for $H_0(C)$ and representatives $x_1, \dots, x_r \in C_0$. Let C'_0 be the free R-module on generators $y_1, \dots, y_r$ and let $\varphi_0(y_k) = x_k$. Since R is assumed Noetherian, the kernel B'_0 of the composite map

$$C'_0 \xrightarrow{\varphi_0} C_0 \to H_0(C),$$

being a submodule of the finitely generated module C'_0, is also finitely generated. Choose generators $z_1, \dots, z_t$ for B'_0 and let C'_1 be free on generators $w_1, \dots, w_t$. Define d'_1 so that $d'_1(w_k) = z_k$. Then $C'_1 \xrightarrow{d'_1} C'_0$ is a chain complex with $H_0(C') = C'_0/B'_0$ and φ_0 induces an isomorphism on H_0. Since we want φ_0 to be the 0-degree part of a chain map, we need to define φ_1 so that

$$\begin{array}{ccc} C'_1 & \xrightarrow{\varphi_1} & C_1 \\ {\scriptstyle d'_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ C'_0 & \xrightarrow{\varphi_0} & C_0 \end{array}$$

commutes. Since $\varphi_0(z_k)$ goes to 0 in $H_0(C)$, we can choose $u_k \in C_1$, with $d_1(u_k) = \varphi_0(z_k)$. So we let $\varphi_1(w_k) = u_k$ and the condition is satisfied. This completes the first step in the induction.

For the inductive step, assume we've constructed a complex of finite type of free R-modules (C'_j, d_j) for $j \le m$ and a chain map $\varphi : C' \to C$ which is an isomorphism on homology in degrees $< m$. Continuing as before, choose generators $[x_1], \dots, [x_r]$ for $H_m(C)$ and representatives $x_1, \dots, x_r \in Z_m \subseteq C_m$. Replace the old C'_m by its direct sum with the free R-module

on generators $y_1, \ldots, y_r$ and let $\varphi_m(y_k) = x_k$. We keep φ_m the same on the old C'_m. Similarly, we do not change d'_m on the old C'_m and let $d'_m(y_k) = 0$. Then we still have a chain complex and a chain map for $j \leq m$ but now φ_* is surjective on H_m. As before, we choose C'_{m+1} finitely generated and free with $d_{m+1} : C'_{m+1} \to C_m$ mapping onto the kernel of the composite

$$C'_m \xrightarrow{\varphi_m} Z_m \to H_m(C),$$

and define φ_{m+1} as above so that we have a chain map which now is an isomorphism in homology through degree m. We continue by induction until we've constructed a complex of finite type of free R-modules and a chain map $\varphi : C' \to C$ which is an isomorphism on homology in degrees $< n$ and a surjection in homology in degree n. Of course, since everything is zero in degrees $> n$, φ_* is actually an isomorphism on homology in all degrees except n.

Now consider the mapping cone $(C''_\bullet, d'')$ of φ. This is a bounded complex of projective R-modules with non-zero chain modules only in degrees 0 through $n+1$. By the exact sequence (1.7.8) (in our situation C and C' are reversed), C'' has only one non-zero homology module, in degree $n+1$. Repeating the proof of Proposition 1.7.4, we can construct a chain contraction of C'' through degree n, which shows that $d''_{n+1} : C''_{n+1} \to B''_n$ is split surjective and thus that $H_{n+1}(C'') = Z''_{n+1} = \ker d''_{n+1}$ is R-projective and a direct summand in $C''_{n+1} \cong C'_n$. Hence we may replace C'_n by a projective complement to $H_{n+1}(C'')$ and thereby make C' a complex of finite type of projective R-modules and φ_* an isomorphism on homology, hence a chain-homotopy equivalence, by Theorem 1.7.7. □

Remark. This proof demonstrates clearly the origin of Wall's obstruction. At the last step of our induction, we can either make φ_* into a homology isomorphism in degree n at the expense of making C_n a possibly non-free projective module, or we can make C'_n free and φ_* an epimorphism on homology in degree n, but in general we can't take C_n free and at the same time make φ a homotopy equivalence.

Now for some topological applications. Wall's work on finiteness obstructions for chain complexes arose from the question of when a connected space X is homotopy-equivalent to a finite CW-complex. If Y is a finite connected CW-complex, Y is locally simply connected (so that covering space theory applies) and has a finitely presented fundamental group π. (The fundamental group of the 1-skeleton of Y is a finitely generated free group surjecting onto π, and π is obtained from this free group by adding in one relation for each 2-cell.) Thus we may form the universal covering $\tilde{Y}$ of Y, which carries a free cellular action of π. The cellular chain complex of $\tilde{Y}$, while not of finite type over $\mathbb{Z}$, may be viewed as a chain complex of finite type of free R-modules, where $R = \mathbb{Z}\pi$, the integral group ring of π. Alternatively, we may think of this complex as the chain complex of Y with local coefficients. Thus if X is a space which is homotopy-equivalent to Y, it must also have fundamental group π (finitely presented), and its singular chain complex with local coefficients $S_\bullet(X)$, which is a complex of

free R-modules but is very far from being of finite type in general, must be chain-homotopy-equivalent to a complex of finite type of free R-modules.

Theorem 1.7.12 now gives a necessary and sufficient condition for $S_\bullet(X)$ to have this property. Call $S_\bullet(X)$ **finitely dominated** if it is chain-homotopy-equivalent to a complex of finite type of projective R-modules. Theorem 1.7.12 says that a finitely dominated complex has a well-defined **finiteness obstruction** in $\tilde{K}_0(R)$, and is chain-homotopy-equivalent to a complex of finite type of free R-modules if and only if this finiteness obstruction vanishes. If $R = \mathbb{Z}\pi$ happens to be Noetherian, which is not the case for all finitely presented groups π, but is true say if π is a product of a finite group and a free abelian group (the group ring of a finite group is finitely generated as a $\mathbb{Z}$-module, hence Noetherian, and the group ring of $\pi \times \mathbb{Z}^n$ is a Laurent polynomial ring in n variables over the group ring of π), one can apply Theorem 1.7.13 to see that an R-module chain complex C is finitely dominated if and only if it is homologically finite-dimensional and its homology groups are finitely generated.

Wall actually went further than this; he showed that a connected space X with finitely presented fundamental group and the homotopy type of a CW-complex is finitely dominated *if and only if* $S_\bullet(X)$ is finitely dominated, and has the homotopy type of a finite CW-complex *if and only if* $S_\bullet(X)$ is finitely dominated and has vanishing finiteness obstruction. The method of proof for the "if" directions is to inductively construct a sequence Y_n ($n \geq 1$) of finite CW-complexes by attaching cells, along with maps $Y_n \to X$ which are dominations (resp., homotopy equivalences) "through dimension $n-1$." The proof of Theorem 1.7.13 is an abstract version of this technique, in the case where R is Noetherian. In proving homotopy finiteness, the finiteness obstruction is precisely the obstruction to having this inductive process terminate after a finite number of steps.

1.7.14. Example. Let us illustrate a geometric application of Theorems 1.7.12 and 1.7.13. Suppose X^n is a connected non-compact (topological or smooth) manifold and one wants to know whether X is homeomorphic to the interior of a manifold W^n with boundary. Precise necessary and sufficient conditions were found by Siebenmann (provided one stays away from the problem dimensions 3 and 4 by assuming $n \leq 2$ or $n \geq 6$) using surgery theory, but we have done enough now to at least give some interesting necessary conditions.

If W^n exists, then it must have the homotopy type of a finite CW-complex, hence so must X, since the inclusion of X into W is a homotopy-equivalence. Furthermore, for each component N^{n-1} of ∂W, N must have a "collar" neighborhood in W homeomorphic to $N \times [0, 1)$, so the corresponding "end" of $X = W \smallsetminus \partial W$ must be homeomorphic to $N \times (0, 1)$, and in particular must be homotopy-equivalent to the compact manifold N. (For a locally compact Hausdorff space X, a **neighborhood of infinity** may be defined to be the complement of a compact set. An **end** may be defined to be a connected component of $\beta X \smallsetminus X$, where βX is the Stone-Čech or maximal compactification of X (the space of maximal ideals of the

algebra of bounded continuous real-valued functions on X). Equivalently, an end is an equivalence class of components of neighborhoods of infinity. In the present situation, the ends must be in one-to-one correspondence with the components of ∂W.) So the homotopy type of N is determined by that of the corresponding neighborhoods of the associated end of X.

In particular, we now derive a number of necessary conditions for our being able to complete X to a compact manifold with boundary. X must have finitely many ends, and for each end E of X, if X_i is a sequence of connected open neighborhoods of E with $X_i \searrow E$, the fundamental groups of the X_i must stabilize to some finitely presented group $\pi_1(E)$ (in the sense of the Mittag-Leffler condition, that $\varprojlim \pi_1(X_i) = \pi_1(E)$, and for each i, the images in $\pi_1(X_i)$ of the $\pi_1(X_j)$, $j \geq i$, eventually stabilize). Let $R = \mathbb{Z}\pi_1(E)$. Then we obtain an inverse system $(H_\bullet(X_i;\, R))$ of homology groups with local coefficients in R which must also stabilize to what will correspond to $H_\bullet(N;\, R)$. Thus for a suitable open connected neighborhood U of E, $\pi_1(U) = \pi_1(E)$ and $H_\bullet(U;\, R)$ looks like the homology of a compact manifold of dimension $n-1$. If for instance $\pi_1(E)$ is finite (this is not so essential but it already covers an interesting case), R is Noetherian and the homology must be finitely generated by Theorem 1.7.13. By the same Theorem, the cellular chain complex of U with coefficients in R is homotopy-equivalent to a complex of projective modules of finite type. Thus the obstruction $\tilde{\chi}(C_\bullet(U;\, R))$ is defined and must vanish in $\tilde{K}_0(R)$. Siebenmann's Theorem says that once this is satisfied, one can put a boundary on the end E provided at least that $n \neq 3$, 4, or 5. See [Weinberger, §§ 1.5 and 1.6] for a further explanation. The homeomorphism class of the boundary to be added is not always uniquely determined; but the non-uniqueness is also related to K-theory: it is classified by the Whitehead torsion invariant to be studied in §2.4 below.

1.7.15. Remarks. Since this is not a book on topology, we will not prove any purely topological results here. However, in the interests of completeness, let us say a few more words (without proofs) about Wall's original results (in the topological setting) and about one other important area of application, the spherical space form problem.

Wall's work on the finiteness problem was motivated in part by earlier work of Swan [Swan1] on the question of when a finitely dominated space X, with finite fundamental group $\pi = \pi_1(X)$ and universal cover $\tilde{X}$ homotopy-equivalent to a sphere, can be homotopy-equivalent to a finite CW-complex. Swan already realized that at least in this particular situation, an obstruction in $\tilde{K}_0(\mathbb{Z}\pi)$ plays a fundamental role, and he showed how to kill off this obstruction in order to solve a particular geometric problem in which he was interested. This geometric problem was a modified version of what is now known as the **spherical space form problem**: to classify compact manifolds M^n, known as **spherical space forms**, having a sphere as universal cover, $\tilde{M}^n \cong S^n$. Certain obvious examples, such as real projective spaces and lens spaces, arise from free orthogonal actions of finite groups, and the groups that can act in this way are completely known

(see [Wolf]). However, a rather subtle question remains: can there be any examples of spherical space forms not homotopy-equivalent to examples of this type, for instance with fundamental groups (such as the non-abelian group of order pq, where p and q are distinct odd primes with $p|(q-1)$) that cannot have a free orthogonal action on a sphere?

The answer to this latter question turns out to be "yes," and the question of what finite groups can act freely on spheres is now totally understood (see [Madsen]). The relevance of the finiteness obstruction comes from the following method of attack. We begin by looking at n-dimensional CW-complexes X with the desired finite fundamental group π, having the property that the universal cover $\tilde{X}$ is homotopy-equivalent to a sphere S^n. This means of course that $\tilde{X}$ must have vanishing homology in degrees $0 < j < n$, and infinite cyclic homology in degree n, but in fact, by the basic theorems of homotopy theory (the Hurewicz and Whitehead Theorems, to be discussed in §5.1 below), this homology condition is not only necessary but also sufficient. The study of the homology of X shows then that π must be a "group with periodic homology" [CartanEilenberg, pp. 357–358], for which there is an elegant classification theorem [CartanEilenberg, Ch. XII, §11]. It turns out that a necessary and sufficient condition for the existence of such a space X is that each Sylow subgroup of π be either a cyclic group or a generalized quaternion group.

However, one is still faced with another problem: given the X whose universal cover is homotopy-equivalent to a sphere, is X homotopy-equivalent to a (smooth) compact manifold M? If the answer is "yes," then the universal cover of M will be a compact manifold homotopy-equivalent to a sphere. By known results on the Poincaré Conjecture, the universal cover is then actually homeomorphic to a sphere, except perhaps when $n = 3$. (See the remarks following Theorem 2.4.4 below.)

A detailed sketch of how this problem is attacked may be found in Madsen's survey [Madsen]. However, a crucial first step already understood by Swan comes from the well-known fact that any smooth compact manifold is homotopy-equivalent to a finite CW-complex. Thus if M is to exist in the homotopy type of X, the CW-complex X must have vanishing finiteness obstruction in $\tilde{K}_0(\mathbb{Z}\pi)$. This group is known to be finite when π is finite, but is usually quite hard to compute. For cyclic groups of prime order, we began the calculation of this group (in terms of number-theoretic invariants) in Example 1.5.10, and will complete the calculation in Example 3.3.5(b) below.

1.7.16. Exercise (Nontriviality of the finiteness obstruction for bounded complexes of free modules). Let R be a ring with unit, P a finitely generated projective R-module which is not stably free.

(a) Show that there is an R-module homomorphism $\varphi : F \to F$, where F is a **free** R-module of countable infinite rank, that is, split surjective with kernel $\cong P$. (This is attributed to Eilenberg, though the idea may be older. Compare Exercise 1.1.8.)

(b) Deduce that the complex

$$\cdots \to 0 \to 0 \to F \xrightarrow{\varphi} F \to 0 \to 0 \to \cdots$$

is homotopy-equivalent to the complex

$$\cdots \to 0 \to 0 \to P \to 0 \to 0 \to 0 \to \cdots,$$

but not to a complex of finite length consisting of finitely generated free modules.

1.7.17. Exercise. Show that the condition that R be Noetherian in Theorem 1.7.13 is necessary, by exhibiting a non-Noetherian ring R and a chain complex of finite type of free R-modules that does not have finitely generated homology. Hint: find a non-Noetherian commutative ring R (**not** an integral domain) containing an element x whose annihilator in R is not finitely generated.

1.7.18. Exercise (Behavior of the finiteness obstruction under products). It is of interest to know how homotopy finiteness of X and of $X \times Z$ are related, when Z is itself a finite CW-complex, for instance, a sphere or a projective space. The algebraic analogue of this is to take the tensor product of two complexes to obtain a double complex. Note that $\pi_1(X \times Z) \cong \pi_1(X) \times \pi_1(Z)$, so that the relevant ring for the geometrical problem is

$$\mathbb{Z}\pi_1(X \times Z) \cong \mathbb{Z}[\pi_1(X) \times \pi_1(Z)] \cong \mathbb{Z}\pi_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}\pi_1(Z).$$

(a) Show that if $(C^1_\bullet, d^1)$ and $(C^1_\bullet, d^1)$ are complexes of projective R-modules and S-modules, respectively, then

$$\begin{cases} C_j = \bigoplus_{k=-\infty}^{\infty} C^1_{j-k} \otimes_{\mathbb{Z}} C^2_k, \\ d_j = d^1 \otimes id + (-1)^p id \otimes d^2 \quad \text{on } C^1_p \otimes_{\mathbb{Z}} C^2_q \end{cases}$$

defines a complex of projective $R \otimes_{\mathbb{Z}} S$-modules, called the **total complex** of the double complex $C^1_\bullet \otimes_{\mathbb{Z}} C^2_\bullet$.

(b) Show that if $(C^1_\bullet, d^1)$ is homotopy-equivalent to $(\bar{C}^1_\bullet, \bar{d}^1)$ and if $(C^2_\bullet, d^2)$ is homotopy-equivalent to $(\bar{C}^2_\bullet, \bar{d}^2)$, then the total complex of $C^1_\bullet \otimes_{\mathbb{Z}} C^2_\bullet$ is homotopy-equivalent to the total complex of $\bar{C}^1_\bullet \otimes_{\mathbb{Z}} \bar{C}^2_\bullet$. (You can either carry the homotopies around, or else use a mapping cone argument and Theorem 1.7.7 to reduce to the case where one of the complexes is contractible.)

(c) Deduce that if $\chi(C^1)$ and $\chi(C^2)$ are well defined, so is $\chi(C)$, and that if either $\chi(C^1)$ or $\chi(C^2)$ vanishes, so does $\chi(C)$.

(d) Suppose $S = \mathbb{Z}$ (this is the algebraic analogue of taking Z to be simply connected in the geometrical problem). Show that when $\chi(C^1)$ is well defined and C^2 is of finite type, then

$$\tilde{\chi}(C) = \tilde{\chi}(C^1)\chi(C^2),$$

where $\chi(C^2) \in K_0(\mathbb{Z}) = \mathbb{Z}$. The topological version of this exercise shows that if X is dominated by a finite CW-complex and Z is a finite complex with $\chi(Z) = 0$, then $X \times Z$ is homotopy-equivalent to a finite complex, and that if Z is simply connected with $\chi(Z) \neq 0$, then the Wall obstruction of $X \times Z$ is $\chi(Z)\times$ (the Wall obstruction of X). In particular, taking a product with S^1 kills finiteness obstructions, and taking a product with S^2 multiplies them by 2.

1.7.19. Exercise (Algebraic finite domination) [Ranicki]. Recall that a space X is called finitely dominated if up to homotopy it is a retract of a finite CW-complex; in other words, if there is a finite CW-complex Y and there are maps $f : X \to Y$, $g : Y \to X$ with $g \circ f \simeq id_X$. Now if X is literally a retract of a finite CW-complex, in other words, if we can arrange to have $g \circ f = id_X$, then obviously the singular chain complex of X is a direct summand in the singular chain complex of Y, which in turn is homotopy-equivalent to the cellular chain complex of Y, which is of finite type. Thus in this case it is clear that the singular chain complex of X satisfies the hypothesis of Theorem 1.7.12. However, it is perhaps not immediately apparent that the same holds true if we only have $g \circ f \simeq id_X$, for then the singular chain complex $C_\bullet$ of X is only a direct summand of the singular chain complex $D_\bullet$ of Y "up to homotopy."

The following trick for dealing with the general case is due to Ranicki.

(1) Suppose $C_\bullet$ is a chain complex of projective R-modules, bounded below (say non-zero only in non-negative degrees) which is a "direct summand up to homotopy" of a complex of finite type $D_\bullet$ of free R-modules. In other words, we assume we are given chain maps $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to C_\bullet$, as well as a chain homotopy h satisfying $id_C - g \circ f = d \circ h + h \circ d$. Note that $f \circ g - (f \circ g)^2 = d \circ fhg + fhg \circ d$, so that fhg gives a chain homotopy between $f \circ g$ and $(f \circ g)^2$. Show that the endomorphism p of $\bigoplus_{i=0}^{\infty} D_i$ given by the matrix

$$\begin{pmatrix} fg & -d & 0 & 0 & \cdots \\ -fhg & 1-fg & d & 0 & \cdots \\ -fh^2g & fhg & fg & -d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is an idempotent, so that its image is a finitely generated projective module over R.

(2) Let $C'_i = \bigoplus_{j=0}^{i} D_j$ and define a map $d' : C'_i \to C'_{i-1}$ by the $(i-1)\times i$ matrix

$$\begin{pmatrix} fg & -d & 0 & 0 & \cdots \\ -fhg & 1-fg & d & 0 & \cdots \\ -fh^2g & fhg & fg & -d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if i is even,

$$\begin{pmatrix} 1-fg & d & 0 & 0 & \cdots \\ fhg & fg & -d & 0 & \cdots \\ fh^2g & -fhg & 1-fg & d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if i is odd. Show that $(d')^2 = 0$, so that $(C'_\bullet, d')$ is a chain complex.

(3) Define maps $\varphi : C_i \to C'_i$ and $\psi : C'_i \to C_i$ by

$$\varphi(x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{pmatrix} \in C'_i = D_0 \oplus D_1 \oplus \cdots \oplus D_i$$

and by

$$\psi \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_i \end{pmatrix} = h^i g(x_0) + h^{i-1} g(x_1) + \cdots + hg(x_{i-1}) + g(x_i) \in C_i.$$

Show that φ and ψ are chain maps and that they give a chain homotopy equivalence between $C_\bullet$ and $C'_\bullet$. (Hint: $\psi \circ \varphi = g \circ f$, which we already know is chain homotopic to the identity. The homotopy between $\varphi \circ \psi$ and the identity is given by a simple "shift" map.)

(4) Suppose $D_\bullet$ is of "dimension n," in other words, that $D_i = 0$ for $i > n$. Thus $C'_i \cong \bigoplus_{j=0}^{n} D_j$ for all $i \geq n$. Show by "truncating" $C'_\bullet$ that its finiteness obstruction (and thus the finiteness obstruction of $C_\bullet$) is well defined, and equal to the class in $\tilde{K}_0(R)$ of the image of p from (1).

1.7.20. Exercise [Swan1, §6]. The work of Swan discussed in Remarks 1.7.15 above leads to some interesting examples of finitely generated projective modules over group rings. Suppose G is a finite group of order n and let $R = \mathbb{Z}G$, the integral group ring. Define the **norm element** of R by $N = \sum_{g \in G} g$. Observe that for any $g \in G$, $gN = Ng = N$, so N is central in R and $N^2 = nN$. Let $r \in \mathbb{Z}$ be prime to n, and let P_r be the ideal of R generated by r and N. (It doesn't matter whether one takes the ideal to be one-sided or two-sided, since N and r are both central.) Obviously P_1 is just R itself.

(1) Show that P_r is the universal R-module defined by two generators u and v and the relations $gv = v$ all $g \in G$, $Nu = rv$. (Here u corresponds to r and v corresponds to N.)
(2) Show that $P_r \cong P_{r'}$ provided $r \equiv r' \mod n$. (Use (1) and define the isomorphism by $v \mapsto v'$, $u \mapsto u' + hv'$, where $r - r' = hn$.)
(3) Show that $R \oplus P_{rr'} \cong P_r \oplus P_{r'}$. Note the suggestive analogy with Lemma 1.4.11! (Again use (1). If u'' and v'' are the generators of $P_{rr'}$, send $(0, v'') \mapsto (v, 0)$, $(1, 0) \mapsto (u, au' + bv')$, and $(0, u'') \mapsto (r'u, c(nu' - r'v'))$, where a, b, $c \in \mathbb{Z}$ are suitably chosen.)
(4) Choose r and r' in (3) to be multiplicative inverses of each other mod n, and deduce that $P_r \oplus P_{r'} \cong R^2$, hence that P_r and $P_{r'}$ are projective modules whose images in $\tilde{K}_0(R)$ are the negatives of each other. In particular, we find that if $n = 8$ and $r = 3$, then since $3^2 \equiv 1 \mod 8$, P_3 defines an element of $\tilde{K}_0(R)$ which must be either trivial or of order 2. It is known to be of order 2 and to be a generator of $\tilde{K}_0(R)$ when $G = Q_8$ is the quaternion group. The projective modules P_r naturally arise in the study of the finiteness obstructions coming up in the spherical space form problem (as explained above in 1.7.15).

2

K_1 of Rings

1. Defining K_1

Most courses in linear algebra begin with a discussion of vector spaces and dimension, and then go on to a study of automorphisms of vector spaces, *i.e.*, linear transformations and their invariants (determinants, canonical forms, and so on). The usual development of K-theory for rings follows the same pattern. One begins by studying projective modules and their stable classification via K_0, and then goes on to the study of the stable classification of automorphisms of free and projective modules, in other words, to invariants of (invertible) matrices, which are given by the functor K_1.

We will begin with the classical approach to K_1 via matrices, and in the next chapter will describe a more category-theoretic approach via the study of the category of finitely generated projective modules.

2.1.1. Definition. Let R be a ring (with unit). Recall the definitions of $M(R)$ and $GL(R)$ from 1.2.2. We call an $n \times n$ matrix **elementary** if it has 1's on the diagonal and at most one non-zero off-diagonal entry. More precisely, if $a \in R$ and $i \neq j$, $1 \leq i, j \leq n$, we define the elementary matrix $e_{ij}(a)$ to be the $(n \times n)$ matrix with 1's on the diagonal, with an a in the $(i,\ j)$-slot, and with 0's elsewhere. The subgroup of $GL(n,\ R)$ generated by such matrices is denoted $E(n,\ R)$. Via the usual embedding of $GL(n,\ R)$ in $GL(n+1,\ R)$ (see 1.2.2), $E(n,\ R)$ embeds in $E(n+1,\ R)$. The infinite union of the $E(n,\ R)$ is denoted $E(R)$, and is usually called (by slight abuse of language) the **group of elementary matrices**.

The following lemma, which summarizes some easy matrix identities, is only needed in part at the moment, but is included here for future reference.

2.1.2. Lemma. *The elementary matrices over a ring R satisfy the relations*

$$\begin{cases} e_{ij}(a)e_{ij}(b) = e_{ij}(a+b); & (a) \\ e_{ij}(a)e_{kl}(b) = e_{kl}(b)e_{ij}(a), \quad j \neq k \text{ and } i \neq l; & (b) \\ e_{ij}(a)e_{jk}(b)e_{ij}(a)^{-1}e_{jk}(b)^{-1} = e_{ik}(ab), \quad i, j, k \text{ distinct}; & (c) \\ e_{ij}(a)e_{ki}(b)e_{ij}(a)^{-1}e_{ki}(b)^{-1} = e_{kj}(-ba), \quad i, j, k \text{ distinct}. & (d) \end{cases}$$

Furthermore, any upper-triangular or lower-triangular matrix with 1's on the diagonal belongs to $E(R)$.

Proof. The relations are easily checked by matrix multiplication. Suppose $A = (a_{ij}) \in GL(n, R)$ is upper-triangular with 1's on the diagonal. Then

$$A' = (a'_{ij}) = Ae_{12}(-a_{12})e_{23}(-a_{23}) \cdots e_{n-1,n}(-a_{n-1,n})$$

still is upper-triangular with 1's on the diagonal and has 0's on the super-diagonal $j - i = 1$. Let

$$A'' = (a''_{ij}) = A'e_{13}(-a'_{13})e_{24}(-a'_{24}) \cdots e_{n-2,n}(-a'_{n-2,n}).$$

This now is upper-triangular with 1's on the diagonal and has 0's on the super-diagonals $j - i = 1, 2$. Continuing by induction, we construct a sequence

$$A,\ A',\ A'',\ \ldots,\ A^{(n-1)}$$

of matrices in $GL(n, R)$, each differing from the previous one by an element of $E(n, R)$, each upper-triangular with 1's on the diagonal, and with the additional property that $a^{(k)}_{ij}$ vanishes for $0 < j-i \leq k$. Thus $A^{(n-1)} = 1_n$, the $n \times n$ identity matrix, so $A \in E(n, R)$. The lower-triangular case is similar. □

2.1.3. Corollary. *For any matrix $A \in GL(n, R)$, the $2n \times 2n$ matrix $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ lies in $E(2n, R)$.*

Proof. Apply the identity

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

from the proof of Lemma 1.5.4. By Lemma 2.1.2, the first three factors on the right lie in $E(2n, R)$. And

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

hence the last factor on the right is also in $E(2n, R)$, by Lemma 2.1.2 again. □

2.1.4. Proposition (Whitehead's Lemma). *For any ring R, the commutator subgroups of $GL(R)$ and of $E(R)$ coincide with $E(R)$. In particular. $E(R)$ is normal in $GL(R)$ and the quotient $GL(R)/E(R)$ is the maximal abelian quotient $GL(R)_{\text{ab}}$ of $GL(R)$.*

Proof. Since $E(R) \subseteq GL(R)$, $[E(R), E(R)] \subseteq [GL(R), GL(R)]$. Furthermore, relation (c) of Lemma 2.1.2 shows that

$$e_{ij}(a) = [e_{ik}(a), e_{kj}(1)]$$

provided i, j, and k are distinct. Thus each generator of $E(R)$ is a commutator of two other generators and $[E(R), E(R)] = E(R)$. We need only show that $[GL(R), GL(R)] \subseteq E(R)$. Let A, $B \in GL(n, R)$. We embed $GL(n, R)$ in $GL(2n, R)$ and compute that

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

By Corollary 2.1.3, all the factors on the right lie in $E(2n, R)$, so

$$ABA^{-1}B^{-1} \in E(R). \quad \square$$

2.1.5. Definition. If R is a ring (with unit), we define $K_1(R)$ to be $GL(R)_{\text{ab}} = GL(R)/E(R)$. Note that $R \rightsquigarrow K_1(R)$ defines a functor from rings to abelian groups, for if $\varphi : R \to S$ is a (unit-preserving) ring homomorphism, φ induces a map from $GL(R)$ to $GL(S)$ and hence from $GL(R)_{\text{ab}}$ to $GL(S)_{\text{ab}}$.

If A, $B \in GL(R)$, the product of the corresponding classes $[A]$, $[B] \in K_1(R)$ may be represented in two convenient ways. On the one hand, $[A] \cdot [B] = [AB]$. On the other hand, one may form the "block sum" $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and since

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix},$$

Corollary 2.1.3 shows that

$$[A \oplus B] = [AB \oplus 1] = [AB].$$

One may also interpret $K_1(R)$ as the group of canonical forms for invertible matrices over R under elementary row or column operations (in the usual sense of linear algebra). For if $A \in M(n, R)$, $e_{ij}(a)A$ is the matrix obtained from A by adding a times the j-th row to the i-th row (an elementary row operation), and $Ae_{ij}(a)$ is the matrix obtained from A by adding a times the i-th column to the j-th column (an elementary column operation). Vanishing of $K_1(R)$, for instance, would mean that every matrix in $GL(R)$ can be row-reduced or column-reduced to the identity matrix.

2.1.6. Exercise: behavior of K_1 under Cartesian products. Let $R = R_1 \times R_2$, a Cartesian product of rings. By using the obvious decomposition $GL(R) = GL(R_1) \times GL(R_2)$, show that $K_1(R) \cong K_1(R_1) \times K_1(R_2)$. Generalize to arbitrary finite products. (Compare Exercise 1.2.8.)

2.1.7. Exercise: a ring with vanishing K_1. Let k be a field and let V be an infinite-dimensional vector space over k. Let $R = \mathrm{End}_k(V)$. Show that $K_1(R) = 1$. Hint: V is isomorphic to an infinite direct sum of copies of itself. Thus if $A \in GL(R)$, one can form

$$\infty \cdot A = \begin{pmatrix} A & 0 & \\ 0 & A & \\ & & \ddots \end{pmatrix}$$

and regard it also as an element of $GL(R)$. Show that $A \oplus (\infty \cdot A)$ is conjugate to $(\infty \cdot A)$, hence that A represents the identity in $K_1(R)$. (Compare Example 1.2.6.)

2.1.8. Exercise: Morita invariance of K_1. In analogy with Theorem 1.2.4, show that $K_1(M_n(R)) \cong K_1(R)$, for any ring R and any positive integer n.

2.1.9. Exercise: K_1 of a direct limit. Show by an argument somewhat similar to the proof of Theorem 1.2.5 that if $(R_\alpha)_{\alpha \in I}$, $(\theta_{\alpha\beta} : R_\alpha \to R_\beta)_{\alpha<\beta}$ is a direct system of rings and $R = \varinjlim R_\alpha$ is the direct limit of the system, then $K_1(R) \cong \varinjlim K_1(R_\alpha)$.

2. K_1 of division rings and local rings

We now begin to compute K_1 for rings of practical interest. In the case of a commutative ring, the determinant gives us our first piece of information.

2.2.1. Proposition. *If R is a commutative ring and $R^\times = GL(1, R)$ is its group of units, the determinant* $\det : GL(n, R) \to R^\times$ *extends to a split surjection $GL(R) \to R^\times$ and thus gives a split surjection $K_1(R) \to R^\times$.*

Proof. Note that $\det(A \oplus 1) = \det A$, so that the determinants on $GL(n, R)$ for various n are compatible with the embeddings of $GL(n, R)$ in $GL(m, R)$ for $n < m$. Since $\det(AB) = \det(A)\det(B)$, we obtain a homomorphism $GL(R) \to R^\times$ which must factor through a map $GL(R)_{\mathrm{ab}} \to R^\times$ (since $R^\times$ is commutative). There is a splitting defined by $R^\times = GL(1, R) \hookrightarrow GL(R)$. □

Remark. When R is commutative, it is standard to denote the matrices of determinant 1 in $GL(n, R)$ by $SL(n, R)$ and in $GL(R)$ by $SL(R)$. The notations $GL(R)$ and $SL(R)$ stand for the **general linear** group and **special linear** group of R, respectively. Note that since each elementary matrix has determinant 1, $E(R) \subseteq SL(R)$. The quotient $SL(R)/E(R)$ is denoted $SK_1(R)$.

Now it is easy to compute K_1 in the case of a field.

2.2.2. Proposition. *If F is a (commutative) field, then $SK_1(F)$ is trivial, i.e., the determinant induces an isomorphism* $\det : K_1(F) \to F^{\times}$.

Proof. This is basically a classical theorem of linear algebra, that any matrix can be row-reduced to a triangular matrix. If $A = (a_{ij}) \in GL(n, F)$, then the first column of A can't consist entirely of zeroes, since then the matrix couldn't be invertible. So $a_{i1} \neq 0$ for some n. If $i = 1$, fine. If not, as in the proof of (2.1.3),

$$e_{1i}(1)e_{i1}(-1)e_{1i}(1) = \begin{pmatrix} 0 & \dots & 1 & \dots \\ \vdots & 1 & \vdots & \\ -1 & 0 & 0 & \dots \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

so premultiplying A by $e_{1i}(1)e_{i1}(-1)e_{1i}(1)$ puts something non-zero into the (1, 1)-slot. So we may as well assume $a_{11} \neq 0$. Adding $-a_{i1}a_{11}^{-1}$ times the first row to the i-th row for $i \neq 1$, we can now kill off all the other entries in the first column. This reduces A to the form $\begin{pmatrix} a_{11} & * \\ 0 & A' \end{pmatrix}$ with A' an $(n-1) \times (n-1)$ matrix, and of course $\det A = a_{11} \det A'$.

We now repeat the same procedure for A', thus changing A by elementary row operations to the form

$$\begin{pmatrix} a_{11} & * & * \\ 0 & a_{22} & * \\ 0 & 0 & A'' \end{pmatrix}$$

with A'' an $(n-2)\times(n-2)$ matrix. Continuing by induction, we see that A can be changed into an invertible upper-triangular matrix via elementary row operations.

Now assume A is an invertible upper-triangular matrix. Adding multiples of the last row to the other rows, we can kill off all the entries in the last column except for a_{nn}. Then adding multiples of the $(n-1)$-th row to the other rows, we can kill off all the entries in the $(n-1)$-th column except for $a_{n-1,n-1}$. Continuing by induction, we can row-reduce A to an invertible diagonal matrix $D = (d_{ij})$. Since elementary row operations don't change the determinant, this diagonal matrix D has the same determinant as our original matrix A.

Finally, we have to transform D into a diagonal matrix with at most one diagonal entry different from 1. This can be done using Lemma 2.1.3, which shows that matrices of the form

$$\operatorname{diag}(1, \dots, 1, a, a^{-1}, 1, \dots, 1)$$

are elementary. Premultiplying D by such matrices, we transform D into a diagonal matrix with at most one diagonal entry, say the one in the (1, 1)-slot, different from 1. This entry must be the same as the determinant, so if A had determinant 1, we see that it can be transformed by elementary row

operations into the identity matrix. In other words, $SL(n, F) = E(n, F)$ and $SK_1(F)$ is trivial. □

Remark. Note that the proof above still works to some extent if F is replaced by a non-commutative division ring R. The one thing that is different is that there is no good definition of a determinant for matrices over general non-commutative rings, so that the argument only proves the following.

2.2.3. Proposition. *If R is a division ring, the inclusion*

$$R^\times = GL(1, R) \hookrightarrow GL(R)$$

induces a surjection $R^\times_{\text{ab}} \twoheadrightarrow K_1(R)$.

Proof. Exactly the same proof shows that every matrix in $GL(n, R)$ can be transformed by elementary row operations into a diagonal matrix of the form $\text{diag}(a, 1, \dots, 1)$, in other words, into the image of $GL(1, R)$ in $GL(n, R)$. Since $K_1(R)$ is abelian, the resulting surjection $R^\times \twoheadrightarrow K_1(R)$ factors through $R^\times_{\text{ab}} = R^\times/[R^\times, R^\times]$. □

In fact, the same proof works in still greater generality.

2.2.4. Proposition. *If R is a local ring (not necessarily commutative), the inclusion $R^\times = GL(1, R) \hookrightarrow GL(R)$ induces a surjection $R^\times_{\text{ab}} \twoheadrightarrow K_1(R)$.*

Proof. In the proof above, we only used the fact that R is a division ring to show that each row and column of a matrix $A = (a_{ij}) \in GL(n, R)$ must contain an invertible element. However, this is still true over a local ring since the non-units constitute the radical. Indeed, if A were to contain a row or column all of whose entries were in the radical, then it's obvious A couldn't be invertible. (For example, if the i-th row of A had all its entries in the radical, then the same would be true for AB for any matrix B, so A couldn't have a right inverse. Similarly, if the j-th column had all its entries in the radical, then the same would be true for BA for any B, and A couldn't have a left inverse.) □

Now we get to the main theorem of this section, which is a calculation of $K_1(R)$ when R is a local ring or division ring. Since we already have an upper bound on the size of $K_1(R)$ from Proposition 2.2.4, we need a lower bound, in other words, a homomorphism **out** of $K_1(R)$ into some abelian group, akin to the determinant. The main idea of the construction is due to Dieudonné; we have followed the exposition in [Srinivas, (1.6)] (with small variations).

2.2.5. Theorem. *Let R be a local ring, not necessarily commutative. Then there exists a unique "determinant" map $GL(R) \to R^\times_{\text{ab}}$ with the following properties:*

(a) *The determinant is invariant under elementary row operations. In other words, if $A \in GL(n, R)$ and A' is obtained from A by adding a (left) multiple of one row to another row, then* $\det A' = \det A$.

(b) *The determinant of the identity matrix is* 1.

(c) *If* $A \in GL(n, R)$ *and if* $a \in R^\times$, *and if* A' *is obtained from* A *by (left-)multiplying one of the rows of* A *by* a, *then* $\det A' = (\bar{a})(\det A)$, *where* $\bar{a}$ *denotes the image of* a *in* $R^\times_{\text{ab}}$.

The determinant also has the following additional properties:

(d) *If* $A, B \in GL(n, R)$, *then* $\det(AB) = (\det A)(\det B)$.

(e) *If* $A \in GL(n, R)$ *and if* A' *is obtained from* A *by interchanging two of its rows, then* $\det A' = (-\bar{1})(\det A)$.

(f) *The determinant is invariant under taking the transpose of a matrix.*

Proof. First we check the uniqueness and the fact that (d)–(f) follow from (a)–(c). Then we prove the existence by an induction argument. Suppose a map det exists satisfying (a)–(c). By Proposition 2.2.4 and its proof, any matrix in $GL(n, R)$ can be row-reduced to one of the form $\text{diag}(a, 1, \ldots, 1)$. Hence by (a), the determinant is determined by its value on such matrices. But by (c), $\det(\text{diag}(a, 1, \ldots, 1)) = \bar{a}(\det 1)$, which by (b) is just $\bar{a}$. Hence (a)–(c) determine det uniquely. Furthermore, if $E \in E(n, R)$ and $EA = \text{diag}(a, 1, \ldots, 1)$, then we have $\det A = \bar{a}$, while $\det(AB) = \det(EAB)$ by (a), which can be rewritten as $\det((EA)B)$. Since premultiplying a matrix by $\text{diag}(a, 1, \ldots, 1)$ amounts to left-multiplying the first row by a, we have by (c) that

$$\det(AB) = \det((EA)B) = \bar{a}(\det B) = (\det A)(\det B),$$

proving (d). To check (e), note that if $A \in GL(n, R)$ and $i < j \le n$, then interchanging the i-th and j-th rows can be accomplished in two steps: premultiplying by the elementary matrix $e_{ij}(1)e_{ji}(-1)e_{ij}(1)$, and then multiplying the i-th row by -1. Hence (e) follows from (a) and (c). Finally, to check (f), note that by (a) the determinant is equal to 1 on elementary matrices, whereas by (d) it is multiplicative. Hence the determinant is unchanged under **post**multiplication by elementary matrices, in other words, elementary column operations. Furthermore, condition (c) implies that for $a \in R^\times$,

$$\det\left(\text{diag}(1, \ldots, 1, a, 1, \ldots, 1)\right) = \bar{a}.$$

Now consider the map $\det' : A \mapsto \det(A^t)$, where A^t is the transpose of A. This clearly satisfies (b), and since the transpose of an elementary matrix is elementary, $\det'$ is also equal to 1 on elementary matrices. Furthermore, we have

$$\begin{aligned}\det'(AB) &= \det\left((AB)^t\right) = \det(B^t A^t) = \det(B^t)\det(A^t)\\ &= \det'(B)\det'(A) = \det'(A)\det'(B),\end{aligned}$$

since the determinant takes values in an abelian group. So $\det'$ satisfies (d), and since it is 1 on elementary matrices and $\bar{a}$ on $\text{diag}(1, \ldots, 1, a, 1, \ldots, 1)$, it satisfies (a) and (c) as well. By the uniqueness of a map satisfying (a)–(c), $\det'$ must coincide with det, proving (f).

Now we proceed to the existence proof. We define $\det_n(A)$ for $A \in GL(n, R)$ by induction on n, in such a way that $\det_{n+m}(A \oplus 1_m) = \det_n A$, so that we get a well-defined map on $GL(R)$. Clearly when $n = 1$ we define $\det_1(a) = \bar{a}$, and properties (a)–(c) are satisfied. So this starts the induction. Assume now that we've defined $\det_k$ for $k < n$ with properties (a)–(c) and compatibility for varying k, and let's define $\det_n$ and show that it satisfies (a)–(c) and compatibility with $\det_k$ for $k < n$. Let $A \in GL(n, R)$, and denote the rows of A by $A_1, \ldots, A_n$. Let $b_1, \ldots, b_n$ be the entries of the first row of A^{-1}. Since $A^{-1}A = 1_n$, expanding out the matrix product gives the relation

$$b_1 A_1 + \cdots + b_n A_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}.$$

In particular, if we write $A_j = \begin{pmatrix} a_{j1} & B_j \end{pmatrix}$, where $B_j \in R^{n-1}$, then $\sum_j b_j B_j = 0$.

By an argument already used before in the proof of Proposition 2.2.4, an entire row of A^{-1} can't consist of elements of $\operatorname{rad} R$, so at least one of the b_j's is invertible, say the i-th one. We then obtain

$$b_i^{-1} b_1 B_1 + \cdots + b_i^{-1} b_{i-1} B_{i-1} + B_i + b_i^{-1} b_{i+1} B_{i+1} + \cdots + b_i^{-1} b_n B_n = 0.$$

So adding multiples of the other rows to A_i row-reduces A to the form

$$\begin{pmatrix} a_{11} & B_1 \\ \vdots & \vdots \\ a_{i-1,1} & B_{i-1} \\ b_i^{-1} & 0 \\ a_{i+1,1} & B_{i+1} \\ \vdots & \vdots \\ a_{nn} & B_n \end{pmatrix},$$

since

$$\begin{aligned} a_{i1} + \sum_{j \neq i} b_i^{-1} b_j a_{j1} &= b_i^{-1} b_i a_{i1} + \sum_{j \neq i} b_i^{-1} b_j a_{j1} \\ &= b_i^{-1} \sum_j b_j a_{j1} = b_i^{-1}. \end{aligned}$$

If relations (a)–(c) are to hold, we see that we must therefore take

$$\det_n A = (-1)^{i-1} \bar{b}_i^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_i} \\ \vdots \\ B_n \end{pmatrix},$$

and we therefore adopt this as our definition. The only problem is to show that this is independent of the choice of i (subject to the condition $b_i \in R^\times$). Suppose $i < j$ and $b_i, b_j \in R^\times$. We need to show that

$$(-1)^i \bar{b}_i^{-1} \det_{n-1} C_i = (-1)^j \bar{b}_j^{-1} \det_{n-1} C_j,$$

where

$$C_i = \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_i} \\ \vdots \\ B_n \end{pmatrix}, \quad C_j = \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_j} \\ \vdots \\ B_n \end{pmatrix}.$$

Now C_j can be obtained from C_i by first permuting the order of the rows to get

$$C = \begin{pmatrix} B_1 \\ \vdots \\ B_{i-1} \\ B_j \\ B_{i+1} \\ \vdots \\ \widehat{B_j} \\ \vdots \\ B_n \end{pmatrix},$$

then changing the i-th row from B_j to B_i. Now going from C_i to C involves cyclically permuting the $j-i$ rows $(B_{i+1}, \ldots, B_j)$. Hence by condition (e) for $\det_{n-1}$, $\det_{n-1} C = (-1)^{j-i-1} \det_{n-1} C_i$. And $B_i = -b_i^{-1} b_j B_j +$(a linear combination of other rows), so by conditions (a) and (c) for $\det_{n-1}$, $\det_{n-1} C_j = -\bar{b}_i^{-1} \bar{b}_j \det_{n-1} C$ and

$$\begin{aligned}(-1)^j \bar{b}_j^{-1} \det_{n-1} C_j &= (-1)^{j-1} \bar{b}_j^{-1} \bar{b}_i^{-1} \bar{b}_j \det_{n-1} C \\ &= (-1)^i \bar{b}_j^{-1} \bar{b}_i^{-1} \bar{b}_j \det_{n-1} C_i \\ &= (-1)^i \bar{b}_i^{-1} \det_{n-1} C_i,\end{aligned}$$

as required. Thus $\det_n$ is well-defined.

To complete the proof, we only need to show that $\det_n$ satisfies (a)–(c) and agrees with $\det_{n-1}$ on matrices of the form $B \oplus 1$, B any $(n-1) \times (n-1)$ invertible matrix. Condition (b) is trivially true from the definition. As for (a), suppose A' with rows $A'_1, \ldots, A'_n$ has $A'_j = A_j$ for $j \neq i$, $A'_i = A_i + aA_k$, where $a \in R^\times$ and $i \neq k$. Then $A' = e_{ik}(a)A$, hence $(A')^{-1} = A^{-1} e_{ik}(-a)$ and the elements $b'_1, \ldots, b'_n$ of the first row of $(A')^{-1}$ are the same as $b_1, \ldots, b_n$ except for $b'_k = b_k - b_i a$. If $b_j \in R^\times$ for some

$j \neq k$, then we have

$$\det_n A' = (-1)^{j-1}\bar{b}_j^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ B_{i-1} \\ B_i + aB_k \\ B_{i+1} \\ \vdots \\ \widehat{B_j} \\ \vdots \\ B_n \end{pmatrix},$$

so by (a) for $\det_{n-1}$, $\det_n A' = \det_n A$. The only case we haven't covered is where b_k and $b'_k = b_k - b_i a$ are both invertible and b_i lies in $\operatorname{rad} R$ for $i \neq k$. In this case,

$$\det_n A' = (-1)^{k-1}\bar{b'}_k^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_k} \\ \vdots \\ B_n \end{pmatrix},$$

and

$$\det_n A = (-1)^{k-1}\bar{b}_k^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_k} \\ \vdots \\ B_n \end{pmatrix},$$

whereas $\bar{b'}_k = \bar{b}_k$ (since $b_i a \in \operatorname{rad} R$) and again $\det_n A' = \det_n A$. So this confirms property (a) for $\det_n$.

Now we check (c). Suppose A' with rows $A'_1, \dots, A'_n$ has $A'_j = A_j$ for $j \neq i$, $A'_i = aA_i$, where $a \in R^\times$. Then $A' = d_i(a)A$, with $d_i(a)$ the diagonal matrix with all 1's on the diagonal except for an a in the (i, i)-slot. Hence $(A')^{-1} = A^{-1}d_i(a^{-1})$ and the elements $b'_1, \dots, b'_n$ of the first row of $(A')^{-1}$ are the same as $b_1, \dots, b_n$ except for $b'_i = b_i a^{-1}$. Again there are two cases. If $b_j \in R^\times$ for some $j \neq i$, then we have

$$\det_n A' = (-1)^{j-1}\bar{b}_j^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ B_{i-1} \\ aB_i \\ B_{i+1} \\ \vdots \\ \widehat{B_j} \\ \vdots \\ B_n \end{pmatrix},$$

so by (c) for $\det_{n-1}$, $\det_n A' = \bar{a} \det_n A$. The only case we haven't covered is where b_i is invertible and b_j lies in $\operatorname{rad} R$ for $j \neq i$. In this case,

$$\det_n A' = (-1)^{i-1} \bar{b'}_i^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_i} \\ \vdots \\ B_n \end{pmatrix},$$

and

$$\det_n A = (-1)^{i-1} \bar{b}_i^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_i} \\ \vdots \\ B_n \end{pmatrix},$$

whereas $\bar{b'}_i = \bar{b}_i \bar{a}^{-1}$ and so again $\det_n A' = \det_n A$. So this confirms property (c) for $\det_n$. For compatibility with $\det_{n-1}$, note that $B \oplus 1$ can be transformed into $1 \oplus B$ by cyclically permuting both the rows and columns, hence by (d) and (f), which follow from (a)–(c), $\det_n(B \oplus 1) = \det_n(1 \oplus B)$. The latter is trivially the same as $\det_{n-1} B$ by our definition. So this completes the proof. □

2.2.6. Corollary. *If R is a local ring, not necessarily commutative, then the determinant of (2.2.5) induces an isomorphism*

$$K_1(R) \xrightarrow{\cong} R^\times_{\text{ab}}.$$

Proof. This is immediate from 2.2.4 and 2.2.5, since the composite $GL(1, R) \hookrightarrow GL(R) \overset{\det}{\twoheadrightarrow} R^\times_{\text{ab}}$ is just the quotient map $R^\times \twoheadrightarrow R^\times_{\text{ab}}$. □

2.2.7. Exercise. (Compare Exercise 1.3.14.) Compute $K_1(\mathbb{Z}/(m))$ in terms of m, for any integer $m > 0$. (Split into local rings and use Exercise 2.1.6 and Corollary 2.2.6.)

2.2.8. Exercise. Compute $K_1(k[t]/(t^m))$, for any field k and for any integer $m > 0$.

2.2.9. Exercise (another approach to a determinant over the quaternions). Let $\mathbb{H}$ be the usual ring of quaternions $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and $ij = k$, $i^2 = j^2 = k^2 = -1$. Recall that one defines $\overline{a + bi + cj + dk} = a - bi - cj - dk$.

a) Show that if one defines $N(z) = z\bar{z}$, then N gives a surjective homomorphism $\mathbb{H}^\times \to \mathbb{R}^\times_+$. In particular, the commutator subgroup of $\mathbb{H}^\times$ must lie in the kernel of N.
b) Show that the kernel of N is exactly the commutator subgroup of $\mathbb{H}^\times$. (Hint: show that $ie^{j\theta}i^{-1} = e^{-j\theta}$, and similarlywith i, j, k cyclically permuted. Deduce that$e^{2j\theta_1}$, $e^{2j\theta_2}$, and $e^{2j\theta_3}$ are all

commutators. Show that these generate an open neighborhood of 1 in $N^{-1}(1) \cong S^3$. But S^3 is connected.) Thus $\mathbb{H}^\times_{\text{ab}} \cong \mathbb{R}^\times_+$.

c) Since $\mathbb{H}$ is a vector space over $\mathbb{R}$ of dimension 4, $\mathbb{H}$ may be embedded in $M_4(\mathbb{R})$ by the left regular representation, and $GL_n(\mathbb{H}) \hookrightarrow GL_{4n}(\mathbb{R})$. Composing with the determinant gives a homomorphism $\det_{\mathbb{R}} : GL_n(\mathbb{H}) \to \mathbb{R}^\times$. Relate this to the Dieudonné determinant and to N, and show that $N : K_1(\mathbb{H}) \xrightarrow{\cong} \mathbb{R}^\times_+$.

2.2.10. Exercise (Some rings of interest in operator theory). Here is an exercise dealing with some rings (actually, algebras over $\mathbb{C}$) of great importance in operator theory and functional analysis. While they are not themselves local rings, we will study a "determinant" somewhat similar to that which we have constructed above in Theorem 2.2.5, and we will make a connection with local rings in the next exercise.

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space with an orthonormal basis $e_1, e_2, \ldots$. A bounded operator on $\mathcal{H}$ is called **compact** if it sends the unit ball to a pre-compact set, or equivalently, if it is a limit (in norm) of operators of finite rank. It is a well-known fact that the spectrum of any compact operator consists of 0 and of a sequence of eigenvalues tending to 0. (This is to be interpreted to mean "counting multiplicities," in the sense that no non-zero eigenvalue has infinite multiplicity. Zero itself may or may not be an eigenvalue.) A compact normal operator is diagonalizable. We denote by $\mathcal{K}(\mathcal{H})$ the Banach space of all compact operators with the operator norm:

$$\|T\|_\infty = \sup_{\|\xi\|\le 1} \|T\xi\|.$$

This is a closed two-sided ideal in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators.

Now if S is a **positive** bounded operator on $\mathcal{H}$, its **trace** is defined by

$$\operatorname{Tr} S = \sum_{i=1}^{\infty} \langle Se_i, e_i\rangle \in [0, \infty].$$

The trace is independent of the choice of orthonormal basis, for if the sum converges and $e'_1, e'_2, \ldots$ is another orthonormal basis, then

$$\begin{aligned}\sum_{i=1}^{\infty} \langle Se_i, e_i\rangle &= \sum_{i=1}^{\infty} \left\langle \sum_{j=1}^{\infty} \langle Se_i, e'_j\rangle\, e'_j,\, e_i \right\rangle \\ &= \sum_{i,j=1}^{\infty} \langle Se_i, e'_j\rangle \langle e'_j, e_i\rangle \\ &= \sum_{i,j=1}^{\infty} \overline{\langle Se'_j, e_i\rangle \langle e_i, e'_j\rangle}\end{aligned}$$

$$= \sum_{j=1}^{\infty} \left\langle \overline{\sum_{i=1}^{\infty} \langle Se'_j, e_i \rangle e_i, e'_j} \right\rangle$$
$$= \sum_{j=1}^{\infty} \langle Se'_j, e'_j \rangle.$$

An immediate consequence is that if U is a unitary operator, $\operatorname{Tr} S = \operatorname{Tr}(U^*SU)$. (Compute the trace of U^*SU using the original basis and the trace of S using the basis $\{Ue_j\}$.) If $S = T^*T$ is a positive operator and the trace $\operatorname{Tr}(S)$ is finite, then $\langle Se_i, e_i \rangle \to 0$, *i.e.*, $\|Te_i\| \to 0$, so that $\{Te_i\}$ is norm-convergent to 0. Thus T is compact and S is compact.

If $1 \le p < \infty$, the **Schatten p-class** of $\mathcal{H}$ is the Banach space $\mathcal{L}^p(\mathcal{H})$ of operators T for which

$$\|T\|_p = (\operatorname{Tr}(|T|^p))^{\frac{1}{p}} < \infty,$$

where $|T| = (T^*T)^{\frac{1}{2}}$. The Schatten classes consist of compact operators since this condition implies $|T|^p$ is compact, hence $|T|$ and T are compact. It turns out that $\| \ \|_p$ is a norm and that $\mathcal{L}^p(\mathcal{H})$ is complete in this norm. Furthermore, $\mathcal{L}^p(\mathcal{H}) \subseteq \mathcal{L}^{p'}(\mathcal{H})$ for $p \le p'$, since (for T compact) $T \in \mathcal{L}^p(\mathcal{H})$ if and only if the sequence of eigenvalues of $|T|$ lies in l^p. When $p = 2$,

$$\|T\|_2 = (\operatorname{Tr}(T^*T))^{\frac{1}{2}} = \sum_{i=1}^{\infty} \langle T^*Te_i, e_i \rangle = \sum_{i=1}^{\infty} \langle Te_i, Te_i \rangle,$$

so $\|T\|_2 = \langle T, T \rangle_{\mathrm{HS}}$, where the inner product $\langle \ , \ \rangle_{\mathrm{HS}}$ is defined by

$$\langle T, S \rangle_{\mathrm{HS}} = \sum_{i=1}^{\infty} \langle Te_i, Se_i \rangle.$$

Thus in this case $\mathcal{L}^2(\mathcal{H})$ is a Hilbert space, called the Hilbert space of Hilbert-Schmidt operators. In general note that clearly $\|\lambda T\|_p = |\lambda| \|T\|_p$ and $\|T\|_p \ge 0$. If $\|T\|_p = 0$, then the positive quadratic form defined by $|T|^p$ vanishes on all the e_i, hence everywhere, so $|T|^p = 0$, $|T| = 0$, and $T = 0$. The triangle inequality can be verified by showing first that

$$\|T\|_p = \sup_{\substack{F \text{ of finite rank} \\ \|F\|_q \le 1}} |\operatorname{Tr}(TF)|, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where if $p = 1$ we interpret $\|F\|_q$ to mean the operator norm of F. (Since TF has finite rank, its trace is well defined in the usual sense.) One can also check easily that $\mathcal{L}^p(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ (though not closed in the operator norm).

The space $\mathcal{L}^1(\mathcal{H})$ is called the space of **trace-class** operators. If $T \in \mathcal{L}^1(\mathcal{H})$, the sum $\sum \langle Te_i, e_i \rangle$ converges absolutely, and defines a linear functional $\operatorname{Tr} T$ independent of the choice of orthonormal basis (just as before). Hence, once again $\operatorname{Tr}(U^*TU) = \operatorname{Tr}(T)$ for U unitary.

Now let $\tilde{\mathcal{K}}(\mathcal{H}) = \mathbb{C}\cdot 1_{\mathcal{H}} + \mathcal{K}(\mathcal{H})$, and similarly let $\tilde{\mathcal{L}}^p(\mathcal{H}) = \mathbb{C}\cdot 1_{\mathcal{H}} + \mathcal{L}^p(\mathcal{H})$. Each of these rings has a unique maximal two-sided ideal, of codimension one. (For instance, $\mathcal{K}(\mathcal{H})$ is a two-sided ideal in $\tilde{\mathcal{K}}(\mathcal{H})$ of codimension one, so it is maximal even as either a left ideal or right ideal.)

(1) Complete the proof that $\|\ \|_1$ is a norm and that $\mathcal{L}^1(\mathcal{H})$ is complete, by showing that for $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^1(\mathcal{H})$,

$$|\operatorname{Tr}(AB)| \leq \|A\|_\infty \|T\|_1.$$

Hint: Use the polar decomposition $T = U|T|$ to split AB as $(AU|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}})$ and use the Cauchy-Schwarz inequality for $\langle\ ,\ \rangle_{\mathrm{HS}}$. Then if T, $S \in \mathcal{L}^1(\mathcal{H})$, write $T + S = U|T + S|$ (polar decomposition) and estimate $\operatorname{Tr}(|T + S|)$ as $\operatorname{Tr}(U^*(T + S)) = \operatorname{Tr}(U^*T) + \operatorname{Tr}(U^*S)$ via the above estimate.

(2) Show that if T or S is of trace class, then $\operatorname{Tr}(TS) = \operatorname{Tr}(ST)$. Hint: if T is of trace class and S is unitary, this follows from invariance of the trace under conjugation by S. Now get the result for all S (with T still of trace class) by taking linear combinations.

(3) Show that $\tilde{\mathcal{K}}(\mathcal{H})$ and $\tilde{\mathcal{L}}^p(\mathcal{H})$ have split surjections onto $\mathbb{C}$ inducing surjections on K_1.

(4) (**The operator determinant**) Let $R = \tilde{\mathcal{L}}^1(\mathcal{H})$, the trace-class operators with identity adjoined. Let $R_1^\times = \ker(R^\times \to \mathbb{C}^\times)$, and call this the group of **determinant-class** operators. Construct a homomorphism $\det : R_1^\times \to \mathbb{C}^\times$ with the property that

$$(*) \qquad \det(e^T) = e^{\operatorname{Tr}(T)} \quad \text{for } T \in \mathcal{L}^1(\mathcal{H}).$$

(Here the exponential of an operator is constructed via the usual exponential power series.)

Hint: First show that every determinant-class operator D is an exponential of a trace-class operator. One can do this by noting that every element of the spectrum of D, except perhaps for 1, is an eigenvalue of finite multiplicity, and that 1 is the only accumulation point of the spectrum. Hence, if V_1 is the span of the generalized eigenspaces for D corresponding to the eigenvalues λ with $|\lambda - 1| \geq 1$, one obtains a (not necessarily orthogonal) direct sum decomposition of $\mathcal{H}$ into two invariant subspaces V_1 and V_2 for D, where V_1 is finite-dimensional and the spectral radius of $(D-1)|_{V_2}$ is < 1. Then one can take a logarithm of $D|_{V_2}$ using the usual power series

$$\log z = (z-1) - \frac{1}{2}(z-1)^2 + \cdots$$

and choose any logarithm for the invertible operator $D|_{V_1}$ of finite rank (using, say, the Jordan canonical form).

Next, observe that if T and S are both of trace class and $e^T = e^S = D$, then if T has eigenvalues λ_j and S has eigenvalues μ_k, the

set $\{e^{\lambda_j}\}$ must coincide with the set $\{e^{\mu_k}\}$, and the multiplicities must match up. On the other hand, $\lambda_j \to 0$ and $\mu_k \to 0$. One can see from this that again one can find a (not necessarily orthogonal) direct sum decomposition of $\mathcal{H}$ into two invariant subspaces V_1 and V_2 for both T and S, where V_1 is finite-dimensional and $e^T|_{V_1} = e^S|_{V_1}$, and where $T|_{V_2} = S|_{V_2}$. In particular,

$$\mathrm{Tr}(T) - \mathrm{Tr}(S) = \mathrm{Tr}(T|_{V_1}) - \mathrm{Tr}(S|_{V_1}) \in 2\pi i\mathbb{Z}, \quad \text{so } e^{\mathrm{Tr}(T)} = e^{\mathrm{Tr}(S)}.$$

This shows that $(*)$ gives a well-defined definition of det.

Finally, show that the determinant is multiplicative, *i.e.*, that if T and S are of trace class, then $\det(e^T e^S) = \det(e^T)\det(e^S)$. On can do this using the Campbell-Baker-Hausdorff formula

$$e^{tT}e^{sS} = \exp\left\{tT + sS + \frac{1}{2}ts[T,S] + \frac{1}{12}t^2 s[T,[T,S]] + \frac{1}{12}ts^2[S,[S,T]] + \cdots\right\}$$

and the fact ((2) above) that Tr vanishes on commutators.

(5) Extend the definition of det to a homomorphism defined on

$$\ker[GL(R) \to GL(\mathbb{C})].$$

(Hint: if $T \in GL(n, R)$ and $T \mapsto 1 \in GL(n, \mathbb{C})$, then T may be viewed as a determinant-class operator on $\mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}^n$.)

2.2.11. Exercise (A local ring in operator theory). In this exercise, we pursue the use of K-theory in operator theory in the context of local rings. Let $\mathcal{H}$ be a complex Hilbert space as in the last exercise and let $\mathcal{A}$ be some algebra of bounded operators on $\mathcal{H}$, not necessarily with unit. Thus $\mathcal{A}$ could be $\mathcal{B}(\mathcal{H})$ or $\mathcal{L}^1(\mathcal{H})$. Let R be the ring of formal operator-valued power series $a_0 \cdot 1 + zA_1 + z^2A_2 + \cdots$, where $A_j \in \mathcal{A}$ for $j \geq 1$ and the constant term $a_0 \cdot 1$ is a scalar multiple of the identity operator.

(1) Show that if $a_0 \neq 0$, then $a_0 \cdot 1 + zA_1 + z^2A_2 + \cdots$ has an inverse in R. Deduce that R is a local ring, with radical the power series without constant term.
(2) If $\mathcal{A}$ is a Banach algebra, show that the same holds for R' if we define R' similarly using only those power series with a positive radius of convergence in z, in other words, with germs at $z = 0$ of analytic operator-valued functions in place of formal power series.
(3) Let $A \in \mathcal{B}(\mathcal{H})$. Then $1 - zA$ has an inverse in R, which is essentially (except for the change of variable $z \mapsto z^{-1}$) what is called in operator theory the **resolvent** of A. Show that the power series for $(1 - zA)^{-1}$ converges for $|z| < \|A\|^{-1}$.
(4) Let $\mathcal{A} = \mathcal{L}^1(\mathcal{H})$. Show that the determinant of the last exercise defines a homomorphism from $(R')^\times$ to the group of units in the commutative local ring of germs of analytic functions around 0.

Show also the following useful fact: if A is a trace-class operator, $f_A(z) = \det(1 - zA)^{-1}$ extends to a function of z analytic in the whole complex plane except perhaps for countably many isolated singularities, and that if z_0 is a zero or singularity of f_A, then $z_0^{-1} \in \operatorname{Spec} A$. (Actually, more is true; f_A is entire analytic, and $f_A(z_0) = 0$ if and only if $z_0^{-1} \in \operatorname{Spec} A$. See [Ringrose, Ch. 3] for more details.)

3. K_1 of PIDs and Dedekind domains

As we did in Chapter 1 in studying K_0, we shall proceed from the study of K_1 of division rings and local rings to the study of K_1 of the most elementary examples of non-local commutative rings. Of particular interest are the sorts of rings that occur in algebraic geometry and number theory. Here we shall discuss PIDs and Dedekind domains; polynomial rings will be dealt with in the next chapter.

The easiest examples to treat are Euclidean rings. These include $\mathbb{Z}$, the Gaussian integers $\mathbb{Z}[i]$, $\mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]$, the rings of integers in a few other special number fields, and the polynomial ring $k[t]$ in one variable over a field k. To fix notation, we remind the reader of the basic definition.

2.3.1. Definition. A (commutative) integral domain R is called a **Euclidean ring** or **Euclidean domain** if there is a **norm function** $|\ | : R \to \mathbb{N}$ with the following properties:

(i) If $a \in R$, $|a| = 0$ if and only if $a = 0$.
(ii) If a, $b \in R$, $|ab| = |a||b|$.
(iii) **(Euclidean algorithm)** If a, $b \in R$, $b \neq 0$, then there exist q, $r \in R$, called the **quotient** and **remainder**, respectively, such that $a = qb + r$ and $0 \leq |r| < |b|$.

In the examples $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]$, and $k[t]$, the norm function is given by the usual absolute value, by $|a + bi| = a^2 + b^2$, by $|a + b\frac{-1+i\sqrt{3}}{2}| = a^2 - ab + b^2$, and by $|f(t)|=2^{\deg f}$ (with the convention that $\deg 0 = -\infty$), respectively.

2.3.2. Theorem. *If R is a Euclidean ring, then $SK_1(R)$ vanishes and $K_1(R) \cong R^\times$. In fact, for each n, $SL(n, R) = E(n, R)$.*

Proof. Let $A = (a_{ij}) \in GL(n, R)$. We try to proceed roughly as in the proof of Proposition 2.2.2, but the problem is of course that there is no guarantee that there will be an invertible entry in a given row or column of A. However, the norm function on R gives us a mechanism for doing an induction. To illustrate, start with the first column of A. Not all elements of this column can be zero, so there is some $a_{i1} \neq 0$ and with $|a_{i1}|$ minimal subject to this condition. If $|a_{i1}| = 1$, then a_{i1} must be a unit. (By the Euclidean algorithm, $1 = qa_{i1} + r$ with $0 \leq |r| < 1$, hence with $|r| = 0$, so $r = 0$ by (i) of (2.3.1).) If $|a_{i1}| > 1$, then a_{i1} is not a unit, and so

generates a proper ideal (a_{i1}). On the other hand, since A is invertible, the ideal generated by the elements of the first column must be all of R, and so there is some $j \neq i$ with $a_{j1} \notin (a_{i1})$. Applying the division algorithm gives $a_{j1} = qa_{i1} + r$, where $|r| < |a_{i1}|$. Since $a_{j1} \notin (a_{i1})$, $r \neq 0$ and thus $|r| > 0$. So by subtracting $q \times$ (i-th row of A) from the j-th row, we can row-reduce A to decrease the minimal norm of a non-zero element in the first column. Once we've shown this, then iterating the reduction procedure enables us to reduce to the case where there's a unit in the first column. So then we can proceed as in the case of R a field and row-reduce A to the form $\begin{pmatrix} a_{11} & * \\ 0 & A' \end{pmatrix}$, where a_{11} is a unit and A' is of size $(n-1) \times (n-1)$ and invertible. Then we repeat the whole process with A', etc. The rest of the proof is identical to that of Proposition 2.2.2. □

2.3.3. Corollary. *$K_1(\mathbb{Z}) \cong \{1, -1\}$, $K_1(\mathbb{Z}[i]) \cong \{1, i, -1, -i\}$, $K_1(\mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]) \cong$ {6-th roots of 1}, and $K_1(k[t]) \cong k^\times$.*

Proof. In the examples of 2.3.1, it's easy to see which elements have norm 1. □

Theorem 2.3.2 naturally raises the question of whether the same statement is true or not for more general PIDs or Dedekind domains. Unfortunately, the answer is "no"; there are PIDs with non-zero SK_1, though they are not so easy to find. (For examples, see [Ischebeck] and [Grayson].) Thus it seems the idea of the proof of Theorem 2.3.2 cannot be pushed any further. However, there is one general result about K_1 of Dedekind domains that arises as a special case of Bass's general theory of "stable range." One may view the vanishing of SK_1 for a commutative ring R as the statement that $K_1(R)$ is generated by the image in $GL(R)$ of $GL(1, R)$. When this doesn't hold, the next best thing would be for $K_1(R)$ to be generated by the image in $GL(R)$ of $GL(2, R)$. Instead of trying to explain the general theory (for which one can consult [Bass]), which gives for a ring R an estimate on the smallest value of n for which $K_1(R)$ is generated by the image in $GL(R)$ of $GL(n, R)$, we will give a simplified proof of the one case we need. We begin with a lemma which will also be used in Section 5 of this chapter. Because of Corollary 2.1.3, Lemma 1.5.4 is just a special case of the following.

2.3.4. Lemma. *Let R be a ring (with unit) and I a two-sided in R. Then for any n, the natural map $E(n, R) \to E(n, R/I)$ is surjective.*

Proof. By definition, $E(n, R/I)$ is generated by elementary matrices $e_{ij}(\dot{a})$, where $\dot{a}$ is the image in R/I of $a \in R$. Such a matrix clearly lifts to the elementary matrix $e_{ij}(a) \in E(n, R)$. □

2.3.5. Theorem. *Let R be a Dedekind domain. Then $K_1(R)$ is generated by the image in $GL(R)$ of $GL(2, R)$ (in fact, by the images in $GL(R)$ of $GL(1, R)$ and of $SL(2, R)$).*

Proof. Let $A \in GL(n, R)$ and suppose $n \geq 3$. We will show that A can be row-reduced to a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & A' \end{pmatrix}$, where A' is of size $(n-1) \times (n-1)$ and invertible. Subtracting $a_{1i} \times$ (the first column of A) from the i-th column then reduces A to the form $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$ with $B \in GL(n-1, R)$, so $[A] \in K_1(R)$ lies in the image of $GL(n-1, R)$. Induction on n then gives the result of the theorem. (We already know the image of $GL(2, R)$ is generated by $GL(1, R)$ and by $SL(2, R)$.)

Now consider the first column of A. Since A is invertible, the ideal generated by its entries is all of R. We will show we can do elementary row operations on A to put at least one zero in the first column. One this is done, the ideal generated by the remaining entries in the column is all of R, so adding multiples of the other rows to the row with the zero, we can change the zero to a 1. Then if necessary, we may premultiply by $e_{1i}(1)e_{i1}(-1)e_{1i}(1)$ to put the 1 in the $(1, 1)$-slot. Subtracting multiples of the first row from the other rows then reduces A to the desired form.

Let I be the ideal generated by $a_{31}, \ldots, a_{n1}$. If $I = 0$, then $a_{31} = 0$ and we're already done. If $I = R$, then subtracting a linear combination of rows 3 through n from the first row puts a zero in the $(1, 1)$-slot, and we're again done. So we may assume I is a proper non-zero ideal. By Theorem 1.4.7, we may factor I uniquely into a product of maximal ideals. By the Chinese Remainder Theorem, this gives a corresponding factorization of R/I into a product of local rings of the form R/P^k, where P is a maximal ideal. By Proposition 2.2.4, $SK_1(R/P^k) = 0$, so by Exercise 2.1.6, $SK_1(R/I) = 0$. In fact, by the method of proof, we know that $SL(m, R/I) = E(m, R/I)$ for any m. We will use this fact for $m = 2$.

For each element $a \in R$, let $\dot{a}$ be its image in R/I. Since $Ra_{11} + \cdots + Ra_{n1} = R$, dividing by I gives that $(R/I)\dot{a}_{11} + (R/I)\dot{a}_{21} = R/I$. In other words, we can find x_1 and x_2 in R such that $\dot{x}_1\dot{a}_{11} + \dot{x}_2\dot{a}_{21} = \dot{1}$, or

$$\det \begin{pmatrix} \dot{x}_1 & \dot{x}_2 \\ -\dot{a}_{21} & \dot{a}_{11} \end{pmatrix} = \dot{1}.$$

So we have a matrix in $SL(2, R/I) = E(2, R/I)$. By Lemma 2.3.4, it lifts to an elementary matrix $\begin{pmatrix} x_1 & x_2 \\ -b_2 & b_1 \end{pmatrix}$ in $SL(2, R)$, and $b_1x_1 + b_2x_2 = 1$ (here we may have to change the original x_1 and x_2 within their I-cosets). But on the other hand, $x_1a_{11} + x_2a_{21} - 1 \in I$, so there exist $x_3, \ldots, x_n \in R$ with $\sum_{i=1}^n x_ia_{i1} = 1$. For $i \geq 3$, we have $x_i = x_i(b_1x_1 + b_2x_2)$. So we get the equation

$$x_1a_{11} + x_2a_{21} + (x_3b_1x_1a_{31} + x_3b_2x_2a_{31}) + \cdots + (x_nb_1x_1a_{n1} + x_nb_2x_2a_{n1}) = 1$$

or

$$x_1(a_{11} + x_3b_1a_{31} + \cdots + x_nb_1a_{n1}) + x_2(a_{21} + x_3b_2a_{31} + \cdots + x_nb_2a_{n1}) = 1.$$

This says exactly that by adding $(x_3b_1) \times$ (the 3rd row) $+ \cdots + (x_nb_1) \times$ (the nth row) to the first row, and by adding $(x_3b_2) \times$ (the 3rd row) $+ \cdots +$

$(x_n b_2) \times$ (the nth row) to the second row, we can change A so that the ideal generated by the new a_{11} and a_{21} is all of R. Then subtracting a linear combination of the first and second rows from the last row, we can achieve the desired zero. □

The above theorem suggests studying, for commutative rings R and especially Dedekind domains, the subgroup of $SK_1(R)$ generated by the image of $SL(2, R)$. The convenient way to do this is in terms of so-called **Mennicke symbols.**

2.3.6. Theorem. *Let R be a commutative ring.*

(1) *For $a, b \in R$ with $Ra + Rb = R$, choose $c, d \in R$ with $ad - bc = 1$. Then the class in $SK_1(R)$ of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$ is independent of the choice of c and d, hence can be denoted $[a\ \ b]$ without possibility of confusion. Such an element of $SK_1(R)$ is called a Mennicke symbol, and if R is a Dedekind domain, all elements of $SK_1(R)$ are of this form.*

(2) *$[a\ \ b] = 1$ if $a \in R^\times$, $b \in R$.*

(3) *For $a, b \in R$ relatively prime, the Mennicke symbols satisfy the relations $[a\ \ b] = [b\ \ a]$ and $[a\ \ b] = [a + b\lambda\ \ b]$ for any $\lambda \in R$.*

(4) *If $Ra_1a_2 + Rb = R$, then $[a_1\ \ b] \cdot [a_2\ \ b] = [a_1a_2\ \ b]$.*

Proof. (1) The assertion that the class of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is independent of the choice of c and d follows immediately from the calculation that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} a & b \\ c' & d' \end{pmatrix} \in SL(2, R)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b \\ -c' & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ cd' - c'd & 1 \end{pmatrix}.$$

The Mennicke symbols clearly exhaust the image of $SL(2, R)$ in $K_1(R)$, so by Theorem 2.3.5, they exhaust $SK_1(R)$ if R is a Dedekind domain.

(2) is clear from the fact that if $a \in R^\times$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$, then we can subtract $ca^{-1} \times$ (1st row) from the second row to change A to the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. Then multiplying by the elementary matrix $\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$ makes the matrix strictly upper-triangular, hence elementary.

For (3), note first that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix},$$

so $[a\ \ b] = [-b\ \ a]$. When we verify (4), it will follow that

$$[a\ \ b] = [-b\ \ a] = [b\ \ a][-1\ \ a] = [b\ \ a] \quad \text{(by (2))}.$$

Furthermore,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} a+b\lambda & b \\ c+d\lambda & d \end{pmatrix},$$

so $[a\ \ b] = [a+b\lambda\ \ b]$.

To check (4), assume $Ra_1a_2 + Rb = R$. Then if $\begin{pmatrix} a_1 & b \\ c_1 & d_1 \end{pmatrix}$, $\begin{pmatrix} a_2 & b \\ c_2 & d_2 \end{pmatrix}$ have determinant 1,

$$\begin{aligned} &\begin{pmatrix} a_1 & b & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & b \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} a_1a_2 & b & a_1b \\ c_1a_2 & d_1 & c_1b \\ c_2 & 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} a_1a_2 & b & 0 \\ c_1a_2 & d_1 & -1 \\ c_2 & 0 & d_2 \end{pmatrix}. \end{aligned}$$

Premultiplying by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ now keeps the first row the same and puts a 1 in the (3, 3)-slot, and further elementary operations reduce the matrix to the form

$$\begin{pmatrix} a_1a_2 & b & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $[a_1\ \ b][a_2\ \ b] = [a_1a_2\ \ b]$. □

2.3.7. Corollary. *If R is a Dedekind domain and R/P is a finite field for each non-zero prime ideal of R, then $SK_1(R)$ is a torsion group.*

Proof. Consider a Mennicke symbol $[a\ \ b]$. If $b = 0$, then $a \in R^\times$ so $[a\ \ b] = 1$ by (2) of the theorem. Similarly, $[a\ \ b] = 1$ if $b \in R^\times$. If neither is the case, (b) is a non-zero proper ideal of R and so is a product of non-zero primes ideals P_j by Theorem 1.4.7. Since each R/P_j is finite, it follows that $R/(b)$ is finite (cf. the beginning of the proof of Theorem 1.4.19). Since the image of a in $R/(b)$ is a unit and $(R/(b))^\times$ is a finite group, there is some k with $a^k \equiv 1 \mod (b)$, and then by (4) of the theorem,

$$[a\ \ b]^k = [a^k\ \ b] = [1+b\lambda\ \ b] \quad (\text{for some } \lambda) \quad = [1\ \ b] = 1$$

by (3) and then (2) of the theorem. □

This is about as much as one can say about general Dedekind domains. However, for the examples of greatest interest in number theory, namely the rings R of algebraic integers in number fields (finite extensions of $\mathbb{Q}$), it turns out that one can explicitly compute $R^\times$ and also show that $SK_1(R)$

vanishes. The computations for these cases are also of great interest in topology because of Example 1.5.10(b), which shows that the group ring of a cyclic group of order p is closely related to $\mathbb{Z}[e^{2\pi i/p}]$, and the following Section 4 of this chapter, which shows that K_1 of group rings is of great importance in topology. We proceed to the calculation of $R^\times$, which is a famous classical result of Dirichlet.

2.3.8. Theorem (Dirichlet Unit Theorem). *Let F be a number field, i.e., a finite algebraic extension of $\mathbb{Q}$, and let R be the ring of algebraic integers in F, that is, the integral closure of $\mathbb{Z}$ in F. Then $R^\times$ is finitely generated, with torsion subgroup the finite cyclic group of roots of unity in F, and with torsion-free part a free abelian group of rank r_1+r_2-1, where $r_1+2r_2 = n = [F:\mathbb{Q}]$ and where r_1 is the number of distinct embeddings of F into $\mathbb{R}$, and r_2 is the number of distinct conjugate pairs of embeddings of F into $\mathbb{C}$ with image not contained in $\mathbb{R}$. In particular, $R^\times$ is infinite if and only if F is not $\mathbb{Q}$ or an imaginary quadratic field.*

Proof. We begin by recalling that by elementary Galois theory, if $[F:\mathbb{Q}] = n$, then F must have n distinct embeddings σ_j into $\mathbb{C}$. In general, a certain number of these, say $\sigma_1, \dots, \sigma_{r_1}$, will have image contained in $\mathbb{R}$. The rest occur in complex conjugate pairs; let these be

$$\sigma_{r_1+1}, \dots, \sigma_{r_1+r_2}, \bar{\sigma}_{r_1+1}, \dots, \bar{\sigma}_{r_1+r_2}.$$

So $r_1 + 2r_2 = n$. Define a map λ, called the **logarithmic embedding**, from $F^\times$ to $\mathbb{R}^{r_1+r_2}$ by

$$\begin{aligned}\lambda(a) &= (\lambda_1(a), \dots, \lambda_{r_1+r_2}(a)) \\ &= (\log(|\sigma_1(a)|), \dots, \log(|\sigma_{r_1}(a)|), \\ &\quad 2\log(|\sigma_{r_1+1}(a)|), \dots, 2\log(|\sigma_{r_1+r_2}(a)|)),\end{aligned}$$

and note that since $\mathrm{N}_{F/\mathbb{Q}}(a) = \prod_{j=1}^{n} \sigma_j(a)$, we have the relation

$$\log(|\,\mathrm{N}_{F/\mathbb{Q}}(a)|) = \sum_{j=1}^{r_1+r_2} \lambda_j(a).$$

(Incidentally, λ is not injective since $\lambda(-1) = 0$, but we will see shortly that λ does give an embedding of the **torsion-free** part of $R^\times$.) Furthermore, by multiplicativity of the usual absolute value on $\mathbb{C}$ or $\mathbb{R}$ and the additivity of the logarithm for products, $\lambda : F^\times \to \mathbb{R}^{r_1+r_2}$ is a group homomorphism (the group operation is multiplication in the left-hand group, addition in the right-hand one). In particular, since $\mathrm{N}_{F/\mathbb{Q}}(a)$ is a unit in $\mathbb{Z}$, hence ± 1, for $a \in R^\times$, λ restricts to a homomorphism (which we will also denote by λ) from $R^\times$ to the hyperplane

$$V = \{(x_1, \dots, x_{r_1+r_2}) : \sum_{j=1}^{r_1+r_2} x_j = 0\},$$

a real vector space of dimension $r_1 + r_2 - 1$. Now a bound on

$$(|\sigma_1(a)|, \dots, |\sigma_{r_1+r_2}(a)|)$$

implies a bound on the absolute values of the elementary symmetric functions of the $\sigma_j(a)$, which are the coefficients of a monic polynomial equation satisfied by a, and are ordinary integers. So the inverse image under λ of any given ball of $\mathbb{R}^{r_1+r+2}$ is finite, which shows that $\lambda(R^\times)$ is discrete and the kernel of λ is finite.

The kernel of λ therefore consists of $a \in R$ for which $a^q = 1$ for some q, in other words, of roots of unity. On the other hand, since λ maps into a torsion-free group, all roots of unity in F must lie in the kernel of λ, and the kernel coincides with the group of roots of unity in F, the torsion subgroup of $R^\times$. If $F = \mathbb{Z}$, then obviously $R^\times$ is just $\{\pm 1\}$, and coincides with the kernel of λ. If F is an imaginary quadratic field, then $r_2 = 1$, $r_1 = 0$, and $V = 0$, so again $R^\times = \ker \lambda$. Furthermore, for general F, since the image of λ is a discrete subgroup of a real vector space of dimension $r_1 + r_2 - 1$, $\lambda(R^\times)$ is free abelian of rank $\le r_1 + r_2 - 1$, and $R^\times$ is finitely generated.

It remains only to show that the rank of $\lambda(R^\times)$ is **precisely** $r_1 + r_2 - 1$. This is the hard part of the proof, since for general F, there may not be any obvious elements of $R^\times$ other than the roots of unity, even if $r_1 + r_2 - 1$ is large. Since $\lambda(R^\times)$ is a discrete subgroup of the real vector space V, to show that $\lambda(R^\times)$ has rank equal to the dimension of V is equivalent to showing that $V/\lambda(R^\times)$ is compact, or to showing that there is some compact subset K of V whose translates under $\lambda(R^\times)$ cover V.

To show this, we first recall that by the proof of Theorem 1.4.18, $\prod_{j=1}^{r_1+r_2} \sigma_j$ gives an embedding σ of the **additive** group of R as a lattice (discrete cocompact subgroup) in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. In particular, the volume (in the sense of n-dimensional Lebesgue measure) of $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})/\sigma(R)$ is some finite positive constant, say C_1. Now if $v = (v_1, \dots, v_{r_1+r_2}) \in V$, let

$$e^v = (e^{v_1}, \dots, e^{v_{r_1+r_2}}) \in \mathbb{R}^{r_1+r_2} \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

Note also that since $\sum v_j = 0$, the product of the coordinates e^{v_j} is 1. Hence $e^v \cdot \sigma(R)$ (where $\cdot$ denotes coordinatewise multiplication) is again a lattice in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ of covolume C_1. So if Q is a closed cube or ball of volume $> C_1$ centered at the origin in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, its image in the quotient by $e^v \cdot \sigma(R)$ must have smaller volume, hence there had to be two points x_1 and x_2 in Q with the same image. In other words, $x_1 - x_2 \in e^v \cdot \sigma(R)$, so that $2Q$ (the cube or ball with dimensions twice as big) contains a point of $e^v \cdot \sigma(R)$. Let K' be the compact image of $2Q$ under the map $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \longrightarrow \mathbb{R}^{r_1+r_2}$ defined by taking the logarithm of the absolute value of each coordinate. Then we have shown that for all points $v \in V$, $v + \lambda(R \smallsetminus \{0\})$ meets K'.

This is almost, but not quite, what we want, since we are interested in $\lambda(R^\times)$, not $\lambda(R \smallsetminus \{0\})$ (which is a semigroup but not a group). However, if C_2 denotes the maximum L^1-norm of a point in K', in other words, the

maximum value of the sum of the coordinates, then any $a \in (R \smallsetminus \{0\})$ with $e^v \cdot \sigma(a) \in 2Q$ must satisfy

$$|\mathrm{N}_{F/\mathbb{Q}}(a)| \leq e^{C_2}.$$

However, as observed in the proof of Theorem 1.4.19, there are only finitely many integral ideals in R of norm $\leq e^{C_2}$ (for any C_2), and so **up to units** there are only finitely many possibilities for a, say $a_1, \dots, a_k$. Thus we have shown that for any $v \in V$, there is a unit $u \in R^\times$ such that $v + \lambda(a_j) + \lambda(u)$ meets K' for some $j \leq k$. Thus there is a compact set K independent of v such that $v + \lambda(u)$ meets K for some $u \in R^\times$, and this proves the theorem. (Take $K = \bigcup_{j=1}^{k}(K' - \lambda(a_j))$.) □

It has been shown in [BassMilnorSerre] and in [Milnor, §16] that in fact $SK_1(R)$ vanishes when R is the ring of algebraic integers in a number field, so that Theorem 2.3.8 gives the complete calculation of $K_1(R)$ in this case. However, this is not an easy theorem and there doesn't seem to be an elementary proof. With less effort, one can prove somewhat less, for instance, that $SK_1(R)$ is finite. There are quite a number of proofs available, though all seem to require some additional tools. One method is to first show that $SL(2, R)$ is finitely generated, for instance, by constructing an explicit fundamental domain for $SL(2, R)$ as a discrete subgroup of a product G of r_1 copies of $SL(2, \mathbb{R})$ and of r_2 copies of $SL(2, \mathbb{C})$. It then follows from Theorem 2.3.6 and Corollary 2.3.7 that $SK_1(R)$ is finite.

An alternative argument in [Kazhdan] uses representation theory. One can show that for each n, $SL(n, R)$ is a discrete subgroup of a product $G(n)$ of r_1 copies of $SL(n, \mathbb{R})$ and of r_2 copies of $SL(n, \mathbb{C})$, and that the quotient $G(n)/SL(n, R)$ has finite invariant measure. On the other hand, Kazhdan shows that for $n \geq 3$, the locally compact group $G(n)$ has **property T**, *i.e.*, its trivial one-dimensional representation is an isolated point in the space of all irreducible unitary representations of the group. Kazhdan also observes that property T inherits to discrete subgroups of cofinite volume and to quotients thereof. Therefore the abelianization $SL(n, R)_{\mathrm{ab}}$ has property T. However, for a locally compact abelian group A, the irreducible unitary representations are just the continuous homomorphisms into $\mathbb{T}$, the circle group, so property T means that $\hat{A} = \mathrm{Hom}(A, \mathbb{T})$ is discrete. For A discrete, $\hat{A}$ is compact, so the only way it can also be discrete is if it is finite. So $SL(n, R)_{\mathrm{ab}}$ is finite for $n \geq 3$. In particular, $SK_1(R)$, which we have seen is a quotient of $SL(3, R)_{\mathrm{ab}}$, is finite.

2.3.9. Exercise (Finite generation of $E(n)$ and $SL(n)$).

(1) Show using Lemma 2.1.2(a) that if a ring R is finitely generated as a $\mathbb{Z}$-module, then $E(n, R)$ is finitely generated as a group. Deduce from Theorem 2.3.2 and Corollary 2.3.3 that $SL(n, \mathbb{Z})$, $SL(n, \mathbb{Z}[i])$, and $SL(n, \mathbb{Z}[\frac{-1+i\sqrt{3}}{2}])$ are finitely generated groups for all n. (This is not so easy to show directly.)

(2) Show using Lemma 2.1.2(c) that for any ring R, $E(n, R)$ is its own commutator subgroup (*i.e.*, is a **perfect group**) for $n \geq 3$.

Also use Lemma 2.1.2(c) to strengthen the result of (1): if a ring R is finitely generated as a **$\mathbb{Z}$-algebra**, then $E(n, R)$ is finitely generated as a group for $n \geq 3$.

(3) Show that $SL(2, \mathbb{Z}) = E(2, \mathbb{Z})$ is **not** its own commutator subgroup, by exhibiting a homomorphism onto an abelian group. Hint: what is $SL(2, \mathbb{Z}/(2))$?

2.3.10. Exercise (Stabilization of $GL(n)/E(n)$ for Dedekind domains).

(1) Let R be any ring. Show using the proof of Proposition 2.1.4 that $[GL(2, R), GL(2, R)] \subseteq E(4, R)$ (when $GL(2)$ is embedded in $GL(4)$ as usual).

(2) Again let R be any ring. Show that the image of $GL(2, R)$ in $GL(n, R)$ normalizes $E(n, R)$ if $n \geq 3$. Hint: first note that the image of $GL(2, R)$ normalizes the subgroup E_1 generated by the $e_{ij}(a)$ with $i \leq 2$ and $j \geq 3$, the subgroup E_1 generated by the $e_{ij}(a)$ with $j \leq 2$ and $i \geq 3$, and the subgroup E_3 generated by the $e_{ij}(a)$ with $i, j \geq 3$. Then use Lemma 2.1.2(c) to show E_1, E_2, and E_3 generate all of $E(n, R)$.

(3) Now let R be a Dedekind domain. By the proof of Theorem 2.3.5, if $n \geq 3$, $GL(n, R)$ is generated by $E(n, R)$ and by the image of $GL(2, R)$. Deduce from this fact and from (1) and (2) above that for any $n \geq 3$, $E(n, R)$ is normal in $GL(n, R)$, and that for any $n \geq 4$, $GL(n, R)/E(n, R)$ is the abelianization of $GL(n, R)$. (In fact there are cases where $E(2, R)$ is not normal in $GL(2, R)$. With somewhat more work, one can show that $GL(n, R)/E(n, R)$ is already abelian for $n = 3$.)

(4) Deduce from (3) and from part (2) of Exercise 2.3.9 the following theorem about finite generation of $SL(n, R)$: if R is a Dedekind domain which is finitely generated as a $\mathbb{Z}$-algebra, and if $SL(2, R)_{\text{ab}}$ is finitely generated, then $SL(n, R)$ is finitely generated as a group for all $n \geq 4$. (As remarked in (3), this can be strengthened to $n \geq 3$.)

2.3.11. Exercise (Non-triviality of Mennicke symbols). The following famous example from [BassMilnorSerre] shows there are Dedekind domains with non-trivial Mennicke symbols. Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, the ring of polynomial functions on the circle. This is a Noetherian integral domain with field of fractions $F = \mathbb{R}(x, y)/(x^2 + y^2 - 1)$.

(1) Show that R is a Dedekind domain. (This part of the exercise also appeared in Exercise 1.4.23. There are several possible arguments, such as checking the original definition or showing that R is integrally closed in F and applying Theorem 1.4.17.)

(2) Observe that $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in SL(2, R)$ and that for any $n \geq 2$, the

associated function $S^1 \to SL(n, \mathbb{R})$, defined via the formula

$$(x, y) \mapsto \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & 1_{n-2} \end{pmatrix},$$

represents a non-trivial element of $\pi_1(SL(n, \mathbb{R})) \cong \pi_1(SO(n))$ (see Example 1.6.13 for the calculation of this fundamental group).

(3) Argue on the other hand that if $g(x, y) \in E(n, R)$, then the matrix-valued function $(x, y) \mapsto g(x, y) \in SL(n, \mathbb{R})$ must represent 0 in $\pi_1(SL(n, \mathbb{R}))$. Hint: it's enough to check this for elementary matrices, for which there's an obvious homotopy to a trivial loop.

(4) Deduce that there's a homomorphism $SK_1(R) \to \widetilde{KO}^0(S^2) = \mathbb{Z}/(2)$ sending $[x \;\; y]$ to the non-zero element of $\mathbb{Z}/(2)$.

(5) Show that in fact $[x \;\; y]$ is an element of order 2 in $SK_1(R)$ by using Theorem 2.3.6 to show $[x \;\; y]^2 = 1$.

4. Whitehead groups and Whitehead torsion

For applications of K_1 to topology, just as in the case of the Wall obstruction, the rings of interest are integral group rings $\mathbb{Z}G$, where G is a group which in the applications is the fundamental group of some topological space. Note that $K_1(\mathbb{Z}G)$ always contains certain "obvious" elements, namely the images of the units $\pm g$, $g \in G$. We therefore focus attention on the "non-obvious" part of $K_1(\mathbb{Z}G)$.

2.4.1. Definition. *If G is a group, its* **Whitehead group** Wh(G) *is the quotient of $K_1(\mathbb{Z}G)$ by the image of $\{\pm g : g \in G\} \subseteq (\mathbb{Z}G)^\times$.*

Thus if G is the trivial group, $\mathrm{Wh}(G) = K_1(\mathbb{Z})/\{\pm 1\}$ is trivial by Corollary 2.3.3. The rings $\mathbb{Z}G$ are in general quite complicated from the ring-theoretic point of view; for instance, in what would appear to be the simplest non-trivial case, if G is the cyclic group of two elements with generator t, the map $a+bt \mapsto (a+b, a-b)$ embeds $\mathbb{Z}G$ into the Cartesian product $\mathbb{Z}\times\mathbb{Z}$ as what we called in Definition 1.5.1 the double $D(\mathbb{Z}, (2))$ of $\mathbb{Z}$ along the ideal (2). The units ± 1, $\pm t$ of $\mathbb{Z}G$ correspond in $D(\mathbb{Z}, (2))$ to $\pm(1, 1)$ and to $\pm(1, -1)$, which are all the units of $\mathbb{Z}\times\mathbb{Z}$, so $\mathrm{Wh}(G) = SK_1(D(\mathbb{Z}, (2)))$. One can show that this vanishes (see Theorem 2.4.3 below), but to do this from scratch is a bit involved, and this only handles the case of the simplest non-trivial group! Thus the computation of Whitehead groups is usually not easy. Nevertheless, the Whitehead groups of finite groups are now thoroughly understood, and we refer the reader to [Oliver] for a complete treatment. Here we content ourselves with a few elementary results.

Since it may not be apparent from Definition 2.4.1 that Whitehead groups are ever non-zero, we begin with an example.

2.4.2. Example. Let G be a cyclic group of order 5, with generator t. We shall exhibit an element of infinite order in $\mathrm{Wh}(G)$. Let $a = 1 - t - t^{-1}$ and note that

$$\begin{aligned}(1-t-t^{-1})\cdot(1-t^2-t^3) &= 1-t-t^{-1}-t^2+t^3+t-t^3+t^{-1}+t^2\\ &= 1,\end{aligned}$$

so that $a \in (\mathbb{Z}G)^\times$. Under the homomorphism $\alpha : \mathbb{Z}G \to \mathbb{C}$ defined by sending $t \mapsto e^{2\pi i/5}$, $\{\pm g : g \in G\}$ maps into the roots of unity and in particular into the complex numbers $\mathbb{T}$ of absolute value 1. So $b \mapsto |\alpha(b)|$ defines a homomorphism from $\mathrm{Wh}(G)$ to $\mathbb{R}_+^\times$. Since

$$|\alpha(a)| = |1 - e^{2\pi i/5} - e^{-2\pi i/5}| = |1 - 2\cos\frac{2\pi}{5}| \approx 0.4,$$

we deduce that α gives an element of infinite order in $\mathrm{Wh}(G)$.

The example may be generalized. Suppose G is **any** group and we are given a homomorphism $\alpha : G \to U(n)$, the unitary $n \times n$ matrices over $\mathbb{C}$. This group homomorphism clearly extends to a ring homomorphism $\alpha : \mathbb{Z}G \to M_n(\mathbb{C})$, and thus induces a homomorphism

$$\alpha_* : K_1(\mathbb{Z}G) \to K_1(M_n(\mathbb{C})) \cong K_1(\mathbb{C}) \cong \mathbb{C}^\times.$$

(Here we have used Morita invariance, Exercise 2.1.8.) But $\alpha(\pm G) \subseteq U(n)$, which maps to $\mathbb{T}$ in $K_1(\mathbb{C})$ under the determinant. Hence the absolute value of the determinant gives a homomorphism $\alpha_* : \mathrm{Wh}(G) \to \mathbb{R}_+^\times$ which can be used to detect elements of infinite order in the Whitehead group. Detecting elements of finite order in $\mathrm{Wh}(G)$ is trickier and requires more sophisticated methods. Nevertheless, the technique of Example 2.4.2 in fact detects all of $\mathrm{Wh}(G)$ for many groups of practical interest, for instance for cyclic groups, though we aren't prepared to prove this at the moment. To give an idea of what can be done by brute force, we show that the Whitehead group of a cyclic group of order two is trivial. (More powerful methods of computation use the exact sequences of the next section and Chapter 4.)

2.4.3. Theorem. *The Whitehead group of a cyclic group of order two is trivial.*

Proof. We have seen above that this is equivalent to proving that $SK_1(D(\mathbb{Z}, (2)))$ vanishes. Suppose $(A, B) \in SL(n, D(\mathbb{Z}, (2)))$. This means $A, B \in SL(n, \mathbb{Z})$ and $A - B \equiv 0 \mod 2$. By Theorem 2.3.2, $A \in E(n, \mathbb{Z})$. Thus clearly $(A, A) \in E(n, D(\mathbb{Z}, (2)))$. Multiplying (A, B) by $(A, A)^{-1}$, we see that we may assume $A = 1_n$, the $n \times n$ identity matrix. So suppose $A = 1_n$ and $B \equiv 1_n \mod 2$. If we could row-reduce $B = (b_{ij})$ to the identity matrix by elementary operations involving adding **even** multiples of one row to another row, then it would be clear that $(1, B) \in E(n, D(\mathbb{Z}, (2)))$.

So we try to apply the division algorithm as in the proof of Theorem 2.3.2. Let $(b_{21}, \ldots, b_{n1}) = (b_1)$. Then b_1 is even and b_{11} is odd. We show that we can reduce B by elementary operations of the allowable sort so that $b_{11} = \pm 1$, $b_1 = 0$, *i.e.*, $B = \begin{pmatrix} \pm 1 & * \\ 0 & B' \end{pmatrix}$. Then we repeat the same procedure with B', and so on. Eventually we come down to the case where B is upper-triangular with ± 1's on the diagonal and even entries above. More allowable elementary operations now reduce B to a diagonal matrix with ± 1's on the diagonal, and since $\det B = 1$, the number of -1's is even. To finish the argument, we only have to see what to do with the case $n = 2$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (since after renumbering of the rows and columns B is a direct sum of blocks of this type and of some identity matrix). In fact the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is **not** contained in the subgroup of $SL(2, \mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and by $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$; however,

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = (1, -1) \oplus (1, -1)^{-1}$$

is elementary as a matrix over $D(\mathbb{Z}, (2))$ by Corollary 2.1.3. So this completes the argument except for the step about reducing b_{11} to ± 1 and b_1 to 0.

For this we note that if $|b_{11}| = 1$, we can subtract even multiples of the first row of B from the other rows and thereby reduce $|b_1|$ to 0. If $|b_1| = 0$, then since $(b_{11}) + (b_1) = \mathbb{Z}$, we must have $|b_{11}| = 1$. If $|b_{11}| > 1$ and $|b_1| > 0$, there are two cases, depending on which of these is larger. If $|b_{11}| < |b_1|$, then by the division algorithm we can write $b_1 = qb_{11} + r$ with $0 < |r| < |b_{11}|$ (r can't be 0 since b_{11} and b_1 are relatively prime). If q is even, then we may reduce the size of $|b_1|$ by adding even multiples of the first row to the other rows. If q is odd, then r is odd and we write instead $b_1 = (q \pm 1)b_{11} + (r \mp b_{11})$. With the correct choice of the sign, we have $0 < |r \mp b_{11}| < |b_{11}|$, but $q \pm 1$ is even so we can argue as before.

In the other case, $|b_{11}| > |b_1|$. Again we apply the division algorithm and obtain $b_{11} = qb_1 + r$ with $0 < |r| < |b_1|$ and r odd. If q is even, this means we can subtract even multiples of other rows from the first row to reduce the absolute value of b_{11}. If q is odd, we use the same trick as before and write $b_{11} = (q \pm 1)b_1 + (r \mp b_1)$ with the sign chosen so that $0 < |r \mp b_1| < |b_1|$. Again we can subtract even multiples of other rows from the first row to reduce the absolute value of b_{11}. After repeating the algorithm finitely many times, we eventually come down to the case where $|b_{11}| = 1$. □

The reader will presumably agree after seeing this proof that computing Whitehead groups from scratch is not very practical. But at least we know now that $\mathrm{Wh}(G)$ is trivial for some finite groups and infinite for others.

In the rest of this section, we will give a brief exposition of the concept of **Whitehead torsion**, which provides the motivation for introducing the Whitehead groups. Whitehead torsion gives an algebraic obstruction for homotopy equivalences between certain topological spaces to be "simple," or of the "obvious" sort. Since for present purposes a homeomorphism is to be viewed as an "obvious" sort of homotopy equivalence, Whitehead torsion can be used to distinguish homotopy-equivalent spaces which are not homeomorphic.

The most famous application of Whitehead torsion is the "s-cobordism theorem," which is the main tool in classifying manifolds in dimension ≥ 5. So that the reader can appreciate the importance of the Whitehead groups for topological problems, we will give the statement here. However, we shall not discuss the proof as it will take us too far afield. For details, see [MilnorHCT] for the simply connected case and [RourkeSanderson, Ch. 6] and [Kervaire1] for the general case.

2.4.4. Theorem ("s-cobordism theorem"—Barden, Mazur, Stallings). *Let M^n be a connected compact n-manifold of dimension ≥ 5 with fundamental group π, and consider the family $\mathcal{F}$ of all "h-cobordisms" built on M. These are connected compact manifolds W^{n+1} with exactly two boundary components, one of which is M^n and the other of which is some other manifold M'^n, such that W has deformation retractions onto both M and M'. There is a map $\tau : \mathcal{F} \to \mathrm{Wh}(\pi)$, called the "Whitehead torsion," and τ induces a natural one-to-one correspondence from $\mathcal{F}/\sim$ to $\mathrm{Wh}(\pi)$, where $\sim$ is the equivalence relation induced by homeomorphisms $W \to W'$ which are the identity on M. If W is the "trivial" h-cobordism $W = M \times [0, 1]$, then $\tau(W) = 1$.*

2.4.5. Corollary. *If M^n is a connected compact n-manifold of dimension ≥ 5 with fundamental group π, and if $\mathrm{Wh}(\pi) = 1$ (for instance, if M is simply connected or if π is of order 2), then every h-cobordism built on M is homeomorphic (*rel *M) to a product $M \times [0, 1]$. In particular, the other boundary component M' is homeomorphic to M.*

Remarks. We have been deliberately vague about what category of manifolds we are dealing with here. In fact, the theorem is valid in all three of the major categories of manifolds: topological manifolds and continuous maps, PL manifolds and PL maps, and smooth manifolds and C^∞ maps. In the last of these, "homeomorphism" in the theorem is to be interpreted as "diffeomorphism."

One of the main applications of the Corollary, as noticed by Smale, is the proof of the Poincaré conjecture: that in dimension $n \geq 6$ (this can be reduced to 5 with a little more work), any manifold Σ^n homotopy-equivalent to S^n is (topologically) homeomorphic to S^n. Furthermore, the set of diffeomorphism classes of **smooth** homotopy spheres Σ^n is in one-to-one correspondence with the group $\mathrm{Diff}_0(S^{n-1})$ of isotopy classes of diffeomorphims of S^{n-1}. To prove this, cut out two small disks from Σ^n, viewed as the "polar caps" of the homotopy sphere. What remains is a manifold W^n with the homotopy type of a cylinder and with two

boundary components each homeomorphic to S^{n-1}. Since $n-1 \geq 5$ and S^{n-1} is simply connected, the hypotheses of the Corollary are satisfied and there is a homeomorphism (or diffeomorphism, if Σ is a smooth manifold) from W to $S^{n-1} \times [0, 1]$ which is the identity on the boundary component corresponding to the south polar cap. Hence we can glue the south polar cap back in and deduce that $\Sigma^n \cong B^n \cup_f B^n$, a union of two balls glued by a homeomorphism (if we're in the topological category) or diffeomorphism (if we're in the smooth category) f from S^{n-1} to itself. In addition, it's clear that any such f defines a homotopy sphere $B^n \cup_f B^n$. The equivalence class of this homotopy sphere only depends on the isotopy class of f, since an isotopy of f's gives an h-cobordism of the corresponding homotopy spheres and we can apply the Corollary again. Conversely, if there is an orientation-preserving diffeomorphism from $B^n \cup_f B^n$ to the standard sphere, it is not hard to see that there must be an isotopy from f to the identity. This explains why the smooth homotopy spheres are parameterized by $\mathrm{Diff}_0(S^{n-1})$. In the topological category, since B^n is the cone on S^{n-1}, any self-homeomorphism f of S^{n-1} extends to a self-homeomorphism F of B^n by the simple formula

$$F(rx) = rf(x), \quad r \in [0, 1], x \in S^{n-1}.$$

(This is the "Alexander trick.") This yields a homeomorphism from $B^n \cup_f B^n$ to S^n, proving the Poincaré conjecture.

The most elementary context in which to discuss "simplicity" of homotopy equivalences is that of a finite relative CW-complex (X, A). In other words, we assume A is a (Hausdorff) topological space and that X is obtained from A by attaching finitely many cells, so that k-cells are always attached before $(k+1)$-cells and the inclusion $A \hookrightarrow X$ is a homotopy equivalence. We assume as well that A and X are both path-connected and locally simply connected, with the same fundamental group π (computed with respect to some basepoint x_0 in A). Let $\tilde{X}$ and $\tilde{A}$ be the universal covers of X and A, which carry free actions of π by covering transformations, and let $R = \mathbb{Z}\pi$ be the group ring of π. In this situation, the relative homology groups $H_\bullet(X, A; \mathbb{Z}\pi) = H_\bullet(\tilde{X}, \tilde{A}; \mathbb{Z})$ must vanish. However, these may be computed from the cellular chain complex $C_\bullet(X, A; \mathbb{Z}\pi) = C_\bullet(\tilde{X}, \tilde{A}; \mathbb{Z})$, which is the direct sum of one free rank-one R-module in degree k for each k-cell added in obtaining X from A. The hypothesis that $A \hookrightarrow X$ is a homotopy equivalence means (by the Whitehead and Hurewicz theorems) exactly that this chain complex of finite type is acyclic. The Whitehead torsion of the homotopy equivalence will be an invariant of the chain complex $C_\bullet(X, A; \mathbb{Z}\pi)$ defined using one extra piece of structure—a choice of basis elements for the free modules $C_k(X, A; R)$. Since the k-chain module contains one free rank-one R-module for each geometric k-cell, there is a choice of a basis which is canonical up to an element of $\{\pm g : g \in \pi\}$ for each cell. Namely, we choose a basis element for the free cyclic submodule corresponding to each cell in $X \smallsetminus A$, and it only depends on a choice of orientation for this cell (hence the $\pm$ sign) and

on a choice of a lift of this cell to a cell in $\tilde{X} \smallsetminus \tilde{A}$ (hence the element of the covering group). If there are only cells in two consecutive dimensions, $k-1$ and k, then once we have fixed our basis elements, the differential $d_k : C_k(X, A; R) \to C_{k-1}(X, A; R)$ must be given by an invertible $n \times n$ matrix over R, where n is the number of k-cells or $(k-1)$-cells. (The number of cells must be the same in both dimensions since $H_\bullet(X, A; \mathbb{Q})$ must vanish, hence $\dim C_{k-1}(X, A; \mathbb{Q}) = \dim C_k(X, A; \mathbb{Q})$.)

2.4.6. Definition. The **Whitehead torsion** $\tau(X, A)$ of the homotopy equivalence $A \hookrightarrow X$ is the image in $\mathrm{Wh}(\pi)$ of the matrix of d_k in $GL(n, R)$ if k is even, or the inverse thereof if k is odd. Note that while the matrix of d_k is not well defined as it depends on the choice of basis, the torsion is well defined since we have divided out by all possible ambiguities.

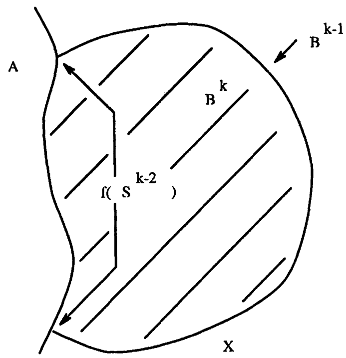
Now consider the general case where $C_\bullet(X, A; R)$ is allowed to be any acyclic chain complex of finite type of free R-modules, starting in degree 0, with bases chosen for each chain module. By the argument in the proof of Theorem 1.7.12, one may increase the ranks of the chain modules (adding "cancelling pairs" of cells in consecutive dimensions) so that $B_k = Z_k =_{\text{def}} \ker d_k$ is free for each k. Then d_k defines an isomorphism $C_k/B_k \to B_{k-1}$. We choose bases for the non-zero B_k's, taking the basis for $B_0 = C_0$ to be the basis we already have for C_0, and idempotents $p_k : C_k \to B_k$. Then $p_k \oplus d_k : C_k \to B_k \oplus B_{k-1}$ is given by an invertible matrix with entries in R, and we let $[d_k]$ be its class in $\mathrm{Wh}(\pi)$. (We can suppress the p_k because if p'_k is another projection from C_k onto B_k, then $p_k - p'_k$ vanishes on B_k and hence factors through d_k. But the matrix of $(p_k + s \circ d_k) \oplus d_k$ differs from that of $p_k \oplus d_k$ by an elementary matrix, so their classes in $K_1(R)$ are the same.) The **Whitehead torsion** $\tau(X, A)$ of the homotopy equivalence $A \hookrightarrow X$ is then defined to be the alternating product (since we're writing Whitehead groups multiplicatively) $\prod_k [d_k]^{(-1)^k}$.

This is independent of the choice of bases for the B_k's, since if we change the choice of basis for B_k by an invertible matrix P, this multiplies the matrix for d_{k+1} by P and the matrix for d_k by P^{-1}, so that we get cancellation in the alternating product. Notice also that this agrees with our previous definition when $C_p = 0$ for $p \neq k$, $k-1$, since $B_k = 0$ and $B_{k-1} = C_{k-1}$, so that we can use the same basis for B_{k-1} as for C_{k-1}. Note finally that the fact that we had to stabilize to make all the B_k's free, by adding on "cancelling pairs" of cells in consecutive dimensions, does not matter, since this kind of geometric stabilization corresponds to passage to the limit from $GL(n, R)$ to $GL(R)$ in the definition of K_1.

There is a geometric definition that corresponds to the algebraic condition of vanishing torsion.

2.4.7. Definition. The homotopy equivalence $A \hookrightarrow X$ is called **elementary**, or given by an **elementary collapse**, written $X \searrow_e A$, if X is obtained from A by attaching two cancelling cells in adjacent dimensions; in other words, if for some k, $X = (A \cup_f B^{k-1}) \cup_g B^k$. Here $f : S^{k-2} \to A$ is the attaching map for the $(k-1)$-cell and we suppose $g : S^{k-1} \to (A \cup_f B^{k-1})$

maps one hemisphere identically onto the $(k-1)$-cell and the other hemisphere of S^{k-1} into A. This is illustrated in the following picture.



2.4.8. Figure: An elementary collapse

Note that if f collapses S^{k-2} to a point a, this just means that $X = A \vee_a B^k$ and one can obviously collapse B^k to the attaching point a. In the general case, f extends to a map $\bar{f} : B^{k-1} \to A$ and X has a deformation retraction down to A collapsing the k-cell down to $\bar{f}(B^{k-1})$, as one can see in Figure 2.4.8.

More generally, we say X **collapses to** A or A **expands to** X and write $X \searrow A$ or $A \nearrow X$ if

$$X \searrow_e X_1 \searrow_e X_2 \searrow_e \cdots \searrow_e A,$$

and say the homotopy equivalence $A \hookrightarrow X$ is **simple** if it is in the equivalence relation generated by $\searrow$, *i.e.*, if $X \nearrow X_1 \searrow X_2 \nearrow \cdots \searrow A$ (with all the collapses and expansions fixing A pointwise).

2.4.9. Theorem (Geometric characterization of Whitehead torsion). *In the above context of a finite CW-pair (X, A) with A and X Hausdorff, path-connected, and locally simply connected, and where the inclusion $A \hookrightarrow X$ is a homotopy equivalence, the inclusion is simple if and only if $\tau(X, A) = 1$ in $\mathrm{Wh}(\pi)$. In particular, if $\mathrm{Wh}(\pi) = 1$, for instance if X and A are simply connected or π is of order 2, then every such homotopy equivalence $A \hookrightarrow X$ is simple.*

Furthermore, for fixed A and a fixed element $\alpha \in \mathrm{Wh}(\pi)$, there exists a finite CW-pair (X, A) such that the inclusion $A \hookrightarrow X$ is a homotopy equivalence with $\tau(X, A) = \alpha$.

Proof (Sketch). If $X \searrow_e A$, then $\tau(X, A) = 1$ since the boundary map in the cellular chain complex just corresponds to the 1×1 matrix (1), as one can see from Figure 2.4.8. Next observe that if

$$A = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots X_n = X$$

and all the inclusions are of finite CW-pairs and are homotopy equivalences, then

$$\tau(X, A) = \tau(X, X_{n-1}) \cdots \tau(X_1, A).$$

This follows from the fact that

$$C_\bullet(X, A; R) = C_\bullet(X, X_{n-1}; R) \oplus \cdots \oplus C_\bullet(X_1, A; R)$$

and the matrix defining $[d_k]$ for (X, A) differs from the direct sum of those defining the $[d_k]$ for the successive pairs (X_j, X_{j-1}) by an elementary matrix. It follows that the torsion vanishes if $X \searrow A$. The same principle also shows the torsion vanishes if $A \hookrightarrow X$ is simple, for if for instance $X_1 \searrow X \supseteq A$ and $X_1 \searrow X_2 \supseteq A$, then $\tau(X_1, A) = \tau(X_1, X)\tau(X, A) = \tau(X_1, X_2)\tau(X_2, A)$, so $\tau(X, A) = \tau(X_2, A)$. The general case follows from the same argument by iteration.

The existence part of the theorem is a direct construction. Given $\alpha \in \mathrm{Wh}(\pi)$, realize it by a matrix $B \in GL(n, R)$. Then let

$$X_1 = A \vee \underbrace{S^2 \vee \cdots \vee S^2}_{n \text{ times}}$$

and construct X from X_1 by attaching n 3-cells so that in the universal cover the cellular boundary map is given by

$$B : C_3(X, A; R) \cong R^n \to R^n \cong C_2(X, A; R).$$

This is possible since $\pi_3(\widetilde{X_1}, \tilde{A})$ is a free R-module on n generators. Then (X, A) obviously has the right torsion.

For the last part of the theorem, one needs to note first that if X' differs from X by a homotopy of the attaching maps for the cells rel A, then X can be converted to X' by a sequence of expansions and collapses (rel A). For this it's enough to consider the case of $X = A \cup_{f_0} B^k$ and $X' = A \cup_{f_1} B^k$, where

$$f : S^{k-1} \times [0, 1] \to A$$

is a homotopy of attaching maps. Merely define $W = A \cup_f (B^k \times [0, 1])$, which is defined by attaching $B^k \times [0, 1]$ to A along $S^{k-1} \times [0, 1]$. Then (W, A) is a finite CW-pair: one can first attach two k-cells to A via f_0 and f_1, then glue in a $(k+1)$-cell $B^{k+1} \cong B^k \times [0, 1]$ via f on $S^{k-1} \times [0, 1]$ and via the identity maps to the two k-cells along $B^k \times \{0, 1\}$. But $W \searrow_e X$ and $W \searrow_e X'$ since one can "cancel" the $(k+1)$-cell with either of the two k-cells.

The hardest part of the theorem is to show that if $\tau(X, A) = 1$, then $A \hookrightarrow X$ is simple. For this the idea is to proceed in two steps: first to modify X (rel A) by means of elementary expansions and collapses (which as we have seen do not affect the torsion) so that all the cells added to A to form X are in two consecutive dimensions k and $k-1$, then to show that each elementary matrix operation applied to

$$d_k : C_k(X, A; R) \cong R^n \to R^n \cong C_{k-1}(X, A; R)$$

has a geometric analogue. Here we only deal with the last part; see [Rourke-Sanderson] or [Cohen] for the full argument. Suppose X_1 is obtained from A by attaching n $(k-1)$-cells, and X is obtained from X_1 by attaching n k-cells via an elementary matrix $e_{ij}(a)$. Using the observation about homotopies of attaching maps, one can change X by expansions and collapses so that for $m \neq j$, the m-th k-cell is glued onto the m-th $(k-1)$-cell as in Figure 2.4.8, and the pair of cells collapses down to A. The j-th k-cell is glued onto both the j-th $(k-1)$-cell and the i-th $(k-1)$-cell. But now since the i-th $(k-1)$-cell can be collapsed down to A (along with the i-th k-cell glued onto it), the attaching map for the j-th k-cell can be homotoped through A to "unhook" this cell from the i-th $(k-1)$-cell. So after further expansions and collapses, we can assume each k-cell is glued onto exactly one $(k-1)$-cell as in Figure 2.4.8, and the cells can be collapsed in pairs down to A. □

The concept of Whitehead torsion can be carried over from inclusions $A \hookrightarrow X$ to general homotopy equivalences f from one finite (connected) CW-complex X_1 to another, X_2. To do this, if f is cellular, form the mapping cylinder $X = C_f = X_1 \times [0, 1] \cup_f X_2$ (here we use f to attach $X_1 \times \{1\}$ to X_2). Since we assumed f is cellular, this is a finite CW-complex, and since f was assumed a homotopy equivalence, it has deformation retractions down to the subcomplexes $A = X_1 \times \{0\}$ and X_2. We define $\tau(f) = \tau(X, A)$. Note that if f is actually an inclusion of a finite CW-subcomplex, then the pair (X, A) is an expansion of the pair (X_2, X_1) and so $\tau(f)$ agrees with our existing definition of $\tau(X_2, X_1)$. Furthermore, if two homotopy equivalences f_0 and $f_1 : X_1 \to X_2$ are homotopic to one another, then C_{f_1} is obtained from C_{f_0} by a homotopy of attaching maps, and hence by the proof of Theorem 2.4.9, their torsions are the same.

This makes it possible to define $\tau(f)$ for a homotopy equivalence f which isn't cellular. We homotope f to a cellular map f_0 (this is possible by the "cellular approximation theorem") and define $\tau(f) = \tau(f_0)$. The result is well defined since if we homotope f to a different cellular map f_1, then $f_0 \simeq f_1$ and so $\tau(f_1) = \tau(f_0)$. It also turns out that if f is a homeomorphism, then $\tau(f) = 1$, but this is a hard theorem [Chapman] unless f is cellular, in which case it's a triviality. (If f is a cellular homeomorphism, then C_f is cellularly isomorphic to $X_1 \times [0, 1]$, which clearly collapses to X_1.)

For further discussions of the various guises and applications of Whitehead torsion, see [MilnorWT] and [Weinberger, Ch. 1].

2.4.10. Exercise. Extend the proof of Theorem 2.4.3 to show that the Whitehead group of any elementary abelian 2-group (product of finitely many cyclic groups of order 2) is trivial.

2.4.11. Exercise (Behavior of Whitehead torsion under products). This exercise is in some sense the K_1-parallel of Exercise 1.7.18.

(a) Suppose $(C^1_\bullet, d^1)$ and $(C^1_\bullet, d^1)$ are complexes of finite type of based free R-modules and S-modules, respectively, with $C^1_\bullet$ acyclic (so that $\tau(C^1_\bullet)$ is defined). Show that the total complex of the double

complex $C^1_\bullet \otimes_{\mathbb{Z}} C^2_\bullet$ of free $R \otimes_{\mathbb{Z}} S$-modules,

$$\begin{cases} C_j = \displaystyle\bigoplus_{k=-\infty}^{\infty} C^1_{j-k} \otimes_{\mathbb{Z}} C^2_k, \\ d_j = d^1 \otimes id + (-1)^p id \otimes d^2 \quad \text{on } C^1_p \otimes_{\mathbb{Z}} C^2_q \end{cases}$$

is also based and acyclic.

(b) Suppose that in the situation of (a), $S = \mathbb{Z}$. Show that

$$\tau(C) = \tau(C^1)\chi(C^2),$$

where $\chi(C^2) \in K_0(\mathbb{Z}) = \mathbb{Z}$.

(c) Suppose $A \hookrightarrow X$ is a homotopy equivalence satisfying the hypotheses of Theorem 2.4.9, so that its torsion is defined, and let Z be a finite connected and simply connected CW-complex. Show using (b) that $A \times Z \hookrightarrow X \times Z$ is also a homotopy equivalence satisfying the hypotheses of Theorem 2.4.9, and that $\tau(X \times Z, A \times Z) = \tau(X, A)\chi(Z)$. Thus if $Z = S^3$, deduce that $A \times Z \hookrightarrow X \times Z$ is always simple.

(d) Show also that in the situation of (a), if $\chi(C^2) = 0$, then $\tau(C) = 0$ regardless of what S is. Deduce that if $Z = S^1$, then $A \times Z \hookrightarrow X \times Z$ is always simple.

5. Relative K_1 and the exact sequence

As with K_0, we want to be able to relate K_1 of a quotient ring R/I to $K_1(R)$ and to some invariants of the ideal I (and the way it is embedded in R). In this section, we will define the relative group $K_1(R, I)$ and show that the three-term exact sequence of Section 1.5 extends to a six-term exact sequence relating K_0 and K_1. This will provide us with some more computational tools for computing K-groups.

2.5.1. Definition. Let R be a ring (with unit) and let I be a two-sided ideal in R. We define $D(R, I)$ as in 1.5.1 and define the **relative K_1-group** of the ring R and the ideal I to be

$$K_1(R, I) = \ker\left((p_1)_* : K_1(D(R, I)) \to K_1(R)\right).$$

Note that this is the exact parallel of Definition 1.5.3. Since it's convenient to have another definition closer in spirit to Definition 2.1.5, we now prove a relative version of Whitehead's Lemma and rework the definition of $K_1(R, I)$ into a more usable form.

2.5.2. Definition. Let R be a ring (with unit) and let I be a two-sided ideal in R. We define $GL(R, I)$ to be the kernel of the map $GL(R) \to GL(R/I)$ induced by the quotient map $R \to R/I$. We define $E(R, I)$ to be the smallest normal subgroup of $E(R)$ containing the elementary matrices $e_{ij}(a)$, $a \in I$. Note that since each such elementary matrix is congruent to the identity matrix modulo I, $E(R, I) \subseteq GL(R, I)$.

2.5.3. Theorem (Relative Whitehead Lemma). *Let R be a ring (with unit) and let I be a two-sided ideal in R. Then $E(R, I)$ is normal in $GL(R, I)$ and in $GL(R)$,*

$$GL(R, I)/E(R, I) \cong K_1(R, I),$$

and $GL(R, I)/E(R, I)$ is the center of $GL(R)/E(R, I)$. Furthermore, $E(R, I) = [E(R), E(R, I)] = [GL(R), E(R, I)]$.

Proof. The first assertion follows from the fact that if $A \in GL(n, R)$ and $B \in E(n, R, I)$, then

$$\begin{pmatrix} ABA^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix}.$$

Since $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ is elementary by Corollary 2.1.3 and by its definition $E(R, I)$ is normal in $E(R)$, the right-hand side lies in $E(R, I)$.

Next suppose $(A_1, A_2) \in GL(D(R, I)) \subseteq GL(R \times R)$ and maps to the identity element of $K_1(R)$ under $(p_1)_*$. This means of course that $A_1 \in E(R)$. But then $(A_1, A_1) \in E(D(R, I))$, since if $A_1 = \prod_k e_{i_k j_k}(a_k)$,

$$(A_1, A_1) = \prod_k e_{i_k j_k}(a_k, a_k).$$

Multiplying (A_1, A_2) by $(A_1, A_1)^{-1}$ changes it to the form $(1, B)$ with $B \in GL(R)$ but without changing its class in K_1. Since $(1, B) \in GL(D(R, I))$, $B \equiv 1 \mod I$ and $B \in GL(R, I)$. Conversely, every $B \in GL(R, I)$ defines a class in $GL(D(R, I))$. So to show $GL(R, I)/E(R, I) \cong K_1(R, I)$, we need only check that if $B \in GL(R, I)$, then $(1, B) \in E(D(R, I))$ if and only if $B \in E(R, I)$. For one direction, note that $E(R, I)$ is generated by matrices of the form $Se_{ij}(a)S^{-1}$ with $a \in I$ and $S \in E(R)$. But

$$(1, Se_{ij}(a)S^{-1}) = (S, S)e_{ij}(0, a)(S^{-1}, S^{-1})$$

and all three factors on the right lie in $E(D(R, I))$. For the other direction, suppose

$$(1, B) = \prod_{k=1}^{r} e_{i_k j_k}(a_k, b_k) \in E(D(R, I)), \quad \prod_k e_{i_k j_k}(a_k) = 1 \in E(R).$$

Note that for each k,

$$e_{i_k j_k}(a_k, b_k) = e_{i_k j_k}(a_k, a_k) e_{i_k j_k}(0, b_k - a_k) = (S_k, S_k)(1, T_k),$$

where

$$S_k = e_{i_k j_k}(a_k) \in E(R), \quad T_k = e_{i_k j_k}(b_k - a_k), \quad b_k - a_k \in I.$$

Then we have

$$\begin{aligned}\prod_k e_{i_k j_k}(a_k, b_k) &= \prod_k (S_k, S_k T_k) \\ &= (S_1, S_1 T_1 S_1^{-1})(S_2, S_1 S_2 T_2 S_2^{-1} S_1^{-1}) \\ &\qquad \cdots (S_r, S_1 S_2 \cdots S_r T_r) \\ &= (1, (S_1 T_1 S_1^{-1})(S_1 S_2 T_2 S_2^{-1} S_1^{-1}) \\ &\qquad \cdots (S_1 S_2 \cdots S_r T_r S_r^{-1} \cdots S_2^{-1} S_1^{-1})),\end{aligned}$$

since $S_1 S_2 \cdots S_r = 1$, and we've written our element B as a product of generators of $E(R, I)$.

Since $E(R, I)$ is normal in $GL(R, I)$ and in $GL(R)$, $[E(R), E(R, I)] \subseteq [GL(R), E(R, I)] \subseteq E(R, I)$. Equality holds since $E(R, I)$ is generated by matrices of the form $Se_{ij}(a)S^{-1}$ with $a \in I$ and $S \in E(R)$, and

$$\begin{aligned}Se_{ij}(a)S^{-1} &= [S, e_{ij}(a)]e_{ij}(a) = [S, e_{ij}(a)][e_{ik}(1), e_{kj}(a)] \\ &\in [E(R), E(R, I)], \quad k \neq i, j.\end{aligned}$$

It remains only to show that $GL(R, I)/E(R, I)$ is the center of $GL(R)/E(R, I)$. Note first that if $A \in GL(R, I)$,

$$\begin{aligned}\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} &= \begin{pmatrix} 1 & A-1 \\ 0 & 1 \end{pmatrix} \\ &\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -A^{-1}(A-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \right\} \begin{pmatrix} 1 & 0 \\ -(A-1) & 1 \end{pmatrix},\end{aligned}$$

and since $A - 1$ has its entries in I, $\begin{pmatrix} 1 & A-1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -A^{-1}(A-1) \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ -(A-1) & 1 \end{pmatrix}$ lie in $E(R, I)$, hence this calculation shows that $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ lies in $E(R, I)$. So if $B \in GL(R)$,

$$\begin{aligned}\begin{pmatrix} ABA^{-1}B^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \left[\begin{pmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B^{-1} \end{pmatrix} \right] \\ &\in [E(R, I), E(R)] = E(R, I).\end{aligned}$$

So $GL(R, I)$ and $GL(R)$ commute modulo $E(R, I)$. On the other hand, the center of $GL(R)/E(R, I)$ must map (under the homomorphism induced by the quotient map $R \twoheadrightarrow R/I$) to the center of $GL(R/I)$, which is trivial. (A central matrix must be diagonal with equal diagonal entries, but since for a matrix in GL all but finitely many of the diagonal entries are 1, $GL(S)$ has trivial center for any S, in particular for $S = R/I$.) Hence the center of $GL(R)/E(R, I)$ is contained in the kernel of the map to $GL(R/I)$, which is $GL(R, I)/E(R, I)$. □

We're now ready for the main theorem of this section, which is an extension to the left of the exact sequence of Theorem 1.5.5.

2.5.4. Theorem. *Let R be a ring and $I \subseteq R$ an ideal. Then there is a natural exact sequence*

$$K_1(R, I) \to K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(R, I) \to K_0(R) \xrightarrow{q_*} K_0(R/I),$$

where q_* is induced by the quotient map $q : R \twoheadrightarrow R/I$ and the maps $K_j(R, I) \to K_j(R)$ are induced by $p_2 : D(R, I) \to R$.

Proof. For simplicity of notation in the proof, if A is an element of R or a matrix with entries in R, we will often denote $q(A)$, the corresponding matrix over R/I, by $\dot{A}$.

We begin by proving exactness of

$$K_1(R, I) \to K_1(R) \xrightarrow{q_*} K_1(R/I).$$

We have seen that any class in $K_1(R, I)$ is represented by

$$(1, B) \in GL(D(R, I)) \subseteq GL(R \times R)$$

with $B \in GL(R, I)$, so $\dot{B} = \dot{1}$ and $q_*[B] = 1$. Conversely, if $B \in GL(R)$ and $q_*([B]) = 1$, then $\dot{B} \in E(R/I)$. Now if $\dot{a} \in R/I$, it comes from some $a \in R$ and $e_{ij}(\dot{a}) = q(e_{ij}(a))$. So each generator of $E(R/I)$ lies in the image of $E(R)$ and hence $E(R/I) = q(E(R))$ (this argument was used in Lemma 1.5.4). So $\dot{B}$ lifts to a matrix $C \in E(R)$, and $q(BC^{-1}) = 1$. Then $(1, BC^{-1}) \in GL(D(R, I))$ and $[B] = [BC^{-1}]$ in $K_1(R)$ comes from $[(1, BC^{-1})] \in K_1(R, I)$.

Next we have to define the boundary map $K_1(R/I) \xrightarrow{\partial} K_0(R, I)$ and prove exactness at $K_1(R/I)$ and at $K_0(R, I)$. Theorem 1.5.5 will then complete the proof. The definition of the boundary map is based on what in topology is called a "clutching" construction. Given $\dot{A} \in GL(n, R/I)$ (the image of some matrix $A \in M_n(R)$, not necessarily invertible), we use A to "clutch" together two free modules to get a projective module over $D(R, I)$. In other words, let

$$R^n \times_{\dot{A}} R^n = \{(x, y) \in R^n \times R^n : \dot{y} = \dot{x}\dot{A}\}.$$

(We are thinking of x and y as $1 \times n$ matrices.) Make this into a module over $D(R, I)$ by letting

$$(r_1, r_2) \cdot (x, y) = (r_1 x, r_2 y).$$

This makes sense since $\dot{r}_1 = \dot{r}_2$, hence

$$q(r_2 y) = \dot{r}_2 \dot{y} = \dot{r}_1(\dot{x}\dot{A}) = q(r_1 x)\dot{A}.$$

Note that if $\dot{A} = q(A)$ with $A \in GL(n, R)$, then

$$(x, y) \mapsto (xA, y) \in R^n \times_{\dot{1}} R^n \cong D(R, I)^n$$

sets up an isomorphism from $R^n \times_{\dot{A}} R^n$ to a free module of rank n. In particular, since we have seen that $E(R/I) = q(E(R))$, $R^n \times_{\dot{A}} R^n$ is free of rank n if $\dot{A}$ is elementary. For a general $\dot{A} \in GL(n, R/I)$, we can always choose $\dot{B} \in GL(n, R/I)$ such that $\dot{A} \oplus \dot{B}$ is elementary (for instance, $\dot{B} = (\dot{A})^{-1}$ works by Lemma 1.5.4 or Corollary 2.1.3), and then

$$(R^n \times_{\dot{A}} R^n) \oplus (R^n \times_{\dot{B}} R^n) \cong R^{2n} \times_{\dot{A}\oplus\dot{B}} R^{2n} \cong D(R, I)^{2n},$$

so that $R^n \times_{\dot{A}} R^n$ is a direct summand in a free module, *i.e.*, a projective module. Thus it makes sense to define

$$\partial[\dot{A}] = [R^n \times_{\dot{A}} R^n] - [D(R, I)^n] \in K_0(D(R, I)).$$

We will show that ∂ is in fact a homomorphism $K_1(R/I) \to K_0(R, I)$. It maps into $K_0(R, I) = \ker(p_1)_*$ since

$$(p_1)_*(\partial[\dot{A}]) = (p_1)_*([R^n \times_{\dot{A}} R^n]) - (p_1)_*([D(R, I)^n]) = [R^n] - [R^n] = 0.$$

It is additive on direct sums of matrices since

$$(R^n \times_{\dot{A}} R^n) \oplus (R^n \times_{\dot{B}} R^n) \cong R^{2n} \times_{\dot{A}\oplus\dot{B}} R^{2n},$$

and it sends classes of elementary matrices to 0 since if $\dot{A}$ is elementary,

$$\partial[\dot{A}] = [R^n \times_{\dot{A}} R^n] - [D(R, I)^n] \cong [D(R, I)^n] - [D(R, I)^n] = 0.$$

More generally, it is well defined on classes in K_1 since if $\dot{A} = \dot{B}\dot{C}$ with $B \in E(R)$, then

$$(x, y) \mapsto (xB, y) \in R^n \times_{\dot{C}} R^n$$

sets up an isomorphism from $R^n \times_{\dot{A}} R^n$ to $R^n \times_{\dot{C}} R^n$. Thus we obtain a well-defined homomorphism $K_1(R/I) \to K_0(R, I)$. Furthermore we have already seen that the composite

$$K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(R, I)$$

is zero. The composite

$$K_1(R/I) \xrightarrow{\partial} K_0(R, I) \to K_0(R)$$

is zero since

$$(p_2)_*(\partial[\dot{A}]) = (p_2)_*([R^n \times_{\dot{A}} R^n]) - (p_2)_*([D(R, I)^n]) = [R^n] - [R^n] = 0.$$

It remains only to check that $\ker \partial \subseteq q_*(K_1(R))$ and that

$$\ker\{(p_2)_* : K_0(R, I) \to K_0(R)\} \subseteq \partial(K_1(R/I)).$$

Suppose $\partial([\dot{A}]) = 0$. This means that $R^n \times_{\dot{A}} R^n$ is stably isomorphic to a free module of rank n, or that for some m,

$$R^n \times_{\dot{A}} R^n \oplus D(R, I)^m \cong D(R, I)^{n+m}.$$

After replacing A by $A \oplus 1_m$, we may assume that in fact

$$R^n \times_{\dot{A}} R^n \cong D(R, I)^n.$$

Choose an isomorphism

$$\varphi : D(R, I)^n = R^n \times_{\dot{1}} R^n \to R^n \times_{\dot{A}} R^n.$$

Then we can define matrices $B, C \in M_n(R)$ by $(e_j B, e_j C) = \varphi(e_j, e_j)$, where e_j is the j-th standard basis vector for R^n, or in other words by taking the j-th rows of B and C to be the first and second coordinates (respectively) of $\varphi(e_j, e_j)$. Then by linearity, $\varphi(u, v) = (uB, vC)$ for any $(u, v) \in D(R, I)^n = R^n \times_{\dot{1}} R^n$, and since for such u and v, $\dot{u} = \dot{v}$, we have $\dot{B}\dot{A} = \dot{C}$. Since φ is invertible, it is clear that B and C are invertible with $\varphi^{-1}(x, y) = (xB^{-1}, yC^{-1})$ for $(x, y) \in R^n \times_{\dot{A}} R^n$. Thus $\dot{A} = q(B^{-1}C)$ and so $\ker \partial \subseteq q_*(K_1(R))$.

Finally, suppose one has a class in $K_0(R, I)$ going to 0 in $K_0(R)$. This means we have a class in $K_0(D(R, I))$ going to 0 under both $(p_1)_*$ and $(p_2)_*$. Represent the class by $[P] - [D(R, I)^n]$, where P is a projective $D(R, I)$-module such that $(p_1)_*(P)$ and $(p_2)_*(P)$ are stably isomorphic to R^n. If necessary, we may add on a free module of rank k to P and replace n by $n + k$ so that $(p_1)_*(P)$ and $(p_2)_*(P)$ are both actually isomorphic to R^n. Then it is clear that P is of the form $R^n \times_{\dot{A}} R^n$, and thus

$$[P] - [D(R, I)^n] = \partial([\dot{A}]).$$

This completes the proof. □

2.5.5. Corollary. *(Cf. Exercise 1.5.11.) Let R be a ring, $I \subseteq R$ an ideal such that the quotient map $q : R \twoheadrightarrow R/I$ splits (in other words, such that there exists a ring homomorphism $s : R/I \to R$ with $q \circ s = id_{R/I}$). Then*

$$0 \to K_0(I) \to K_0(R) \to K_0(R/I) \to 0$$

is split exact.

Proof. Clearly s_* is a splitting for q_*, by functoriality of K_0. We need only show that $K_0(I) \to K_0(R)$ is injective. But this follows from the fact that $s_* : K_1(R/I) \to K_1(R)$ is a splitting for $q_* : K_1(R) \to K_1(R/I)$, hence $\partial = 0$ in the exact sequence of 2.5.4. □

2.5.6. Examples. (Cf. Examples 1.5.10.)

(a) Suppose $R = \mathbb{Z}$ and $I = (m)$, where $m > 0$. Then $K_1(R) \cong \{\pm 1\}$ by Corollary 2.3.3, while $K_1(R/I)$ was computed in Exercise 2.2.7. It is thus possible to compute $K_0(I)$ from the exact sequence. For example, suppose $m = 2$. Then R/I is the field of two elements and $(R/I)^\times = \{1\}$. The exact sequence therefore becomes

$$K_1(R, I) \to \{\pm 1\} \to \{1\} \xrightarrow{\partial} K_0(I) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z},$$

and $K_0(I) = 0$. At the same time, we see that $K_1(R, I)$ must surject onto $\{\pm 1\}$.

Next, suppose $m = p$ is an odd prime. Then R/I is the field $\mathbb{F}_p$ of p elements and $(R/I)^\times$ is cyclic of order $p - 1$. Hence the exact sequence becomes

$$K_1(R, I) \to \{\pm 1\} \to \mathbb{F}_p^\times \xrightarrow{\partial} K_0(I) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z},$$

and $K_0(I) \cong \mathbb{F}_p^\times/\{\pm 1\}$, which is cyclic of order $\frac{p-1}{2}$. In this case, the map $K_1(R, I) \to \{\pm 1\}$ is trivial.

As a third example, suppose $m = 2^r$ is a power of 2 with $r > 1$. Then R/I is a local ring with maximal ideal of index 2, and $(R/I)^\times$ is an abelian group of order 2^{r-1}. Furthermore, ± 1 are distinct elements of this group. For instance, if $m = 8$, then since any odd square is $\equiv 1 \pmod 8$, all elements of $(R/I)^\times$ are of order 2 and $(R/I)^\times$ is a Klein 4-group $(\mathbb{Z}/(2) \times \mathbb{Z}/(2))$. By Corollary 2.2.6, $K_1(R/I) \cong (R/I)^\times$. The exact sequence has the form

$$K_1(R, I) \to \{\pm 1\} \to (R/I)^\times \xrightarrow{\partial} K_0(I) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z},$$

and $K_0(I) \cong (R/I)^\times/\{\pm 1\}$, an abelian group of order 2^{r-2} which is not necessarily cyclic. Again in this case, the map $K_1(R, I) \to \{\pm 1\}$ is trivial.

(b) Suppose G is a cyclic group of prime order p, say with generator t, and $R = \mathbb{Z}G$ is its integral group ring, which may be identified with $\mathbb{Z}[t]/(t^p - 1)$. If $\xi = e^{2\pi i/p}$, a primitive p-th root of unity, and if $S = \mathbb{Z}[\xi]$, then S is the ring of integers in the cyclotomic field $\mathbb{Q}(\xi)$, hence is a Dedekind domain by Theorem 1.4.18. There is a surjective homomorphism $R \twoheadrightarrow S$ defined by sending $t \mapsto \xi$. Since the cyclotomic polynomial $f_p(t) = t^{p-1} + \cdots + t + 1$ is irreducible, any polynomial $g(t) \in \mathbb{Z}[t]$ with $g(\xi) = 0$ must be divisible by f_p. In particular, anything in the kernel I of the map $R \to S$ must be a multiple of f_p. Note that as an element of R, $f_p^2 = pf_p$. Thus I in this example is, as a ring without unit, the same as in the last example if we specialize to the case $m = p$. In particular, $K_0(R, I) = K_0(\mathbb{Z}, (p)) \cong \mathbb{F}_p^\times/\{\pm 1\}$, which is cyclic of order $\frac{p-1}{2}$ by (a). We thus have an exact sequence

$$K_1(R, I) \to K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}[\xi]) \xrightarrow{\partial} \mathbb{F}_p^\times/\{\pm 1\}$$
$$\to \tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}[\xi]).$$

If $p = 2$, then $\mathbb{Z}[\xi] = \mathbb{Z}$ and this specializes to

$$K_1(R, I) \to K_1(\mathbb{Z}G) \to \{\pm 1\} \xrightarrow{\partial} 0 \to \tilde{K}_0(\mathbb{Z}G) \to 0.$$

Thus $\tilde{K}_0(\mathbb{Z}G) = 0$ in this case, and of course we already know by Theorem 2.4.3 that $K_1(\mathbb{Z}G) \cong \{\pm 1\} \times G$, so that the map $K_1(\mathbb{Z}G) \to K_1(\mathbb{Z})$ is surjective with kernel of order two.

If p is an odd prime, the cyclotomic field $\mathbb{Q}(\xi)$ has no real embeddings and $\frac{p-1}{2}$ conjugate pairs of complex embeddings. Thus by the Dirichlet Unit Theorem (Theorem 2.3.8), $(\mathbb{Z}[\xi])^\times$ is the product of the group of roots of unity in $\mathbb{Q}(\xi)$, which is of order $2p$, with a free abelian group of rank $\frac{p-1}{2} - 1 = \frac{p-3}{2}$. Granted the fact that $SK_1(\mathbb{Z}[\xi])$ vanishes (quoted but not proved in Section 2.3, though we know this at least for $p = 3$ by Theorem 2.3.2), we obtain the exact sequence

$$\begin{aligned} K_1(R, I) &\to \mathrm{Wh}(G) \times \{\pm 1\} \times G \to \mathbb{Z}^{\frac{p-3}{2}} \times \{\pm 1\} \times G \\ &\xrightarrow{\partial} \mathbb{F}_p^\times/\{\pm 1\} \to \tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}[\xi]) \end{aligned}$$

or

$$\begin{aligned} K_1(R, I) &\to \mathrm{Wh}(G) \to \mathbb{Z}^{\frac{p-3}{2}} \\ &\xrightarrow{\partial} \mathbb{F}_p^\times/\{\pm 1\} \to \tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}[\xi]). \end{aligned} \tag{2.5.7}$$

If we don't assume the vanishing of $SK_1(\mathbb{Z}[\xi])$, then $\mathbb{Z}^{\frac{p-3}{2}}$ should be multiplied by this group, which we at least know is a torsion group (by Corollary 2.3.7). This is almost, but not quite, enough information to compute $\tilde{K}_0(\mathbb{Z}G)$, the group in which in Wall finiteness obstruction lives, and the Whitehead group $\mathrm{Wh}(G)$. To complete the calculation, we need some information about the map $\mathbb{Z}^{\frac{p-3}{2}} \xrightarrow{\partial} \mathbb{F}_p^\times/\{\pm 1\}$ and also need to extend the exact sequence one step to the left and one step to the right. The extension to the right involves K_{-1}, to be discussed in the next Chapter, and the extension to the left involves K_2, to be discussed in Chapter 4.

2.5.8. Lemma (Rim). *Let $R = \mathbb{Z}G$, G a cyclic group of order p, an odd prime, and let $R/I = \mathbb{Z}[\xi]$, $\xi = e^{2\pi i/p}$ (as in Example 2.5.6(b) above). Then the boundary map $\mathbb{Z}^{\frac{p-3}{2}} \xrightarrow{\partial} \mathbb{F}_p^\times/\{\pm 1\}$ in the exact sequence (2.5.7) is surjective.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} K_1(R) & \longrightarrow & K_1(\mathbb{Z}[\xi]) & \xrightarrow{\partial} & K_0(R, I) & \longrightarrow & \tilde{K}_0(R) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{\pm 1\} & \longrightarrow & \mathbb{F}_p^\times & \xrightarrow{\partial} & K_0(\mathbb{Z}, (p)) & \longrightarrow & 0, \end{array}$$

where the vertical arrows are induced by the homomorphisms $\mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ sending $t \mapsto 1$ and $\mathbb{Z}[\xi] \twoheadrightarrow \mathbb{F}_p$ sending $1 \mapsto 1$, $\xi \mapsto 1$. Since $I \twoheadrightarrow (p)$, the vertical arrow $K_0(R, I) \to K_0(\mathbb{Z}, (p)) \cong \mathbb{F}_p^\times/\{\pm 1\}$ is the excision isomorphism. Thus a diagram chase shows that $K_1(\mathbb{Z}[\xi]) \xrightarrow{\partial} K_0(R, I)$ is surjective if the vertical arrow $K_1(\mathbb{Z}[\xi]) \xrightarrow{\partial} \mathbb{F}_p^\times$ is surjective. Let $1 \le k \le p-1$, so that k represents an element of $\mathbb{F}_p^\times$, and suppose $kl \equiv 1 \pmod{p}$. Let

$$u = \frac{\xi^k - 1}{\xi - 1} = \xi^{k-1} + \cdots + \xi + 1,$$

$$v = \frac{\xi - 1}{\xi^k - 1} = \frac{(\xi^k)^l - 1}{\xi^k - 1} = (\xi^k)^{l-1} + \cdots + (\xi^k) + 1.$$

Then $uv = 1$ in $\mathbb{Z}[\xi]$ and u reduces modulo p to k, which shows $K_1(\mathbb{Z}[\xi]) \xrightarrow{\partial} K_0(R, I)$ is surjective. Furthermore, if $2 \le k \le p-2$, then u is of infinite order in $\mathbb{Z}[\xi]^\times$ since

$$|u| = \frac{|\xi^k - 1|}{|\xi - 1|} > 1$$

in this case (ξ and ξ^{-1} are closer to 1 than the other primitive p-th roots of 1), so $\mathbb{Z}^{\frac{p-3}{2}} \xrightarrow{\partial} \mathbb{F}_p^\times/\{\pm 1\}$ in the exact sequence (2.5.7) is surjective. □

2.5.9. Corollary. *If $R = \mathbb{Z}G$, G a cyclic group of order p, an odd prime, then* Wh(G) *surjects onto $\mathbb{Z}^{\frac{p-3}{2}}$, and $\tilde{K}_0(\mathbb{Z}G)$ injects into the class group $\tilde{K}_0(\mathbb{Z}[\xi])$, $\xi = e^{2\pi i/p}$.*

Proof. This follows immediately from the exact sequence (2.5.7). □

In the last part of this Section, we will now discuss how to find explicit generators for $K_1(R, I)$ for some rings of interest in number theory and topology. This will help us to get more explicit information about the size of this group, and hence to sharpen the information about Whitehead groups in Corollary 2.5.9. The discussion will parallel Theorems 2.3.5 and 2.3.6.

2.5.10. Proposition. *(Cf. Proposition 2.2.1.) Let R be a commutative ring and $I \subseteq R$ an ideal. Then $K_1(R, I)$ splits canonically as*

$$\{a \in R^\times : a \equiv 1 \mod I\} \times SK_1(R, I),$$

where $SK_1(R, I) = SL(R, I)/E(R, I)$ and $SL(R, I)$ is by definition

$$SL(R) \cap GL(R, I).$$

Proof. Clearly the determinant gives a split surjection

$$\det : GL(R, I) \twoheadrightarrow \{a \in R^\times : a \equiv 1 \mod I\}$$

with kernel $SL(R, I)$. Now divide by $E(R, I)$ and use Theorem 2.5.3. □

It is immediately evident that when R is commutative, the first part of the exact sequence of Theorem 2.5.4 splits into two exact sequences

$$1 \to \{a \in R^\times : a \equiv 1 \mod I\} \to R^\times \to (R/I)^\times,$$

$$SK_1(R, I) \to SK_1(R) \to SK_1(R/I).$$

2.5.11. Theorem. *(Cf. Theorem 2.3.5.) Let R be a Dedekind domain and $I \subseteq R$ an ideal. Then $SK_1(R, I)$ is generated by the image in $SL(R, I)$ of $SL(2, R, I)$.*

Proof. The method of proof of Theorem 2.3.5 works here as well provided we can show that given $A = (a_{ij}) \in SL(n, R, I)$ with $n \geq 3$, we can find $t_i \in I$, $1 \leq i \leq n-1$, such that

$$R(a_{11} + t_1 a_{n1}) + \cdots + R(a_{n-1\,1} + t_{n-1} a_{n1}) = R.$$

(Note that adding $t_i \times$ (last row) to the i-th row of A corresponds to multiplying A by a matrix in $E(R, I)$, hence gives a new matrix A' with the same class in $SK_1(R, I)$. Then once we have arranged to have $c_1 a'_{11} + \cdots + c_{n-1} a'_{n-1\,1} = 1$, we can subtract $c_i a_{n1} \times$ (i-th row of A') from the last row (this is also an allowable elementary operation since $a_{n1} \in I$—recall $A' \equiv 1 \mod I$) and kill off the entry in the $(n, 1)$-slot.)

By assumption that $A = (a_{ij}) \in SL(n, R, I)$, we have $a_{11} \equiv 1 \mod I$, $a_{i1} \in I$ for $i > 1$, and $Ra_{11} + \cdots + Ra_{n1} = R$. But then also

$$Ra_{11} + \cdots + Ra_{n-1\,1} + Ra_{n1}^2 = R,$$

since if (a_{n1}) is relatively prime to $(a_{11}, \ldots, a_{n-1\,1})$, so is its square. By the proof of Theorem 2.3.5, we can then find $t'_i \in R$ with

$$R(a_{11} + t'_1 a_{n1}^2) + \cdots + R(a_{n-1\,1} + t_{n-1} a_{n1}^2) = R.$$

Set $t_i = t'_i a_{n1}$ and we're done. □

2.5.12. Theorem. *Let R be a commutative ring, $I \subseteq R$ an ideal.*

(1) *For $a, b \in R$ with $Ra + Rb = R$, $a \equiv 1 \mod I$, $b \in I$, choose c, $d \in R$ with $c \in I$, $d \equiv 1 \mod I$, and with $ad - bc = 1$. (This is possible since if $ad' - bc' = 1$, then automatically $d' \equiv 1 \mod I$, and $abd' - b^2c' = b$, hence $ad' - c'(abd' - b^2c') = ad - bc = 1$ with $d = d'(1 - bc') \equiv 1 \mod I$, $c = -bc'^2 \in I$.) Then the class in $SK_1(R, I)$ of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R, I)$ is independent of the choice of c and d, hence can be denoted $[a\ b]_I$ without possibility of confusion. Such an element of $SK_1(R, I)$ is called a* **relative Mennicke symbol**, *and if R is a Dedekind domain, all elements of $SK_1(R, I)$ are of this form.*

(2) $[a\ b]_I = 1$ *if* $a \in R^\times$, $a \equiv 1 \mod I$, $b \in I$.
(3) *For* $a, b \in R$ *relatively prime with* $a \equiv 1 \mod I$, $b \in I$, *the relative Mennicke symbols satisfy the relations* $[a\ b]_I = [a + b\lambda\ b]_I$ *for any* $\lambda \in R$, *and* $[a\ b]_I = [a\ b{+}a\lambda]_I$ *for any* $\lambda \in I$ *(note the asymmetry).*
(4) *If* a *and* $b \in R$ *are relatively prime with* $a \equiv 1 \mod I$, $b \in I$, *and* $b \equiv \pm 1 \mod a$, *then* $[a\ b]_I = 1$.
(5) *When both sides are defined,* $[a\ b_1]_I \cdot [a\ b_2]_I = [a\ b_1 b_2]_I$.

Proof. (1) The proof that $[a\ b]_I$ is well defined is the same as the corresponding step in the proof of Theorem 2.3.6. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} a & b \\ c' & d' \end{pmatrix} \in SL(2, R, I)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b \\ -c' & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ cd' - c'd & 1 \end{pmatrix},$$

and this lies in $E(R, I)$ since $cd' - c'd \in I$. The Mennicke symbols clearly exhaust the image of $SL(2, R, I)$ in $K_1(R, I)$, so by Theorem 2.5.11, they exhaust $SK_1(R, I)$ if R is a Dedekind domain.

(2) is clear from the fact that if $a \in R^\times$ and $[a\ b]_I$ is defined, then

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix},$$

and both factors on the right lie in $E(2, R, I)$, the first by the proof of Theorem 2.5.3, and the second since $b \in I$, hence $a^{-1}b \in I$.

For (3), suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R, I)$. If $\lambda \in I$, then $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in E(2, R, I)$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + \lambda a \\ c & d + \lambda c \end{pmatrix},$$

so $[a\ b]_I = [a\ b + a\lambda]_I$.

Furthermore, for any $\lambda \in R$,

$$\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} a + b\lambda & b \\ c + d\lambda & d \end{pmatrix},$$

and since $[E(R), E(R, I)] \subseteq E(R, I)$, this shows $[a\ b]_I = [a + b\lambda\ b]_I$.

To check (4), assume $b = \pm 1 + ta$, $t \in R$, and let $q = 1 - a \in I$. Then by (3),

$$\begin{aligned} [a\ b]_I &= [a\ b - ba]_I = [a\ bq]_I \\ &= [a\ bq - a(tq)]_I = [a\ \pm q]_I = [a + q\ \pm q]_I \\ &= [1\ \pm q]_I = 1. \end{aligned}$$

For (5), assume that $[a\ b_1]_I$ and $[a\ b_2]_I$ are defined and that $Rb_1b_2 + Ra = R$. Then if $\begin{pmatrix} a & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\begin{pmatrix} a & b_2 \\ c_2 & d_2 \end{pmatrix}$ lie in $SL(2, R, I)$,

$$\begin{pmatrix} d_2 & 0 & -c_2 \\ 0 & 1 & 0 \\ -b_2 & 0 & a \end{pmatrix}$$

is conjugate to $\begin{pmatrix} a & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ via the elementary matrix $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, and

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ -c_1d_2 & 1 & 0 \\ b_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_2 & 0 & -c_2 \\ 0 & 1 & 0 \\ -b_2 & 0 & a \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & c_2 \\ -c_1d_2 & 1 & -c_1c_2d_2 \\ b_2 & 0 & b_2c_2+1 \end{pmatrix} \begin{pmatrix} ad_2 & b_1 & -ac_2 \\ c_1d_2 & d_1 & -c_1c_2 \\ -b_2 & 0 & a \end{pmatrix} \\
&= \begin{pmatrix} ad_2 - b_2c_2 & b_1 & 0 \\ c_1d_2(-ad_2 + 1 - c_2b_2) & d_1 - b_1c_1d_2 & -c_1c_2 \\ b_2(ad_2 - b_2c_2 - 1) & b_1b_2 & a \end{pmatrix} \\
&= \begin{pmatrix} 1 & b_1 & 0 \\ 0 & d_1 - b_1c_1d_2 & -c_1c_2 \\ 0 & b_1b_2 & a \end{pmatrix}.
\end{aligned}$$

Since $b_1 \in I$, we may eliminate the $-b_1$ from the first row, and with $d = d_1 - b_1c_1d_2 \equiv 1 \mod I$, $[a\ b_1]_I[a\ b_2]_I$ is represented by the class in $SK_1(R, I)$ of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & d & -c_1c_2 \\ 0 & b_1b_2 & a \end{pmatrix}$ and thus of its conjugate by the elementary matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.
Finally, we compute that

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & -c_1c_2 \\ 0 & b_1b_2 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b_1b_2 & 0 \\ -c_1c_2 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that $[a\ b_1]_I[a\ b_2]_I = [a\ b_1b_2]_I$. □

2.5.13. Corollary. *(Cf. Corollary 2.3.7.) If R is a Dedekind domain and R/P is a finite field for each non-zero prime ideal of R, and if I is a proper ideal of R, then $SK_1(R, I)$ is a torsion group.*

Proof. Consider a relative Mennicke symbol $[a\ b]_I$. If $a \in R^\times$, $[a\ b]_I = 1$ by (2) of the theorem. If not, (a) is a non-zero proper ideal of R and so is

a product of non-zero prime ideals P_j by Theorem 1.4.7. Since each R/P_j is finite, it follows that $R/(a)$ is finite (cf. the beginning of the proof of Theorem 1.4.19). Since the image of b in $R/(a)$ is a unit and $(R/(a))^\times$ is a finite group, there is some k with $b^k \equiv 1 \mod (a)$, and then by (4) of the theorem,

$$[a\ b]^k = [a\ b^k] = 1.$$

So $[a\ b]_I$ has order k in $SK_1(R, I)$. But relative Mennicke symbols generate $SK_1(R, I)$ by (1) of the theorem. □

2.5.14. Proposition. *Let $R = \mathbb{Z}[t]/(t^p - 1)$ and $I = (t^{p-1} + \cdots + t + 1)$ be as in Example 2.5.6(b), so that R is the group ring of a cyclic group G of prime order p and $R/I \cong \mathbb{Z}[\xi]$, $\xi = e^{2\pi i/p}$, the ring of integers in the cyclotomic field $\mathbb{Q}(\xi)$. Then $SK_1(R, I) \cong SK_1(\mathbb{Z}, (p))$.*

Proof. Consider the homomorphism $\varphi : R \to \mathbb{Z}$ defined by $t \mapsto 1$. It is obviously surjective and sends I onto (p). We will show it induces an isomorphism $SK_1(R, I) \xrightarrow{\cong} SK_1(\mathbb{Z}, (p))$. For surjectivity, suppose $A \in SL(n, \mathbb{Z}, (p))$. Then $A \in SL(n, \mathbb{Z})$ and $A - 1 \equiv 0 \mod p$, so $A - 1 = pB$ with $B \in M(n, \mathbb{Z})$. Let $g(s) = \det(1 + sB)$. Then g is a polynomial with integer coefficients and $g(0) = g(p) = 1$, so we can write $g(s) = 1 + s(s-p)h(s)$ for some h. Since $f_p^2 = pf_p$ in R, $g(f_p) = 1$ and thus $1 + f_p(t)B$ lies in $SL(n, R, I)$ and maps to A under φ. This shows φ_* is surjective on SK_1.

Now suppose $A \in SL(n, R, I)$ and $[A] \mapsto 1$ in $SK_1(\mathbb{Z}, (p))$. This means $\varphi(A)$ lies in $E(\mathbb{Z}, (p))$. First we show that the map $E(R, I) \to E(\mathbb{Z}, (p))$ is surjective; this will imply that after changing A within the same class in SK_1, we may suppose $\varphi(A) = 1$. Now a typical generator of $E(\mathbb{Z}, (p))$ is $Ce_{ij}(pk)C^{-1}$, where $C \in E(\mathbb{Z})$, $k \in \mathbb{Z}$. We may lift C to a matrix in $E(R)$ by the argument in the beginning of the proof of Theorem 2.5.4, and we may lift $e_{ij}(pk)$ to $e_{ij}(f_p(t)k)$, so each generator of $E(\mathbb{Z}, (p))$ lifts to an element of $E(R, I)$, hence every element may be lifted.

Thus we may assume $\varphi(A) = 1$. But the kernel of φ is the augmentation ideal of R, which is generated by $t-1$, so $A \equiv 1 \mod (t-1)$. On the other hand, we were assuming $A \equiv 1 \mod f_p(t)$. These two facts together give $A = 1$, since $(t-1) \cap I = 0$ in R. So φ_* is also injective. □

2.5.15. Corollary. *If G is a cyclic group of odd prime order p, then the quotient of* Wh(G) *by its torsion subgroup is free abelian of rank exactly $\frac{p-3}{2}$.*

Proof. By Proposition 2.5.14, $SK_1(R, I) \cong SK_1(\mathbb{Z}, (p))$, which by Corollary 2.5.13 is a torsion group. Substituting in (2.5.7), we get the desired result. □

Remark. Note, by the way, that the proof of Corollary 2.5.15 is "elementary" in that it does not depend on the vanishing of $SK_1(\mathbb{Z}[\xi])$. However, to show that the torsion subgroup of Wh(G) is exactly the group of roots of unity in $\mathbb{Z}[\xi]$, which has order $2p$, one needs to prove that $SK_1(\mathbb{Z}[\xi]) = 0$

and that the image of $SK_1(\mathbb{Z}, (p))$ in the Whitehead group vanishes. In fact, one can even show that $SK_1(\mathbb{Z}, (p)) = 0$.

2.5.16. Exercise. Show that if G is a finite abelian group containing an element of order $m > 4$ with $m \neq 6$, then $\mathrm{Wh}(G)$ is infinite. Hint: m must be divisible by 8, by 9, by 12, or by some odd prime $p \geq 5$. First show that for a cyclic group of one of these orders, there is a unit in the integral group ring which under some representation of the group maps to a complex number of absolute value > 1. The proof of Lemma 2.5.8 and Corollary 2.5.9 basically take care of the case of a cyclic group of order an odd prime $p \geq 5$. Thus you need to find units of infinite order in the group rings of cyclic groups of orders 8, 9, and 12. Then reduce the general case to these particular cases using the structure theorem for finite abelian groups.

2.5.17. Exercise [Mennicke]. Show that for any $m > 1$, $SK_1(\mathbb{Z}, (m)) = 0$. Here is an outline. The proof requires use of **Dirichlet's theorem on primes in arithmetic progressions** [SerreCourseArith, §VII.4], which asserts that if a, $b > 0$ and $(a, b) = 1$, then the arithmetic progression $a + kb$, $k \in \mathbb{Z}$, contains infinitely many primes.

Let $R = \mathbb{Z}$, $I = (m)$. Choose any element $[a\ \ b]_I$ of $SK_1(R, I)$. We will show it is the identity. First use Dirichlet's Theorem to choose a prime $p \equiv a \mod b$. Then if ϕ is Euler's phi-function, we have $\phi(p) = p - 1$, and $[a\ \ b]_I = [p\ \ b]_I$ has exponent dividing $p - 1$ by the argument of Corollary 2.5.13. Let $q_1, \dots, q_r$ be the odd prime factors of $p - 1$. Using Dirichlet's Theorem again, choose primes p_1 and p_2 with

$$p_1 \equiv -p \mod b, \quad \mod q_1, \ \dots, \quad \mod q_r,$$

$$p_2 \equiv -1 \mod b, \quad \mod q_1, \ \dots, \quad \mod q_r.$$

Let $a' = p_1 p_2$. Show that $[a\ \ b]_I = [a'\ \ b]_I$; hence it has exponent dividing $\phi(a') = (p_1 - 1)(p_2 - 1)$. Show that this cannot have any q_j as a factor, and hence that the exponent of $[a\ \ b]_I$ can't have an odd prime factor and so is a power of 2.

To finish the argument, first suppose b is not a multiple of 4. Then applying Dirichlet's Theorem at the beginning modulo $4b$ instead of modulo b, we can also suppose $p \equiv 3 \mod 4$. This means $\frac{p-1}{2}$ is odd and some odd power of b is $\equiv -1 \mod p$. Deduce from (4) of Theorem 2.5.12 that $[a\ \ b]_I$ has odd exponent and so is $= 1$.

If b is a multiple of 4, argue similarly, except that if $a \equiv 1 \mod 4$, find a prime $p \equiv 3 \mod 4$ with $-p \equiv a \mod b$.

2.5.18. Exercise. Deduce from Exercise 2.5.17, from Proposition 2.5.14, from the exact sequence of (2.5.7), and from Corollary 2.3.3, that the Whitehead group $\mathrm{Wh}(G)$ vanishes if G is a group of order 3.

2.5.19. Exercise (Relative K_1 for split extensions). (Cf. Exercise 1.5.11.) Show that if

$$0 \to I \to R \to R/I \to 0$$

is split exact (*i.e.*, I is an ideal in a ring R, and there is a splitting homomorphism $s : R/I \to R$), then

$$1 \to K_1(R, I) \to K_1(R) \to K_1(R/I) \to 1$$

is split exact. (Hint: first show that $GL(R)$ is a semidirect product

$$GL(R, I) \rtimes GL(R/I).$$

Then obtain a splitting of $E(R)$.)

2.5.20. Exercise (Failure of excision for K_1). It is **not** true in general that $K_1(R, I)$ only depends on the structure of I as a ring without unit; it also depends on R. Here is a simple counterexample due to Swan [SwanExcision]. Let k be a field and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(k) \right\}, \; R' = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in M_2(k) \right\},$$
$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in M_2(k) \right\}.$$

Note that there are split extensions

$$0 \to I \to R \to k \times k \to 0, \qquad 0 \to I \to R' \to k \to 0.$$

Show that $R' \cong k[t]/(t^2)$, a commutative local ring with maximal ideal I, and use Corollary 2.2.6 and Exercise 2.5.19 to show that $K_1(R', I) \cong k$ (here k is viewed as an additive group).

Show on the other hand that $K_1(R, I) = 1$. Since I is contained in the radical of R, you can apply the method of proof of Proposition 2.2.4 to see that $K_1(R, I)$ is generated by the image of $\{x \in R^\times : x \equiv 1 \mod I\}$. Then you can show that $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in R$ is always a commutator in $R^\times$ (except in the exceptional case where k has only 2 elements, in which case its image in $K_1(R, I)$ is still trivial).

2.5.21. Exercise (The "Congruence Subgroup Problem"). If R is a commutative ring, the famous "Congruence Subgroup Problem" for R asks if every normal subgroup H of $SL(R)$ is one of the "congruence subgroups" $SL(R, I) = \{A \in SL(R) : A \equiv 1 \mod I\}$ for some two-sided ideal I of R. First observe that by Theorem 2.5.3 and Proposition 2.5.10, this can be the case only if $SK_1(R, I) = 1$ for all I (for $I = R$ this says $SK_1(R) = 1$). Prove the converse, by proving the following fact [Bass]:

Theorem (Bass). *If R is a ring and H is a normal subgroup of $GL(R)$, then there exists a unique two-sided ideal I of R such that $E(R, I) \subseteq H \subseteq GL(R, I)$.*

Hint. If $H = 1$, then take $I = 0$. Otherwise, let $H(n) = H \cap GL(n, R)$. This is non-zero and normalized by $E(n, R)$ for some $n \geq 2$. Show by looking at the commutators

$$\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1_n & x \\ 0 & 1 \end{pmatrix} \right], \quad h \in H(n), x \in R^n,$$

that $H(n+1) \supseteq E(n+1, R, I')$ for some non-zero ideal I'. Show that $H \supseteq E(R, I')$. Then let I be the **largest** two-sided ideal of R such that $H \supseteq E(R, I)$. If $H \not\subseteq GL(R, I)$, let H' be the image of H in $GL(R/I)$, repeat the same reasoning with $H' \supseteq GL(R/I)$, and derive a contradiction.

To prove the uniqueness of I, note that if $E(R, I) \subseteq H \subseteq GL(R, J)$, then projecting to R/J, we obtain $E(R/J, (I+J)/J) = 1$, hence $I \subseteq J$. Thus if $E(R, J) \subseteq H \subseteq GL(R, I)$ also, $J \subseteq I$ and $I = J$.

Deduce from Proposition 2.2.2, from Corollary 2.3.3, and from Exercise 2.5.17 that the Congruence Subgroup Problem has an affirmative answer if R is a field or $R = \mathbb{Z}$.

2.5.22. Exercise (Non-triviality of relative Mennicke symbols). Let R be the Dedekind domain of Exercises 1.4.23 and 2.3.11, *i.e.*, $\mathbb{R}[x, y]/(x^2+y^2-1)$. It was shown in the second of these Exercises that $[x\ y]$ represents an element of order 2 in $SK_1(R)$. By Exercise 1.4.23, $I = (y, x-1)$ is a prime ideal in R and $R/I \cong \mathbb{R}$. Deduce from 2.5.10 that $SK_1(R, I) \neq 0$, in fact that $[x\ y]_I \neq 0$ in $SK_1(R, I)$. Is this element also of order 2?

3

K_0 and K_1 of Categories, Negative K-Theory

1. K_0 and K_1 of categories, G_0 and G_1 of rings

For many of the applications of K-theory, it is useful to have the notion of K-theory for categories and not just for rings. In this more general context, the K-theory of a ring R is just the K-theory of the category **Proj** R of finitely generated projective modules over R. Another natural example is the topological K-theory of a compact space X, which is the K-theory of the category **Vect** X of (locally trivial, real, or complex) vector bundles over X. The identification of this with the K-theory of the ring $R = C(X)$ then follows from an equivalence of categories $\mathbf{Proj}\, R \cong \mathbf{Vect}\, X$. But there are also many examples that don't come so directly from rings; for instance, if X is a projective algebraic variety, one can consider in a similar way the category **Vect** X of **algebraic** vector bundles over X. We will see many more examples shortly.

To begin with, we need to place limitations on the sorts of categories we will consider. These are of two sorts. On the one hand, the category needs to have enough structure so that it makes sense to talk about an object as being built up as an extension of smaller objects. There are several ways of ensuring this and we've chosen here what seems to be the most standard choice, though not the most general one. In addition, the category has to be "small" enough to avoid set-theoretic difficulties when we try to make isomorphism classes of objects into a group. Of course, it suffices to require that the category be "small" in the usual sense of category theory (*i.e.*, for its objects and morphisms to constitute sets), but this seems overly restrictive since the natural examples **Proj** R and **Vect** X are **not** small categories. This should explain the following definition. Call a category $\mathcal{A}$ **preadditive** (this term is not entirely standard) if $\mathrm{Hom}\,(A, B)$ is an abelian group for each A, $B \in \mathrm{Obj}\, \mathcal{A}$, and if composition of morphisms is bilinear. Recall first of all that an **additive category** is a preadditive category $\mathcal{A}$ with a distinguished object 0 such that $\mathrm{Hom}\,(A, 0) = 0$,

$\operatorname{Hom}(0, A) = 0$ for each $A \in \operatorname{Obj} \mathcal{A}$, equipped with a binary operation $\oplus$ which is both the categorical product and the categorical coproduct. An **abelian category** is an additive category in which every morphism has a kernel and cokernel, and in which every monomorphism is a kernel and every epimorphism is a cokernel. Any abelian category has a notion of exact sequences for which the Five-Lemma and Snake Lemma are valid. Good general references on abelian categories are [Mac Lane] and [Freyd], though we will need very little of the theory developed in these books.

3.1.1. Definition. A **category with exact sequences** is a full additive subcategory $\mathcal{P}$ of an abelian category $\mathcal{A}$, with the following properties:

(1) $\mathcal{P}$ is closed under extensions, *i.e.*, if

$$0 \to P_1 \to P \to P_2 \to 0$$

is an exact sequence in $\mathcal{A}$ and $P_1, P_2 \in \operatorname{Obj} \mathcal{P}$, then $P \in \operatorname{Obj} \mathcal{P}$.

(2) $\mathcal{P}$ has a small skeleton, *i.e.*, $\mathcal{P}$ has a full subcategory $\mathcal{P}_0$ which is small, *i.e.*, such that $\operatorname{Obj} \mathcal{P}_0$ is a set, and for which the inclusion $\mathcal{P}_0 \hookrightarrow \mathcal{P}$ is an equivalence.

The **exact sequences** in such a category are defined to be the exact sequences in the ambient category $\mathcal{A}$ involving only objects (and morphisms) all chosen from $\mathcal{P}$.

3.1.2. Examples.

(1) Any small abelian category, or more generally any abelian category with a small skeleton, is a category with exact sequences. Examples include the category of finite-dimensional vector spaces over a field F, or the category of finite-dimensional complex representations of a topological group G. To get a small skeleton, take $\{F^n : n \in \mathbb{N}\}$ in the first case, or $\{\operatorname{Hom}(G, GL(n, \mathbb{C})) : n \in \mathbb{N}\}$ in the second case. When $G = \mathbb{Z}$, the category of finite-dimensional complex representations of G may be identified with the category of pairs (V, T), where V is a finite-dimensional complex vector space and $T \in \operatorname{Aut} V$ is the image of the generator of G. Another similar example is the category of finite-dimensional complex representations of the monoid $\mathbb{N}$, which may be identified with the category of pairs (V, T), where V is a finite-dimensional complex vector space and $T \in \operatorname{End} V$.

(2) Let R be a ring. Then $\mathbf{Proj}\, R$, the category of finitely generated projective R-modules, is a category with exact sequences, with small skeleton the set of direct summands in $\{R^n : n \in \mathbb{N}\}$. However, this is usually not an abelian category since the cokernel of a map between projective modules is usually not projective (think of the simple case $R = \mathbb{Z}$, $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$). The category $\mathbf{Proj}\, R$ has the additional property, not true for the category of finite-dimensional complex representations of $\mathbb{Z}$, that every short exact sequence splits.

(3) Let R be a ring and let R-$\mathbf{Mod_{fg}}$ be the category of finitely generated R-modules. This is an additive subcategory of the abelian category of all R-modules, and has as a small skeleton the set of quotient modules of the $\{R^n : n \in \mathbb{N}\}$. If R is not left Noetherian, this is not an abelian category, since the kernel of a map between finitely generated R-modules may fail to be finitely generated. (If I is a left ideal of R that is not finitely generated, then R and R/I are singly generated but the kernel of the quotient map $R \to R/I$ is not finitely generated, so this morphism doesn't have a kernel in the category.) Nevertheless, R-$\mathbf{Mod_{fg}}$ is always a category with exact sequences, since if

$$0 \to M_1 \to M \to M_2 \to 0$$

is an exact sequence of R-modules with M_1 and M_2 finitely generated, one can choose a finite set of elements of M whose images in M_2 generate M_2, and these together with the images of a finite set of generators of M_1 will generate M.

(4) Let R be a ring and let R-$\mathbf{Mod_{fpr}}$ be the category of R-modules with a **finite-type projective resolution**, *i.e.*, R-modules M for which there exists an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to M \to 0 \qquad (3.1.3)$$

with $P_j \in \operatorname{Obj} \mathbf{Proj}\, R$. This is a full additive subcategory of R-$\mathbf{Mod_{fg}}$, and may or may not coincide with R-$\mathbf{Mod_{fg}}$. If it does and R is left Noetherian (so that R-$\mathbf{Mod_{fg}} = R$-$\mathbf{Mod_{fpr}}$ is an abelian category), the ring R is said to be **(left) regular**. For a ring to be left regular, it is sufficient (but not necessary) that it be left Noetherian and have finite global dimension (which means that there exists an N such that every R-module has a projective resolution of length $\leq N$). For the fact that R is left Noetherian implies that every finitely generated R-module has a resolution by finitely generated projective modules, and the global dimension condition then guarantees that every such resolution has length $\leq N$. In particular, any PID is left regular (since any submodule of a free module is free). Any Dedekind domain R is left regular, since R is Noetherian by Theorem 1.4.5, and the proof of Corollary 1.4.6 shows that every submodule of a finitely generated free R-module is projective.

The group rings of non-trivial finite groups are **not** left regular. To see this, note that for a non-trivial finite cyclic group H one has $H_n(H, \mathbb{Z}) \neq 0$ for all odd n, so that the finitely generated $\mathbb{Z}H$-module $\mathbb{Z}$ cannot have a finite projective resolution. Then if G is any non-trivial finite group, we can choose a non-trivial cyclic subgroup $H \subseteq G$, and it follows from "Shapiro's Lemma" that

$$H_n(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}) \cong H_n(H, \mathbb{Z}) \neq 0$$

for all odd n, so that the finitely generated $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}$ cannot have a finite projective resolution.

We will see in Proposition 3.1.4 below that whether or not R is left regular, R-$\mathbf{Mod_{fpr}}$ is a category with exact sequences.

(5) Let $\mathcal{A}$ be an abelian category in which every simple object is isomorphic to an element of some set S of objects. (A **simple** object in an abelian category is the natural generalization of a simple module over a ring; it is an object $M \in \operatorname{Obj} \mathcal{A}$ (with $M \neq 0$) such that any monomorphism $N \rightarrowtail M$ is either 0 or an isomorphism. The definition has a number of immediate consequences. If M is simple, then $\operatorname{End} M$ is a division ring (**Schur's Lemma**), and any non-zero morphism $M \to M'$ is necessarily a monomorphism, since its kernel $N \rightarrowtail M$ can't be an isomorphism, hence must be 0.) Call the simple objects in $\mathcal{A}$ **objects of length one**, and define inductively (for $n \geq 2$) the **objects of length** n to be those objects $M \in \operatorname{Obj} \mathcal{A}$ for which there is an exact sequence in $\mathcal{A}$

$$0 \to M_1 \to M \to M_2 \to 0$$

with M_1 of length $n-1$ and with $M_2 \in S$. We will see in Proposition 3.1.5 below that the full subcategory $\mathcal{A}_{\mathrm{fl}}$ of $\mathcal{A}$ consisting of **objects of finite length**, objects M of length $\leq n$ for some n, is a category with exact sequences. The Jordan-Hölder Theorem holds in this context (with the usual proof), *i.e.*, for M of finite length, the length $\ell(M)$ is well-defined, and the simple objects that occur in a "composition series" for M are unique up to isomorphism and permutation. The category of finite-dimensional representations of a (topological) group G is a good example of a category of objects of finite length.

(6) Let X be a compact Hausdorff space. Then $\mathbf{Vect}\, X$ is a category with exact sequences, equivalent to $\mathbf{Proj}\, R$, $R = C(X)$, by Theorem 1.6.3. Here one can work over either $\mathbb{R}$ or $\mathbb{C}$.

(7) Let X be a projective algebraic variety [Hartshorne, Ch. I, §2] over an algebraically closed field (or more generally a projective scheme—see [Hartshorne, Ch. II]—over a commutative Noetherian ring). Then $\mathbf{Vect}\, X$, the category of algebraic vector bundles over X, is a category with exact sequences. Since a vector bundle is determined by its sections over open sets, $\mathbf{Vect}\, X$ is the same as the category of finitely generated locally free $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is the sheaf of germs of regular (algebraic) functions over X. As such, it may be identified with an additive subcategory of the abelian category of $\mathcal{O}_X$-modules. A major difference between this example and example (6) is that short exact sequences of algebraic vector bundles, unlike short exact sequences of topological vector bundles, do not necessarily split. This is due to the fact that in the algebraic setting, one does not have partitions of unity, and thus it is not possible to mimic the proof of Theorem 1.6.3.

A related (usually slightly larger) category with exact sequences is **CohSh** X, the category of coherent sheaves over X; this is the category of finitely generated $\mathcal{O}_X$-modules with resolutions by modules from **Vect** X. One can show that **CohSh** X is an abelian category. Under suitable regularity assumptions (e.g., X a non-singular variety), resolutions of coherent sheaves by locally free sheaves will have finite length, and the relationship between the two categories **Vect** X and **CohSh** X is then the same as between **Proj** R and R-**Mod**$_{\mathbf{fg}}$ when R is a left regular ring.

3.1.4. Proposition. *Let R be a ring and let*

$$0 \to M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2 \to 0$$

be a short exact sequence of R-modules. If M_1 and M_2 have resolutions of length n by modules in **Proj** R *(of the form (3.1.3)), then so does M. In particular, R-*$\mathbf{Mod}_{\mathbf{fpr}}$ *(as defined in Example 3.1.2(4)) is a category with exact sequences.*

Proof. Choose resolutions

$$0 \to P_n^{(j)} \xrightarrow{\gamma_n^{(j)}} \cdots \xrightarrow{\gamma_1^{(j)}} P_0^{(j)} \xrightarrow{\gamma_0^{(j)}} M_j \to 0, \quad j = 1, 2.$$

By projectivity of $P_0^{(2)}$, there is a map $\delta_0^{(2)} : P_0^{(2)} \to M$ with $\beta \circ \delta_0^{(2)} = \gamma_0^{(2)}$. Then using $\gamma_0^{(1)}$, we can extend this to a surjection

$$\delta_0 : P_0 = P_0^{(1)} \oplus P_0^{(2)} \twoheadrightarrow M$$

since two elements of M with the same image in M_2 differ by an element of $\alpha(M_1)$. Then we have a short exact sequence

$$0 \to \ker \gamma_0^{(2)} \to \ker \delta_0 \to \ker \gamma_0^{(1)} \to 0$$

and we can repeat the process to get a surjection

$$\delta_1 : P_1 = P_1^{(1)} \oplus P_1^{(2)} \twoheadrightarrow \ker \delta_1.$$

Continuing, we eventually get a resolution of M by the $P_j = P_j^{(1)} \oplus P_j^{(2)}$. □

3.1.5. Proposition. *Let $\mathcal{A}$ be an abelian category, for instance the category of R-modules for some ring R, and let*

$$0 \to M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2 \to 0$$

be a short exact sequence in $\mathcal{A}$. Assume M_1 is of length n_1 and M_2 is of length n_2 in the sense of Example 3.1.2(5). Then M is of length $n_1 + n_2$. In particular, $\mathcal{A}_{\mathfrak{fl}}$ is a category with exact sequences.

Proof. The proof is by induction on $n_2 = \ell(M_2)$. If this is 0, the result is obvious, and if it's 1, this is true by definition. Otherwise, assume the result for smaller values of $\ell(M_2)$ and choose an exact sequence

$$0 \to N \to M_2 \to S \to 0$$

with N of length $n_2 - 1$ and S simple (by definition of $\ell(M_2)$). Let $M' = \beta^{-1}(N)$. By inductive hypothesis, $\ell(M') = n_1 + n_2 - 1$, and we have a short exact sequence

$$0 \to M' \to M \to S \to 0,$$

so M is of length $n_1 + n_2$. □

Now that we have a reasonable number of examples to work with, we are ready to define K_0 and K_1 for categories and G_0 and G_1 for rings.

3.1.6. Definition. Let $\mathcal{P}$ be a category with exact sequences with small skeleton $\mathcal{P}_0$. We define $K_0(\mathcal{P})$ to be the free abelian group on $\operatorname{Obj}\mathcal{P}_0$, modulo the following relations:

0-(i) $[P] = [P']$ if there is an isomorphism $P \xrightarrow{\cong} P'$ in $\mathcal{P}$.
0-(ii) $[P] = [P_1] + [P_2]$ if there is a short exact sequence

$$0 \to P_1 \to P \to P_2 \to 0$$

in $\mathcal{P}$.

Here $[P]$ denotes the element of $K_0(\mathcal{P})$ corresponding to $P \in \operatorname{Obj}\mathcal{P}_0$, and 0-(i) is really the special case of 0-(ii) with $P_1 = 0$. Note also that since every $P \in \operatorname{Obj}\mathcal{P}$ is isomorphic to an object of $\mathcal{P}_0$, the notation $[P]$ makes sense (by 0-(i)) for **any** object of $\mathcal{P}$.

We define $K_1(\mathcal{P})$ to be the free abelian group on pairs (P, α), where $P \in \operatorname{Obj}\mathcal{P}_0$ and $\alpha \in \operatorname{Aut} P$, modulo the following relations:

1-(i) $[(P, \alpha)] + [(P, \beta)] = [(P, \alpha\beta)]$.
1-(ii) If there is a commutative diagram in $\mathcal{P}$ with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{\iota} & P & \xrightarrow{\pi} & P_2 & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} & & \| \\
0 & \longrightarrow & P_1 & \xrightarrow{\iota} & P & \xrightarrow{\pi} & P_2 & \longrightarrow & 0,
\end{array}$$

where $\alpha \in \operatorname{Aut} P$, $\alpha_1 \in \operatorname{Aut} P_1$, and $\alpha_2 \in \operatorname{Aut} P_2$, then

$$[(P, \alpha)] = [(P_1, \alpha_1)] + [(P_2, \alpha_2)].$$

If R is a ring (with unit), we define $G_0(R) = K_0(R\text{-}\mathbf{Mod}_{\mathbf{fg}})$, $G_1(R) = K_1(R\text{-}\mathbf{Mod}_{\mathbf{fg}})$.

This definition is justified by the fact that in the case of Example 3.1.2(2), it gives us back our old definitions of K_0 and K_1 for rings.

3.1.7. Theorem. *If R is a ring and $\mathbf{Proj}\,R$ is the category of finitely generated projective modules over R, then $K_0(R)$ may be identified naturally with $K_0(\mathbf{Proj}\,R)$, and $K_1(R)$ may be identified naturally with*

$K_1(\mathbf{Proj}\,R)$. In particular, if R is a division ring, then since $\mathbf{Proj}\,R = R\text{-}\mathbf{Mod}_{\mathbf{fg}}$, $G_0(R) = K_0(R) \cong \mathbb{Z}$ and $G_1(R) = K_1(R) \cong R^{\times}_{\mathrm{ab}}$.

Proof. (1) By their definitions, $K_0(R)$ and $K_0(\mathbf{Proj}\,R)$ are both abelian groups with one generator $[P]$ for each isomorphism class of finitely generated projective modules over R. In $K_0(R)$, $[P] + [Q]$ is defined to be $[P \oplus Q]$, whereas in $K_0(\mathbf{Proj}\,R)$, by relation 0-(ii), $[P] + [Q]$ is given by $[N]$ for any finitely generated projective module N for which there exists a short exact sequence

$$0 \to P \to N \to Q \to 0.$$

Since $N = P \oplus Q$ clearly has this property, the addition operations in the two groups coincide. Finally, we need to see that any relation satisfied in one group is satisfied in the other. By the definition of the Grothendieck group (cf. Theorem 1.1.3), $K_0(R)$ is the free group on the generators $[P]$ modulo the relations $[P] = [P']$ if $P \cong P'$, $[P] + [Q] = [P \oplus Q]$. These relations are satisfied in $K_0(\mathbf{Proj}\,R)$, so we only need check that relation 0-(ii) of Definition 3.1.6 is satisfied in $K_0(R)$. But if

$$0 \to P_1 \to P \to P_2 \to 0$$

is a short exact sequence in $\mathbf{Proj}\,R$, this sequence must split since P_2 is projective, and thus $P \cong P_1 \oplus P_2$, so that

$$[P] = [P_1 \oplus P_2] = [P_1] + [P_2] \quad \text{in } K_0(R),$$

as required.

(2) If $A \in GL(n, R)$, then A defines an automorphism $\alpha \in \mathrm{Aut}(R^n)$, so let us define a map $\varphi : K_1(R) \to K_1(\mathbf{Proj}\,R)$ by $[A] \mapsto [(R^n, \alpha)]$. To show this is well defined, suppose $A' \in GL(n', R)$ defines $\alpha' \in \mathrm{Aut}(R^{n'})$. Recall that $[A] = [A']$ in $K_1(R)$ if and only if there is some $N \geq n, n'$, such that

$$(A \oplus 1_{N-n}) \equiv (A' \oplus 1_{N-n'}) \mod E(N, R).$$

But first of all,

$$[(R^n, \alpha)] = [(R^N, \alpha \oplus 1_{R^{N-n}})] \quad \text{and} \quad [(R^{n'}, \alpha')] = [(R^N, \alpha' \oplus 1_{R^{N-n'}})]$$

in $K_1(\mathbf{Proj}\,R)$ by relation 1-(ii) of Definition 3.1.6. Secondly, if $B \in GL(N, R)$ defines $\beta \in \mathrm{Aut}(R^N)$ and $C \in GL(N, R)$ defines $\gamma \in \mathrm{Aut}(R^N)$, then $BC \in GL(N, R)$ defines $\gamma\beta \in \mathrm{Aut}(R^N)$ (we are letting matrices act on the right), and thus (by 1-(i) of Definition 3.1.6)

$$\varphi([B] \cdot [C]) = \varphi([BC]) = [(R^N, \gamma\beta)] = [(R^N, \gamma)] + [(R^N, \beta)],$$

which is the same as $\varphi([B]) + \varphi([C])$. So to complete the proof that φ is well defined, we need only show that $\varphi([C]) = 1$ if $C \in E(N, R)$. It suffices

to prove this with $C = e_{ij}(a)$, $a \in R$. But note that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^{N-1} & \xrightarrow{\iota} & R^N & \xrightarrow{\pi} & R & \longrightarrow & 0 \\
\| & & \| & & \downarrow{\scriptstyle e_{ij}(a)} & & \| & & \| \\
0 & \longrightarrow & R^{N-1} & \xrightarrow{\iota} & R^N & \xrightarrow{\pi} & R & \longrightarrow & 0,
\end{array}$$

where ι is the obvious map from R^{N-1} to the vectors in R^N with i-th coordinate 0, and π is projection onto the i-th coordinate, so that by relation 1-(ii) of Definition 3.1.6, we have

$$[(R^N, e_{ij}(a))] = [(R^{N-1}, 1_{R^{N-1}})] + [(R, 1_R)] = [(R^N, 1_{R^N})].$$

Thus φ is well defined, and the proof has shown at the same time that it is a homomorphism.

Now let us show that $\varphi : K_1(R) \to K_1(\mathbf{Proj}\, R)$ is an isomorphism. (Note by the way that we are writing $K_1(R)$ multiplicatively and $K_1(\mathbf{Proj}$ $R)$ additively.) To show φ is surjective, it suffices to observe that if $P \in \operatorname{Obj} \mathbf{Proj}\, R$ and $\alpha \in \operatorname{Aut} P$, then there must be (by Theorem 1.1.2) some $Q \in \operatorname{Obj} \mathbf{Proj}\, R$ and $N \in \mathbb{N}$ with $P \oplus Q \cong R^N$. Using relation 1-(ii) of Definition 3.1.6, we have

$$[(P, \alpha)] + [(Q, 1_Q)] = [(P \oplus Q, \alpha \oplus 1_Q)],$$

which, since $P \oplus Q \cong R^N$, lies in the image under φ of $GL(N, R)$. But $[(Q, 1_Q)]$ is the identity element of $K_1(\mathbf{Proj}\, R)$, so this shows $[(P, \alpha)]$ lies in the image of φ.

So it remains only to show injectivity. Suppose $\varphi([C]) = 0$ for some $C \in GL(n, R)$. This means that if γ is the corresponding automorphism of R^n, then $[(R^n, \gamma)]$ lies in the subgroup of the free abelian group on all pairs $[(P, \alpha)]$, $P \in \operatorname{Obj} \mathbf{Proj}\, R$ and $\alpha \in \operatorname{Aut} P$, generated by the relations

$$[(P, \alpha)] + [(P, \beta)] - [(P, \alpha\beta)],$$

$$[(P, \alpha)] - [(P_1, \alpha_1)] - [(P_2, \alpha_2)]$$

associated to 1-(i) and 1-(ii) of Definition 3.1.6. But these relations can all be rewritten as linear combinations of the relations

$$[(P, \alpha)] - [(P \oplus Q, \alpha \oplus 1_Q)]$$

whenever $P \oplus Q \cong R^n$, together with the relations associated to 1-(i) and 1-(ii) with all modules not just projective but free. So we can suppose $[(R^n, \gamma)]$ lies in the subgroup generated by relations associated to finitely generated free modules.

Since we may take our finitely generated free modules to run over the set $\{R^n : n \in \mathbb{N}\}$, we may identify each automorphism of a free module with

the corresponding matrix, and we may suppose that in the free abelian group F on generators $[A, j]$, with $A \in GL(j, R)$, $j \in \mathbb{N}$, $[C, n]$ lies in the subgroup generated by the relations

$$1\text{-}(i') \qquad [A, j] + [B, j] - [BA, j]$$

corresponding to the relations associated to 1-(i), and by the relations

$$1\text{-}(ii') \qquad [A, j+k] - [A_1, j] - [A_2, k]$$

attached to diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^j & \xrightarrow{\iota} & R^{j+k} & \xrightarrow{\pi} & R^k & \longrightarrow & 0 \\ \| & & \downarrow{A_1} & & \downarrow{A} & & \downarrow{A_2} & & \| \\ 0 & \longrightarrow & R^j & \xrightarrow{\iota} & R^{j+k} & \xrightarrow{\pi} & R^k & \longrightarrow & 0, \end{array}$$

corresponding to the relations associated to 1-(ii). We may further rewrite the relations of type 1-(ii') as linear combinations of those of two sorts: relations

$$1\text{-}(ii')\text{-a} \qquad [A, j] - [BAB^{-1}, j]$$

(corresponding to the case $k = 0$ above), allowing for arbitrary changes of basis, and relations

$$1\text{-}(ii')\text{-b} \qquad \left[\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}, j+k\right] - [A_1, j] - [A_2, k]$$

corresponding to the case where the injection $R^j \to R^{j+k}$ is the standard one given by the first j coordinates. The quotient of the free abelian group F by the subgroup generated by relations 1-(i') and 1-(ii')-a is clearly the direct sum $\bigoplus_j GL(j, R)_{\text{ab}}$. Dividing by the subgroup generated by the relations 1-(ii')-b then gives $\varinjlim GL(j, R)_{\text{ab}} = GL(R)_{\text{ab}} = K_1(R)$, divided by the additional relation that

$$\left[\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}\right] = \left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right].$$

However, this relation is already satisfied in $K_1(R)$, so $[C] = 1 \in K_1(R)$ and φ is an isomorphism. □

Let us now examine the meaning of Definition 3.1.6 for the other Examples 3.1.2. When X is a compact Hausdorff space, it is obvious that $K_0(\mathbf{Vect}\, X)$ is the Grothendieck group of the semigroup of isomorphism classes of vector bundles over X, and may be identified with $K^0(X)$. $K_1(\mathbf{Vect}\, X)$ is a less familiar object, but since $\mathbf{Vect}\, X \cong \mathbf{Proj}\, R$ with $R = C(X)$ by Theorem 1.6.3, this is the same as $K_1(C(X))$. It turns out (see Exercise 3.1.23 below) that there are exact sequences of abelian groups

$$0 \to C^{\mathbb{R}}(X) \xrightarrow{\exp} K_1(\mathbf{Vect}_{\mathbb{R}} X) \to KO^{-1}(X) \to 0,$$

$$0 \to C(X, \mathbb{Z}) \xrightarrow{2\pi i} C^{\mathbb{C}}(X) \xrightarrow{\exp} K_1(\mathbf{Vect}_{\mathbb{C}} X) \to KU^{-1}(X) \to 0.$$

The example of finite-dimensional representations of a topological group G is a special case of Example 3.1.2(5), so we turn to this sort of situation next. The following result was pointed out by Grothendieck in his earliest investigations of K-theory.

3.1.8. Theorem ("Devissage"). *Let $\mathcal{A}$ be an abelian category in which every simple object is isomorphic to one and only one element of some set $S \subseteq \operatorname{Obj} \mathcal{A}$. Then*

(1) *$K_0(\mathcal{A}_{\mathrm{fl}})$ is canonically isomorphic to the free abelian group on the set S.*
(2) *$K_1(\mathcal{A}_{\mathrm{fl}})$ is canonically isomorphic to $\bigoplus_{M \in S} K_1(\operatorname{End} M)$. (Since for $M \in S$, $\operatorname{End} M$ is a division ring, we have $K_1(\operatorname{End} M) \cong (\operatorname{End} M)^{\times}_{\mathrm{ab}} = (\operatorname{Aut} M)_{\mathrm{ab}}$ by Corollary 2.2.6.)*

Proof. (1) Clearly there is a homomorphism φ from the indicated free abelian group F to $G = K_0(\mathcal{A}_{\mathrm{fl}})$, defined by sending a generator $[M]$, $M \in S$, to the corresponding generator of G. To define an inverse ψ to this homomorphism, if $M \in \operatorname{Obj} \mathcal{A}$ is of finite length, map $[M] \in G$ to $\sum_i [M_i] \in F$, where the $M_i \in S$ are the composition factors of M (repeated according to their multiplicities), which are well defined by the Jordan-Hölder Theorem. This gives a well-defined map on G since if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence, the composition factors of M (counting multiplicities) are just the union of the composition factors of M' and the composition factors of M''. We have $\psi \circ \varphi = 1_F$ by the construction. To prove that $\varphi \circ \psi = 1_G$, we show $\varphi \circ \psi([M]) = [M]$ for $M \in \operatorname{Obj} \mathcal{A}_{\mathrm{fl}}$ by induction on $\ell(M)$. If $\ell(M) \leq 1$, this is obvious, so assume the result for M' with $\ell(M') < \ell(M)$, and choose a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with $M'' \in S$. By inductive hypothesis, $\varphi \circ \psi([M']) = [M']$ and $\varphi \circ \psi([M'']) = [M'']$. But $[M] = [M'] + [M'']$, so $\varphi \circ \psi([M]) = [M]$, and this completes the inductive step.

(2) Let $\mathcal{A}_{\mathrm{ss}}$ denote the category of semisimple objects in $\mathcal{A}$, *i.e.*, the finite direct sums of simple objects. We will define an isomorphism $\varphi : K_1(\mathcal{A}_{\mathrm{fl}}) \to \bigoplus_{N \in S} K_1(\operatorname{End} N)$ as follows. Given $M \in \operatorname{Obj} \mathcal{A}_{\mathrm{fl}}$ and $\alpha \in \operatorname{Aut} M$, note that the largest semisimple subobject M_1 of M (this is usually called the **socle** of M, denoted $\operatorname{soc} M$) exists and must be non-zero, and is necessarily α-invariant. So there is an α-invariant canonical finite filtration of M with composition factors M_i in $\operatorname{Obj} \mathcal{A}_{\mathrm{ss}}$. (Take the cokernel of $\operatorname{soc} M \rightarrowtail M$, take **its** socle, and keep iterating the construction as many times as necessary.) Let α_i be the automorphism of the composition factor M_i induced by α. By relation 1-(ii) of Definition 3.1.6, we have $[(M, \alpha)] = \sum_i [(M_i, \alpha_i)]$. Now each M_i is isomorphic to a direct sum of simple objects $N \in S$ with certain multiplicities n_i^N, and $\operatorname{End} M_i \cong \prod_{N \in S} M_{n_i^N}(\operatorname{End} N)$. So α_i may be viewed as an element of $\prod_{N \in S} GL(n_i^N, \operatorname{End} N)$ (this is really a finite product), and thus defines an element $[\alpha_i]$ of $\bigoplus_{N \in S} K_1(\operatorname{End} N)$. We let $\varphi([(M, \alpha)]) = \sum_i [\alpha_i]$. This defines a homomorphism from the free abelian group on the pairs (M, α) to $\bigoplus_{N \in S} K_1(\operatorname{End} N)$, and since it is clearly compatible with

relations 1-(i) and 1-(ii) of Definition 3.1.6, it passes to a homomorphism $\varphi : K_1(\mathcal{A}_{\mathrm{fl}}) \to \bigoplus_{N\in S} K_1(\operatorname{End} N)$. Furthermore, φ is clearly surjective, since if $N \in S$ and $a \in \operatorname{Aut} N$, $\varphi([(N, a)]) = [a] \in (\operatorname{Aut} N)_{\mathrm{ab}} = K_1(\operatorname{End} N)$, and thus the image of φ contains a set of generators for $\bigoplus_{N\in S} K_1(\operatorname{End} N)$.

It remains only to show that φ is injective. For this it is enough to note that the proof of surjectivity of φ in fact gives a construction of an inverse, namely, if $N_1, \dots, N_k$ are distinct elements of S and $a_i \in \operatorname{Aut} N_i$,

$$\psi : \sum_i^k [a_i] \mapsto \left[\left(\bigoplus_i^k N_i, \bigoplus_i^k a_i\right)\right].$$

This is well defined since replacing each a_i by a conjugate element of $\operatorname{Aut} N_i$ does not change the K_1-class on the right, and ψ obviously gives a right inverse to φ. To see that ψ gives a left inverse to φ, note that with (M, α) as above,

$$\psi \circ \varphi([(M, \alpha)]) = \sum_i \psi([\alpha_i]) = \left[\left(\bigoplus_i N_i, \bigoplus_i \det \alpha_i\right)\right],$$

which agrees with (M, α) by the proof of the fact that $K_1(\mathbf{Proj} \operatorname{End} N_i) \cong K_1(\operatorname{End} N_i)$ (Theorem 3.1.7). □

The next theorem, also due to Grothendieck, applies to our other main classes of examples, and relates R-$\mathbf{Mod}_{\mathbf{fpr}}$ to $\mathbf{Proj}\, R$ and (in the regular case) $\mathbf{CohSh}\, X$ to $\mathbf{Vect}\, X$. The version in which we state it, taken from [BassHellerSwan], is probably not as general as possible, but will be adequate for our purposes. First we need a simple observation about the functoriality of K_0 and K_1, a simple lemma about the "Euler-Poincaré principle" of §1.7, and a lemma about "resolutions" in a category with exact sequences.

3.1.9. Proposition. *Suppose $\mathcal{P}$ and $\mathcal{M}$ are categories with exact sequences, and $F : \mathcal{P} \to \mathcal{M}$ is an exact functor, i.e., a functor sending short exact sequences to short exact sequences. Then F induces homomorphisms $F_* : K_0(\mathcal{P}) \to K_0(\mathcal{M})$ and $F_* : K_1(\mathcal{P}) \to K_1(\mathcal{M})$. In fact, K_0 and K_1 are functors from the category of all categories with exact sequences and exact functors to the category of abelian groups.*

Proof. This is immediate from the fact that F sends relations 0-(i), 0-(ii), 1-(i), and 1-(ii) for $\mathcal{P}$ to corresponding relations for $\mathcal{M}$. □

3.1.10. Lemma. *Let $\mathcal{M}$ be a category with exact sequences contained in some abelian category $\mathcal{A}$, and assume that if*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$ and $M_2, M_3 \in \operatorname{Obj} \mathcal{M}$, then $M_1 \in \operatorname{Obj} \mathcal{M}$. (In other words, $\mathcal{M}$ contains the kernel of each of its morphisms which is an epimorphism in $\mathcal{A}$.) Then for any exact sequence

$$0 \to M_n \to \cdots \to M_1 \to M_0 \to 0,$$

the Euler characteristic $\sum_j (-1)^j [M_j]$ vanishes in $K_0(\mathcal{M})$.

Proof. This is true by 0-(i) of Definition 3.1.6 if $n = 1$, and by 0-(ii) of Definition 3.1.6 if $n = 2$. So let $n > 3$ and assume by induction on n that the Lemma is true for exact sequences of shorter length. By the assumption on $\mathcal{M}$, the kernel K of $M_1 \twoheadrightarrow M_0$ lies in $\mathcal{M}$, so we can split the given exact sequence into the shorter exact sequences

$$0 \to K \to M_1 \to M_0 \to 0,$$

$$0 \to M_n \to \cdots \to M_2 \to K \to 0.$$

By 0-(ii), $[K] + [M_0] = [M_1]$, and by inductive hypothesis,

$$[K] - \sum_{j=2}^{n} (-1)^j [M_j] = 0.$$

Combining these two equations gives $\sum_{j=0}^{n} (-1)^j [M_j] = 0$. □

3.1.11. Lemma. *Suppose $\mathcal{M}$ and $\mathcal{P}$ are categories with exact sequences, both contained in the same abelian category $\mathcal{A}$, and with $\mathcal{P}$ a full subcategory of $\mathcal{M}$. Also assume:*

(1) *that for each object $M \in \operatorname{Obj}\mathcal{M}$, there is a finite resolution by objects of $\mathcal{P}$, i.e., an exact sequence (3.1.3) in $\mathcal{M}$ of finite length with $P_j \in \operatorname{Obj}\mathcal{P}$;*

(2) *that if*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$ and $M_2, M_3 \in \operatorname{Obj}\mathcal{M}$ (resp., $\operatorname{Obj}\mathcal{P}$), then $M_1 \in \operatorname{Obj}\mathcal{M}$ (resp., $\operatorname{Obj}\mathcal{P}$). (In other words, $\mathcal{M}$ and $\mathcal{P}$ each contain the kernels of each of their morphisms which are epimorphisms in $\mathcal{A}$.)

Then if $M' \xrightarrow{\alpha} M$ is a morphism in $\mathcal{M}$ and

$$0 \to P_n \to \cdots \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is a resolution of M by objects of $\mathcal{P}$, one can complete these to a commuting diagram in $\mathcal{M}$

$$\begin{array}{ccccccccccccc}
0 & \to & \cdots & \to & P'_{n+1} & \to & P'_n & \to & \cdots & \to & P'_0 & \xrightarrow{\varepsilon'} & M' & \to & 0 \\
\| & & & & \downarrow & & \downarrow{\scriptstyle \alpha_n} & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \| \\
0 & \to & \cdots & \to & 0 & \to & P_n & \to & \cdots & \to & P_0 & \xrightarrow{\varepsilon} & M & \to & 0
\end{array}$$

whose rows are finite resolutions by objects of $\mathcal{P}$.

Proof. Note that $P_0 \oplus M' \xrightarrow{(\varepsilon, -\alpha)} M$ is an epimorphism since $P_0 \xrightarrow{\varepsilon} M$ is, hence by hypothesis (2) on $\mathcal{M}$ it has a kernel $B \rightarrowtail P_0 \oplus M'$ in $\mathcal{M}$. (This

is the "pull-back" of ε and α.) Hence by hypothesis (1) on $\mathcal{M}$, there is an epimorphism $P_0' \twoheadrightarrow B$. Composing with the maps $B \to P_0$ and $B \to M'$ we get a commuting diagram

$$\begin{array}{ccccc}
P_0' & \xrightarrow{\varepsilon'} & M' & \longrightarrow & 0 \\
& \searrow \quad \nearrow & & & \\
\alpha_0 \big\downarrow & B & \big\downarrow & & \big\| \\
& \swarrow & & & \\
P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0.
\end{array}$$

The remaining α_j, $j \geq 1$, are constructed by induction on j. Suppose $j \geq 1$ and α_k has been constructed for $0 \leq k < j$. By hypothesis (2) on $\mathcal{M}$, $P_0' \xrightarrow{\varepsilon'} M'$ and $P_0 \xrightarrow{\varepsilon} M$ have kernels Z_0' and Z_0 in $\mathcal{M}$. Then $P_1' \to Z_0'$ and $P_1 \to Z_0$ are epimorphisms, and also have kernels in $\mathcal{M}$. Iterating the argument, we see we have $Z_{j-1}', Z_{j-1} \in \operatorname{Obj} \mathcal{M}$ and a commutative diagram with exact rows

$$\begin{array}{ccccccccccccc}
0 & \to & Z_{j-1}' & \to & P_{j-1}' & \to & \cdots & \to & P_0' & \xrightarrow{\varepsilon'} & M' & \to & 0 \\
\big\| & & {\scriptstyle \operatorname{res} \alpha_{j-1}} \big\downarrow & & {\scriptstyle \alpha_{j-1}} \big\downarrow & & & & {\scriptstyle \alpha_0} \big\downarrow & & {\scriptstyle \alpha} \big\downarrow & & \big\| \\
0 & \to & Z_{j-1} & \to & P_{j-1} & \to & \cdots & \to & P_0 & \xrightarrow{\varepsilon} & M & \to & 0.
\end{array}$$

Now we just repeat the above construction to fill in the commutative diagram

$$\begin{array}{ccccc}
P_j' & \longrightarrow & Z_{j-1}' & \longrightarrow & 0 \\
{\scriptstyle \alpha_j} \big\downarrow & & {\scriptstyle \operatorname{res} \alpha_{j-1}} \big\downarrow & & \big\| \\
P_j & \longrightarrow & Z_{j-1} & \longrightarrow & 0.
\end{array}$$

This completes the inductive step, so the induction gives us a commuting diagram with exact rows

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Z_n' & \longrightarrow & P_n' & \longrightarrow & \cdots & \longrightarrow & P_0' & \xrightarrow{\varepsilon'} & M' & \longrightarrow & 0 \\
\big\| & & \big\downarrow & & {\scriptstyle \alpha_n} \big\downarrow & & & & {\scriptstyle \alpha_0} \big\downarrow & & {\scriptstyle \alpha} \big\downarrow & & \big\| \\
0 & \longrightarrow & 0 & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0.
\end{array}$$

We complete the diagram by using hypothesis (1) to get a finite resolution of Z_n' by objects of $\mathcal{P}$. □

3.1.12. Corollary. *Under the same hypotheses as Lemma 3.1.11, if* $M \in \operatorname{Obj} \mathcal{M}$ *and* $P_\bullet \xrightarrow{\varepsilon} M$, $P_\bullet' \xrightarrow{\varepsilon'} M$ *are two different finite resolutions of* M *by objects of* $\mathcal{P}$, *then* $\sum_j (-1)^j [P_j]_{\mathcal{P}} = \sum_j (-1)^j [P_j']_{\mathcal{P}}$ *in* $K_0(\mathcal{P})$.

Proof. Apply the Lemma to complete the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P''_n & \longrightarrow \cdots \longrightarrow & P''_0 & \xrightarrow{\varepsilon''} & M & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle (\alpha_n, \alpha'_n)} & & \downarrow{\scriptstyle (\alpha_0, \alpha'_0)} & & \downarrow{\scriptstyle \Delta} & & \| \\ 0 & \longrightarrow & P_n \oplus P'_n & \longrightarrow \cdots \longrightarrow & P_0 \oplus P'_0 & \xrightarrow{\varepsilon \oplus \varepsilon'} & M \oplus M & \longrightarrow & 0, \end{array}$$

where Δ is the diagonal map. Consider $\alpha_\bullet$ and $\alpha'_\bullet$ as chain maps of bounded chain complexes in $\mathcal{P}$:

$$\alpha_\bullet : P''_\bullet \to P_\bullet \quad \text{and} \quad \alpha'_\bullet : P''_\bullet \to P'_\bullet.$$

Note that we have cut off the M's at the end, so that $P''_\bullet$, $P_\bullet$, and $P'_\bullet$ are only chain complexes, not exact sequences. Each one is acyclic except at the 0-th slot and has non-vanishing homology $H_0 = M$. Since α_* and α'_* are isomorphisms on homology because of the commutative diagram above, the mapping cones C_α and $C_{\alpha'}$ are acyclic (recall Theorem 1.7.7; we are in a general abelian category rather than the category of modules over a ring, but otherwise the proof is the same). So by Lemma 3.1.10 (applied in the category $\mathcal{P}$), together with the definition of the mapping cone, we have

$$0 = \chi(C_\alpha) = \chi(P_\bullet) - \chi(P''_\bullet), \quad 0 = \chi(C_{\alpha'}) = \chi(P'_\bullet) - \chi(P''_\bullet)$$

in $K_0(\mathcal{P})$. The result follows immediately. □

3.1.13. Theorem ("Resolution theorem"). *Suppose $\mathcal{M}$ and $\mathcal{P}$ are categories with exact sequences, both contained in the same abelian category $\mathcal{A}$, and with $\mathcal{P}$ a full subcategory of $\mathcal{M}$. Also assume:*

(1) *that for each object $M \in \operatorname{Obj} \mathcal{M}$, there is a finite resolution by objects of $\mathcal{P}$, i.e., an exact sequence (3.1.3) in $\mathcal{M}$ of finite length with $P_j \in \operatorname{Obj} \mathcal{P}$;*

(2) *that if*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$ and $M_2, M_3 \in \operatorname{Obj} \mathcal{M}$ (resp., $\operatorname{Obj} \mathcal{P}$), then $M_1 \in \operatorname{Obj} \mathcal{M}$ (resp., $\operatorname{Obj} \mathcal{P}$). (In other words, $\mathcal{M}$ and $\mathcal{P}$ each contain the kernels of each of their morphisms which are epimorphisms in $\mathcal{A}$.)

Then the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}$ induces an isomorphism on K_0.

Proof. If a category with exact sequences $\mathcal{P}$ is a full subcategory of a category with exact sequences $\mathcal{M}$, then the inclusion functor $\iota : \mathcal{P} \hookrightarrow \mathcal{M}$ is exact, so it induces a map ι_* on K_0 and K_1 by Proposition 3.1.9. Using the idea of the Euler characteristic from §1.7, we construct an inverse map $\varphi_0 : K_0(\mathcal{M}) \to K_0(\mathcal{P})$ by $\varphi_0 : [M]_\mathcal{M} \mapsto \sum_j (-1)^j [P_j]_\mathcal{P}$ if

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

is exact in $\mathcal{M}$, with $P_j \in \operatorname{Obj}\mathcal{P}$. This is well defined, *i.e.*, independent of the choice of resolution, by Corollary 3.1.12. By Lemma 3.1.10, $[M]_{\mathcal{M}} = \sum_j (-1)^j [P_j]_{\mathcal{M}}$ in $K_0(\mathcal{M})$, so $\iota_* \circ \varphi_0([M]_{\mathcal{M}}) = [M]_{\mathcal{M}}$. On the other hand it is clear that $\varphi_0 \circ \iota_*([P]_{\mathcal{P}}) = \varphi_0([P]_{\mathcal{M}}) = [P]_{\mathcal{P}}$ for $P \in \operatorname{Obj}\mathcal{P}$, so ι_* is an isomorphism on K_0. □

Next we have the analogue of Theorem 3.1.13 for K_1, again following [BassHellerSwan]. Note that the hypotheses are a bit stronger than those of 3.1.13, though they will still be satisfied for all cases of interest.

3.1.14. Theorem ("Resolution theorem for K_1"). *Suppose $\mathcal{M}$ and $\mathcal{P}$ are categories with exact sequences, both contained in the same abelian category $\mathcal{A}$, and with $\mathcal{P}$ a full subcategory of $\mathcal{M}$. Also assume:*

(1) *that for each object $M \in \operatorname{Obj}\mathcal{M}$, there is an epimorphism $P \twoheadrightarrow M$ in $\mathcal{A}$ with P an object of $\mathcal{P}$, such that every endomorphism of M lifts to an endomorphism of P;*

(2) *that if*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact in $\mathcal{M}$ with $P_j \in \operatorname{Obj}\mathcal{P}$, then $\ker d_n \in \operatorname{Obj}\mathcal{P}$ for some (sufficiently large) n;

(3) *that if*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$ and M_2, $M_3 \in \operatorname{Obj}\mathcal{M}$ (resp., $\operatorname{Obj}\mathcal{P}$), then $M_1 \in \operatorname{Obj}\mathcal{M}$ (resp., $\operatorname{Obj}\mathcal{P}$). (In other words, $\mathcal{M}$ and $\mathcal{P}$ each contain the kernels of each of their morphisms which are epimorphisms in $\mathcal{A}$.)

Then the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}$ induces an isomorphism on K_1.

Proof. First we want to show that every automorphism of an object of $\mathcal{M}$ lifts to an automorphism of some finite resolution of M by objects from $\mathcal{P}$. Then we will be able to apply the same sort of reasoning as in the proof of Theorem 3.1.13.

So let $M \in \operatorname{Obj}\mathcal{M}$, $\alpha \in \operatorname{Aut} M$. Using (1), choose $P \twoheadrightarrow M$ with $P \in \operatorname{Obj}\mathcal{P}$ so that every endomorphism of M lifts to an endomorphism of P. Then consider $\alpha \oplus \alpha^{-1} \in \operatorname{Aut}(M \oplus M)$. By Lemma 1.5.4 (the same argument works in a general abelian category), this factors as a product of "elementary" automorphisms of the form

$$\begin{pmatrix} 1_M & \beta \\ 0 & 1_M \end{pmatrix}, \qquad \begin{pmatrix} 1_M & 0 \\ \gamma & 1_M \end{pmatrix}$$

with β, $\gamma \in \operatorname{End} M$. Lifting β and γ to endomorphisms of P shows that $\alpha \oplus \alpha^{-1}$ lifts to an automorphism of $P \oplus P$. Then $(P \twoheadrightarrow M) \oplus (P \to 0)$ gives us the first step of our desired resolution of M. The kernel of this map $P \oplus P \twoheadrightarrow M$ must be an object of $\mathcal{M}$ by hypothesis (3), so we can

repeat the same construction over and over to get a (potentially infinite) resolution of M by objects of $\mathcal{P}$ so that α lifts to an automorphism of the resolution. Then we can cut off the resolution at some finite stage by hypothesis (2).

The rest of the proof is as in Theorem 3.1.13. We construct an inverse φ_1 to $\iota_* : K_1(\mathcal{P}) \to K_1(\mathcal{M})$ by mapping $[(M, \alpha)] \mapsto \sum_j (-1)^j [(P_j, \alpha_j)]$, where $\alpha_\bullet$ is an automorphism of the finite resolution $P_\bullet$ of M lifting α. To show this is well defined (and independent of the choice of resolution), we use Corollary 3.1.12 applied to the categories of pairs (M, α), $\alpha \in \operatorname{Aut} M$, $M \in \operatorname{Obj} \mathcal{M}$ (resp., $\mathcal{P}$), where the morphisms are commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ M & \xrightarrow{f} & M'. \end{array}$$

It is easy to see that the hypotheses of Corollary 3.1.12 apply to this situation, and we finish the proof as in Theorem 3.1.13. □

In order to apply Theorems 3.1.13 and 3.1.14 to the context of R-modules, we need as a preliminary a familiar fact from homological algebra. (See, for instance, [CartanEilenberg, Proposition VI.2.1].)

3.1.15. Lemma. *Let R be any ring (with unit) and let M be an R-module. Then the following are equivalent:*

(a) *M has a projective resolution of length n.*
(b) *For any R-module N, $\operatorname{Ext}_R^{n+1}(M, N) = 0$.*
(c) *The functor $N \rightsquigarrow \operatorname{Ext}_R^n(M, N)$ is right exact.*
(d) *For* **any** *projective resolution*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

$\operatorname{im} d_n = \ker d_{n-1}$ *is projective, and hence the resolution can be shortened to*

$$0 \longrightarrow \operatorname{im} d_n \longrightarrow P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Proof. (a) ⇒ (b). Suppose $0 \to P_n \to \cdots \to P_0 \to M \to 0$ is a projective resolution of M. Then by definition, $\operatorname{Ext}_R^j(M, N)$ is the j-th homology module group of the complex

$$\operatorname{Hom}_R(P_0, N) \to \cdots \to \operatorname{Hom}_R(P_n, N) \to 0 \to \cdots,$$

so clearly $\operatorname{Ext}_R^{n+1}(M, N) = 0$.

(b) ⇒ (c). Assume $\operatorname{Ext}_R^{n+1}(M, N) = 0$. Given a short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

there is an associated long exact sequence

$$\operatorname{Ext}^n_R(M, N_1) \to \operatorname{Ext}^n_R(M, N_2) \to \operatorname{Ext}^n_R(M, N_3) \to \operatorname{Ext}^{n+1}_R(M, N_1) = 0,$$

and thus the functor $N \rightsquigarrow \operatorname{Ext}^n_R(M, N)$ is right exact.

(c) $\Rightarrow$ (d). To check the projectivity of $\operatorname{im} d_n$, we need to show that given a short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

and a homomorphism $\alpha : \operatorname{im} d_n \to N_3$, α factors through N_2. In other words, we need to show that the natural map $\operatorname{Hom}_R(\operatorname{im} d_n, N_2) \to \operatorname{Hom}_R(\operatorname{im} d_n, N_3)$ is surjective, *i.e.*, that the functor

$$N \rightsquigarrow \operatorname{Hom}_R(\operatorname{im} d_n, N)$$

is right exact. This is immediate from (c) if $n = 0$, so assume $n > 0$. From the short exact sequences

$$\begin{cases} 0 \to \operatorname{im} d_{j+1} = \ker d_j \to P_j \to \operatorname{im} d_j \to 0, \quad j > 0, \\ 0 \to \operatorname{im} d_1 = \ker d_0 \to P_0 \to M \to 0, \end{cases}$$

we obtain exact sequences

$$\begin{aligned} 0 = \operatorname{Ext}^{n-1}_R(P_0, N_j) &\to \operatorname{Ext}^{n-1}_R(\operatorname{im} d_1, N_j) \\ &\to \operatorname{Ext}^n_R(M, N_j) \to \operatorname{Ext}^n_R(P_0, N_j) = 0, \end{aligned}$$

$$\begin{aligned} \operatorname{Ext}^{n-j}_R(P_{j-1}, N_j) \to \operatorname{Ext}^{n-j}_R(\operatorname{im} d_j, N_j) &\to \operatorname{Ext}^{n-j+1}_R(\operatorname{im} d_{j-1}, N_j) \\ &\to \operatorname{Ext}^{n-j+1}_R(P_{j-1}, N_j) = 0,\ 0 < j \le n. \end{aligned}$$

These yield isomorphisms

$$\operatorname{Ext}^n_R(M, N_j) \cong \operatorname{Ext}^{n-1}_R(\operatorname{im} d_1, N_j) \cong \cdots \cong \operatorname{Ext}^1_R(\operatorname{im} d_{n-1}, N_j)$$

and thus an exact sequence

$$\operatorname{Hom}_R(P_{n-1}, N_j) \to \operatorname{Hom}_R(\operatorname{im} d_n, N_j) \to \operatorname{Ext}^n_R(M, N_j) \to 0.$$

Assuming (c) and using projectivity of P_{n-1}, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} \operatorname{Hom}_R(P_{n-1}, N_2) & \longrightarrow & \operatorname{Hom}_R(\operatorname{im} d_n, N_2) & \longrightarrow & \operatorname{Ext}^n_R(M, N_2) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \| \\ \operatorname{Hom}_R(P_{n-1}, N_3) & \longrightarrow & \operatorname{Hom}_R(\operatorname{im} d_n, N_3) & \longrightarrow & \operatorname{Ext}^n_R(M, N_3) & \longrightarrow & 0 \\ \downarrow & & & & \downarrow & & \| \\ 0 & & & & 0 & & 0, \end{array}$$

and surjectivity of $\operatorname{Hom}_R(\operatorname{im} d_n, N_2) \to \operatorname{Hom}_R(\operatorname{im} d_n, N_3)$ follows from a diagram chase.

(d) $\Rightarrow$ (a) is trivial. $\square$

3.1.16. Corollary (Grothendieck). *Let R be a left Noetherian ring (with unit). Then the natural map $K_j(R) \to K_j(R\text{-}\mathbf{Mod_{fpr}})$ (induced by the inclusion $\mathbf{Proj}\,R \hookrightarrow R\text{-}\mathbf{Mod_{fpr}}$ together with Theorem 3.1.7) is an isomorphism for $j = 0, 1$. In particular, if R is left regular, then the natural map $K_j(R) \to G_j(R)$ (induced by the inclusion $\mathbf{Proj}\,R \hookrightarrow R\text{-}\mathbf{Mod_{fg}}$ together with Theorem 3.1.7) is an isomorphism for $j = 0, 1$.*

Proof. We need only check the hypotheses of Theorems 3.1.13 and 3.1.14. It is clear from the definition that every R-module in $R\text{-}\mathbf{Mod_{fpr}}$ has a finite resolution by finitely generated projective modules. Furthermore, given an epimorphism $P \twoheadrightarrow M$ with P projective, and given an endomorphism α of M, we can fill in the diagram

$$\begin{array}{ccccc} P & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \| \\ P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

by projectivity of P to get a lifting $\tilde{\alpha}$ of α to P, which checks hypothesis (1) of Theorem 3.1.14. Next, every epimorphism in $\mathbf{Proj}\,R$ splits, hence has kernel which is a direct summand in a projective module, hence has a kernel in $\mathbf{Proj}\,R$. Hypothesis (2) of Theorem 3.1.14 holds by the implication (a) $\Rightarrow$ (d) of Lemma 3.1.15.

To finish the proof, we need only show that if M and M' have finite projective resolutions of finite type and if $M \overset{\alpha}{\twoheadrightarrow} M'$ is an epimorphism, then $\ker\alpha$ also has a finite projective resolution of finite type. First of all, if M and M' each have projective resolutions of length n, then by the implication (a) $\Rightarrow$ (b) of Lemma 3.1.15, $\mathrm{Ext}_R^{n+1}(M, N) = \mathrm{Ext}_R^{n+1}(M', N) = 0$ for any R-module N. By the long exact sequence associated to the short exact sequence

$$0 \to \ker\alpha \to M \to M' \to 0,$$

$0 = \mathrm{Ext}_R^{n+1}(M, N) \to \mathrm{Ext}_R^{n+1}(\ker\alpha, N) \to \mathrm{Ext}_R^{n+2}(M', N) = 0$ is exact, so $\mathrm{Ext}_R^{n+1}(\ker\alpha, N) = 0$ and $\ker\alpha$ has a projective resolution of length n by the implication (b) $\Rightarrow$ (a) of Lemma 3.1.15. If R is left Noetherian, it is immediate that $\ker\alpha$ in fact has a projective resolution of finite type, since we can start with any resolution of $\ker\alpha$ by finitely generated free modules (such a resolution exists, since any submodule of a finitely generated module is finitely generated) and truncate it using the implication (a) $\Rightarrow$ (d) of Lemma 3.1.15. □

Remark. The same sort of reasoning shows that if X is a nonsingular projective algebraic variety, then the natural map

$$K_j(\mathbf{Vect}\,X) \to K_j(\mathbf{CohSh}\,X)$$

is an isomorphism for $j = 0, 1$. We omit the proof since setting up the necessary machinery requires knowledge of too much algebraic geometry.

Now we are prepared to explain the idea of Grothendieck's original motivation for studying the K-theory of categories, namely, for use in studying the **Riemann-Roch problem.** Grothendieck in fact substantially generalized both this problem and the form of its solution, but for simplicity we will restrict attention here to the classical situation.

For readers who are unfamiliar with it, we begin with a quick review of the terminology of sheaf theory. If X is a topological space, a **presheaf** $\mathcal{F}$ (say of R-modules) over X is a contravariant functor from the category of open sets of X (and morphisms given by inclusions) to the category of R-modules. The notation $\Gamma(U, \mathcal{F})$ is also used for $\mathcal{F}(U)$, and we refer to this module as the **sections of $\mathcal{F}$ over U.** A **sheaf** $\mathcal{F}$ of R-modules is a special kind of presheaf: one which also satisfies the **gluing condition,** that if $\{U_j\}$ is a collection of open subsets of X, the restriction map

$$\Gamma(\bigcup_j U_j, \mathcal{F}) \to \{(f_j) \in \prod_j \Gamma(U_j, \mathcal{F}) : f_j|_{U_j \cap U_k} = f_k|_{U_j \cap U_k} \ \forall j, k\}$$

is a bijection. Typical examples of sheaves are the sheaf of germs of continuous $\mathbb{R}$-valued functions, whose module of sections over U is $C^{\mathbb{R}}(U)$, and the **structure sheaf** $\mathcal{O}_X$ of an algebraic variety, whose module of sections over a Zariski-open set U is the set of regular (algebraic) functions defined in U. These may be viewed as rational functions without poles in U.

In the category of sheaves over X, the **global section functor** $\mathcal{F} \rightsquigarrow \Gamma(X, \mathcal{F})$ is left exact but not right exact. It has derived functors $H^j(X, \mathcal{F})$, with the properties that $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ and that a short exact sequence of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^j(X, \mathcal{F}_1) \to H^j(X, \mathcal{F}_2) \to H^j(X, \mathcal{F}_3) \to H^{j+1}(X, \mathcal{F}_1) \to \cdots .$$

We return now to the classical Riemann-Roch problem. Let X be a nonsingular projective algebraic variety of dimension 1 over $\mathbb{C}$, or for short, a **nonsingular curve.** X is a compact connected complex manifold of complex dimension 1 and real (or topological) dimension 2, or in other words a compact connected Riemann surface, say of genus g. (Recall that the **genus** is a purely topological invariant of the underlying manifold of X that doesn't depend on the algebraic structure. It may be defined as the number of "holes" in X, or more precisely as $\frac{1}{2} \operatorname{rank} H_1(X; \mathbb{Z}) = \frac{1}{2} \dim_{\mathbb{C}} H^1(X; \mathbb{C})$.)

A **divisor** D on X is just a formal finite $\mathbb{Z}$-linear combination $\sum n_j x_j$ of points $x_j \in X$, with $n_j \in \mathbb{Z}$. The divisors D are in bijection with isomorphism classes of algebraic line bundles over X via the map $\sum n_j x_j = D \mapsto \mathcal{L}_D$, where $\mathcal{L}_D$ is the line bundle whose (algebraic) sections over an open set U are the rational functions f over U vanishing to order at least $-n_j$ at x_j (and thus regular at points x where $n_x \le 0$). By convention, we say f vanishes to order 0 at x if $f(x) \in \mathbb{C}^\times$, and f vanishes to order $-k$ at x, $k > 0$, if f has a pole of order k at x. We make the usual identification of

a line bundle with the sheaf of its (algebraic) sections. This is a locally free $\mathcal{O}_X$-module of rank 1; in general, locally free $\mathcal{O}_X$-modules of finite rank correspond to algebraic vector bundles over X. Note that $\mathcal{L}_D^{-1} = \mathcal{L}_{-D}$, in the sense that $\mathcal{L}_D \otimes_X \mathcal{L}_{-D} = \mathcal{O}_X$, with $\otimes_X$ the tensor product for sheaves (computed pointwise over X).

The classical Riemann-Roch problem was to compute the dimension $\ell(D)$ of the space $\Gamma(X; \mathcal{L}_D)$ of global (algebraic) sections of $\mathcal{L}_D$, for any divisor D. We may think of this dimension as a Betti number for sheaf cohomology, namely, as $\dim H^0(X, \mathcal{L}_D)$. For instance, if $D = 0$, $\mathcal{L}_D = \mathcal{O}_X$ and $\ell(D) = 1$ (since any rational function on X without poles must be constant by compactness and the maximum principle for analytic functions). The **Riemann-Roch Theorem** (see for instance [Hartshorne, Ch. IV, §1]) asserts that

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g, \tag{3.1.17}$$

where $\mathcal{L}_K$ is the **canonical sheaf** (the sheaf of algebraic 1-forms $f(x)\,dx$) and the **degree** of a divisor is defined by $\deg \sum n_j x_j = \sum n_j$. The formula (3.1.17) is clear if $D = 0$, since $\ell(0) = 1$, $\deg 0 = 0$, and $\ell(K) = \dim H^0(X, \mathcal{L}_K)$ is the dimension of the space of algebraic 1-forms on X, while by the Hodge Theorem,

$$\begin{aligned} 2g &= \dim H^1(X; \mathbb{C}) \\ &= \dim\{\text{harmonic 1-forms on } X\} \\ &= \dim(\{\text{holomorphic 1-forms on } X\} \\ &\qquad \oplus \{\text{anti-holomorphic 1-forms on } X\}) \\ &= \dim H^0(X, \mathcal{L}_K) + \dim \overline{H^0(X, \mathcal{L}_K)} \\ &= 2 \dim H^0(X, \mathcal{L}_K), \end{aligned}$$

so that $\ell(K) = g$.

Let us now sketch a proof of (3.1.17) using $K_0(\mathbf{CohSh}\,X)$, the K-theory of the category of coherent sheaves on X. Since X is non-singular and of (complex) dimension 1, any coherent sheaf $\mathcal{F}$ over X has a resolution of length 1 by locally free sheaves:

$$0 \to \mathcal{V}_1 \to \mathcal{V}_0 \to \mathcal{F} \to 0. \tag{3.1.18}$$

Furthermore, $H^j(X, \mathcal{F})$ is finite-dimensional for $j = 0, 1$ and vanishes for $j > 1$. One may prove this by using the long exact sequence in sheaf cohomology associated to (3.1.18) to reduce to the case of a vector bundle $\mathcal{V}$. The finite-dimensionality of H^0 comes from compactness of X and Montel's Theorem (which says that the space of holomorphic sections of $\mathcal{V}$ over X must be locally compact, hence finite-dimensional). The **Serre duality theorem** says $\dim H^1(X, \mathcal{V}) = \dim H^0(X, \check{\mathcal{V}} \otimes_X \mathcal{L}_K)$, where $\check{\mathcal{V}}$ is the "dual" bundle to $\mathcal{V}$ (in the case of a line bundle this is just $\mathcal{V}^{-1}$). Hence we have finite-dimensionality of H^1 as well. The vanishing of the

cohomology past the (complex) dimension follows, for instance, from Dolbeault's Theorem, which identifies $H^j(X, \mathcal{V})$ with the j-th cohomology of the complex of antiholomorphic differential forms with values in $\mathcal{V}$.

Thus for $\mathcal{F}$ a coherent sheaf over X, the Euler characteristic

$$\chi(\mathcal{F}) = \sum_{j=0}^{\infty} H^j(X, \mathcal{F})$$

is well defined and given just by $\dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F})$. For a line bundle $\mathcal{L}$, Serre duality gives that

$$\chi(\mathcal{L}) = \dim H^0(X, \mathcal{L}) - \dim H^0(X, \mathcal{L}^{-1} \otimes_X \mathcal{L}_K).$$

In particular,

$$\ell(D) - \ell(K - D) = \dim H^0(X, \mathcal{L}_D) - \dim H^0(X, \mathcal{L}_K \otimes_X \mathcal{L}_D^{-1}) = \chi(\mathcal{L}_D),$$

so the Riemann-Roch Theorem amounts to the statement that

$$\chi(\mathcal{L}_D) - \chi(\mathcal{O}_X) = \deg D. \tag{3.1.19}$$

To prove this, note that by the "Euler-Poincaré Principle" (cf. Proposition 1.7.10), for any short exact sequence of coherent sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

we have additivity of the Euler characteristic: $\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$. (One may see this by taking the corresponding long exact sequence in sheaf cohomology and applying Lemma 3.1.10 in the category of finite-dimensional vector spaces over $\mathbb{C}$.) It follows that the map $\mathcal{F} \mapsto \chi(\mathcal{F})$ preserves relation 0-(ii) of Definition 3.1.6 and thus passes to a homomorphism $\chi : K_0(\mathbf{CohSh}\, X) \to \mathbb{Z}$. It will suffice for us to get a better understanding of this homomorphism. The trick (which was the key contribution of Grothendieck) is that even though we were initially only interested in $\chi(\mathcal{F})$ in the case of line bundles, it pays to study χ in the larger category $\mathbf{CohSh}\, X$ where we have more exact sequences and thus more non-trivial relations to help us.

Let x be a point of X and let D be any divisor. There is a natural monomorphism $\mathcal{L}_D \to \mathcal{L}_{D+x}$ coming from the fact that every section of $\mathcal{L}_D$ is also a section of $\mathcal{L}_{D+x}$. This map is an isomorphism away from x, so the quotient sheaf $\mathcal{S}_x$ is a coherent sheaf supported only at x. Furthermore, if n_x is the coefficient of x in D, then for U small enough, $\Gamma(U, \mathcal{L}_{D+x})/\Gamma(U, \mathcal{L}_D)$ is spanned by

$$z \mapsto \left(\frac{1}{z - x}\right)^{n_x+1}$$

in local coordinates, and hence $\dim \Gamma(U, \mathcal{S}_x) = 1$ if $x \in U$. From this one can see that $\dim H^0(X, \mathcal{S}_x) = 1$, $\dim H^1(X, \mathcal{S}_x) = 0$, so $\chi(\mathcal{S}_x) = 1$. Then from the exact sequence

$$0 \to \mathcal{L}_D \to \mathcal{L}_{D+x} \to \mathcal{S}_x \to 0,$$

we obtain $\chi(\mathcal{L}_{D+x}) = \chi(\mathcal{L}_D)+1$. Reversing the roles of D and $D+x$, we get $\chi(\mathcal{L}_{D-x}) = \chi(\mathcal{L}_D) - 1$. So if $D = \sum n_j x_j$, we get $\chi(\mathcal{L}_D) = \chi(\mathcal{L}_0) + \sum n_j$, which is (3.1.19).

3.1.20. Exercise. Let R be a PID, for instance $\mathbb{Z}$. Show that if M is a finitely generated **torsion** R-module, then $[M] = 0$ in $G_0(R) \cong K_0(R)$ ($\cong \mathbb{Z}$). Is this necessarily true if R is only a Dedekind domain?

3.1.21. Exercise. Show that the analogues of Theorem 1.2.4, Exercise 1.2.8, Exercise 2.1.6 and Exercise 2.1.7 hold for G_0 and G_1. In other words, show that G_j is Morita-invariant and that $G_j(R \times S) \cong G_j(R) \oplus G_j(S)$, for $j = 0, 1$.

3.1.22. Exercise. Consider the categories $\mathbf{Rep}_{\mathbb{Z}}$ of finite-dimensional complex representations of $\mathbb{Z}$, which may be identified with the category of pairs (V, T), where V is a finite-dimensional complex vector space and $T \in \operatorname{Aut} V$ is the image of the generator, and the category $\mathbf{Rep}_{\mathbb{N}}$ of finite-dimensional complex representations of the monoid $\mathbb{N}$, which may be identified with the category of pairs (V, T), where V is a finite-dimensional complex vector space and $T \in \operatorname{End} V$. Determine the simple objects in these categories and use Theorem 3.1.8 to compute K_0 and K_1 for each category.

3.1.23. Exercise [Milnor, §7]. Let X be a compact Hausdorff space, and recall that $K_j(\mathbf{Vect}_{\mathbb{F}} X) \cong K_j(R)$ with $R = C^{\mathbb{F}}(X)$ by Theorem 1.6.3, for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $j = 0, 1$. Show that there are exact sequences of abelian groups

$$0 \to C^{\mathbb{R}}(X) \xrightarrow{\exp} K_1(\mathbf{Vect}_{\mathbb{R}} X) \to KO^{-1}(X) \to 0,$$

$$0 \to C(X, \mathbb{Z}) \xrightarrow{2\pi i} C^{\mathbb{C}}(X) \xrightarrow{\exp} K_1(\mathbf{Vect}_{\mathbb{C}} X) \to KU^{-1}(X) \to 0.$$

Here is a sketch of how to proceed. Recall from Exercise 1.6.15 that

$$K_{\mathbb{F}}^{-1}(X) =_{\text{def}} K_{\mathbb{F}}^0(X \times \mathbb{R}).$$

Let $S = C^{\mathbb{F}}(X \times [0, 1])$ and let I be the closed ideal of functions vanishing on $X \times \{0, 1\}$. Then $I \cong C_0^{\mathbb{F}}(X \times (0, 1))$ (as a Banach algebra without unit) and from the short exact sequence

$$0 \to I \to S \to R \times R \to 0$$

we obtain an exact sequence

$$K_1(S) \to K_1(R) \oplus K_1(R) \to K_0(I) \to K_0(S) \to K_0(R) \oplus K_0(R).$$

By homotopy invariance of K_0 (Corollary 1.6.12), $K_0(S) \cong K_0(R)$, and the map on the right may be identified with the diagonal map $K_0(R) \to K_0(R) \oplus K_0(R)$, which is injective. Furthermore, there is a splitting map from the diagonal copy of R inside $R \times R$ to S (extend a function on X to a function on $X \times \{0, 1\}$ which doesn't depend on the second coordinate), so that the above exact sequence gives the exact sequence

$$K_1(S) \to K_1(\mathbf{Vect}_{\mathbb{F}}X) \to K_{\mathbb{F}}^0(X \times \mathbb{R}) \to 0,$$

where we think of $K_1(\mathbf{Vect}_{\mathbb{F}}X)$ as $K_1(R) \oplus 0 \hookrightarrow K_1(R) \oplus K_1(R)$.

Show that the image of $K_1(S) \to K_1(R) \oplus 0$ can be identified with the classes in $K_1(R)$ represented by matrices in $GL(R) \cong C(X, GL(\mathbb{F}))$ which are homotopic to elementary matrices. Then show that the part of $SK_1(R)$ coming from matrices homotopic to elementary matrices is trivial, and that the classes in $R^\times$ homotopic to the identity coincide with the image of the exponential map $C^{\mathbb{F}}(X) \xrightarrow{\exp} C(X, GL(1, \mathbb{F}))$. (Use the idea of Lemma 1.6.6 to show that an element of $C(X, SL(n, \mathbb{F}))$ (resp., $C(X, GL(1, \mathbb{F}))$) which is close to the identity is an exponential of something in $C(X, SL(n, \mathbb{F}))$ (resp., $C(X, \mathbb{F})$).) Finally, identify the kernel of the exponential map $C^{\mathbb{F}}(X) \xrightarrow{\exp} C(X, GL(1, \mathbb{F}))$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

3.1.24. Exercise. Let p be a prime number and consider the local ring $R = \mathbb{Z}/(p^2)$ with unique maximal ideal $I = (p)$.

(1) Show that R is not left regular, by showing that R/I has a resolution by finitely generated free R-modules for which condition (d) of Lemma 3.1.15 is not satisfied for any n.

(2) Note that R-$\mathbf{Mod}_{\mathbf{fg}}$ is a category in which R/I is the unique simple object (up to isomorphism), and in which every object has finite length. Then use Theorem 3.1.8 to compute $G_0(R)$ and $G_1(R)$. Is the natural map $K_j(R) \to G_j(R)$ an isomorphism for $j = 0$? For $j = 1$?

3.1.25. Exercise (A step toward Grothendieck's generalized Riemann-Roch Theorem). Let X be a non-singular projective algebraic variety over $\mathbb{C}$, now of dimension $n > 1$. In this more general setting, a divisor D on X is defined to be a formal finite $\mathbb{Z}$-linear combination $\sum n_j X_j$ of subvarieties $X_j \subset X$ of codimension 1, with $n_j \in \mathbb{Z}$. The divisors D are again in bijection with isomorphism classes of algebraic line bundles over X via the map $\sum n_j X_j = D \mapsto \mathcal{L}_D$, where $\mathcal{L}_D$ is the line bundle whose (algebraic) sections over an open set U are the rational functions f over U vanishing to order at least $-n_j$ along X_j (and thus regular along subvarieties Y of codimension 1 for which $n_Y \leq 0$). The **generalized Riemann-Roch problem**, solved by Grothendieck, is to give a formula relating $\chi(\mathcal{L}_D)$ to $\chi(\mathcal{O}_X)$, analogous to formula (3.1.19).

(1) Assuming that coherent sheaves over X have finite resolutions by locally free sheaves and thus that the natural map $K_0(\mathbf{Vect}\,X) \to K_0(\mathbf{CohSh}\,X)$ is an isomorphism, and assuming the result of Serre that for $\mathcal{F}$ a coherent sheaf over X, $H^j(X, \mathcal{F})$ is finite-dimensional for $j \leq n$ and vanishes for $j > n$, show as in the one-dimensional case above that the map $\mathcal{F} \mapsto \chi(\mathcal{F})$ preserves relation 0-(ii) of Definition 3.1.6 and thus passes to a homomorphism $\chi : K_0(\mathbf{Vect}\,X) \cong K_0(\mathbf{CohSh}\,X) \to \mathbb{Z}$.

(2) Let Y be an irreducible subvariety of X of codimension 1, taken for simplicity to be non-singular. Show as in the one-dimensional case above that there is a short exact sequence of coherent sheaves

$$0 \to \mathcal{L}_{D-Y} \to \mathcal{L}_D \to \mathcal{S}_{D,Y} \to 0,$$

where the quotient sheaf $\mathcal{S}_{D,Y}$ is a coherent sheaf supported along Y. Note in fact that $\mathcal{S}_{D,Y} \cong \mathcal{L}_D \otimes_X \mathcal{O}_Y$, where we think of the structure sheaf of Y as being extended to a sheaf on X supported along Y. Deduce that $\chi_X(\mathcal{L}_{-Y}) - \chi_X(\mathcal{O}_X) = -\chi_Y(\mathcal{O}_Y)$, or in general that $\chi_X(\mathcal{L}_{D-Y}) - \chi_X(\mathcal{L}_D) = -\chi_Y(\iota^*\mathcal{L}_D)$, where $\iota : Y \hookrightarrow X$ is the inclusion and $\iota^*\mathcal{L}_D$ is the pull-back of $\mathcal{L}_D$ to a line bundle on Y. This suggests a mechanism for proving a generalized Riemann-Roch formula by induction on n.

3.1.26. Exercise (Relative K-groups for categories). Let $\mathcal{A}$ and $\mathcal{B}$ be categories with exact sequences, and let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor, so that it defines homomorphisms $F_* : K_j(\mathcal{A}) \to K_j(\mathcal{B})$, for $j = 0, 1$. Define a relative group $K_0(F)$ to be the free abelian group with generators

$[A_0, A_1, \alpha]$ corresponding to pairs $(A_0, A_1) \in \operatorname{Obj}\mathcal{A}\times\operatorname{Obj}\mathcal{A}$ together with a morphism $\alpha : A_0 \to A_1$ in $\mathcal{A}$ for which $F(\alpha) : F(A_0) \to F(A_1)$ is an isomorphism in $\mathcal{B}$, modulo the relations that

$[A_0, A_1, \alpha] = 0$ if α is an isomorphism in $\mathcal{A}$,

and that if there is a commuting diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_0'' & \longrightarrow & A_0 & \longrightarrow & A_0' & \longrightarrow & 0 \\ \| & & {\scriptstyle\alpha''}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha'}\downarrow & & \| \\ 0 & \longrightarrow & A_1'' & \longrightarrow & A_1 & \longrightarrow & A_1' & \longrightarrow & 0 \end{array}$$

and $F(\alpha)$, $F(\alpha')$, $F(\alpha'')$ are isomorphisms in $\mathcal{B}$, then

$$[A_0, A_1, \alpha] = [A_0', A_1', \alpha'] + [A_0'', A_1'', \alpha''].$$

Define a map $K_0(F) \xrightarrow{\phi} K_0(\mathcal{A})$ by

$$[A_0, A_1, \alpha]_{K_0(F)} \mapsto [A_0]_{K_0(\mathcal{A})} - [A_1]_{K_0(\mathcal{A})}$$

and show that $F_* \circ \phi = 0$.

Assume further that F is what Bass calls "cofinal," in other words that given $B_1 \in \operatorname{Obj}\mathcal{B}$, there is some $B_2 \in \operatorname{Obj}\mathcal{B}$ with $B_1 \oplus B_2 \cong F(A)$ for some $A \in \operatorname{Obj}\mathcal{A}$, and also that one can choose the B_2 so that $F_* : \operatorname{End} A \to \operatorname{End} F(A)$ is surjective (this condition is similar to the first condition in Theorem 3.1.14). Show that there is an exact sequence

$$K_1(\mathcal{A}) \xrightarrow{F_*} K_1(\mathcal{B}) \xrightarrow{\partial} K_0(F) \xrightarrow{\phi} K_0(\mathcal{A}) \xrightarrow{F_*} K_0(\mathcal{B})$$

by imitating arguments from Theorems 1.5.5 and 2.5.4. To define the map ∂, note that if $B_1 \in \operatorname{Obj}\mathcal{B}$ and $\beta_1 \in \operatorname{Aut}(B_1)$, then with B_2 and F as above, $[B_1, \beta] = [B_1 \oplus B_2, \beta_1 \oplus 1_{B_2}]$ in $K_1(\mathcal{B})$ can be replaced by $[F(A), \beta]$ with $A \in \operatorname{Obj}\mathcal{A}$ and $\beta \in \operatorname{Aut} F(A)$. Then if $F_* : \operatorname{End} A \to \operatorname{End} F(A)$ is surjective, β lifts to an endomorphism α of A, and we can define $\partial([F(A), \beta]) = [A, A, \alpha]$. One has to check that this is independent of the choice of A and α.

Check that when $\mathcal{A} = \mathbf{Proj}\, R$, I is an ideal in R, $\mathcal{B} = \mathbf{Proj}\, R/I$, and F is induced by the quotient map $R \twoheadrightarrow R/I$, then the hypotheses on F are satisfied and one recovers the exact sequence of Theorem 2.5.4.

2. The Grothendieck and Bass-Heller-Swan Theorems

In this section, we consider the problem of computing the K-theory of a ring of polynomials or Laurent polynomials over another ring whose K-theory is already known. In the case where $R = C^{\mathbb{F}}(X)$ is the ring of continuous $\mathbb{F}$-valued functions on a compact Hausdorff space X (with

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), the ring of polynomials $R[t]$ is, by the Stone-Weierstrass Theorem, a dense subring of $C^{\mathbb{F}}(X \times [0, 1])$ (via specialization of t to a real number in $[0, 1]$). The homotopy invariance theorem for topological K-theory (Corollary 1.6.11) says that the map $t \mapsto 0$ induces an isomorphism $K_0(C^{\mathbb{F}}(X \times [0, 1])) \to K_0(C^{\mathbb{F}}(X))$. Thus it is reasonable to view the map on K-theory

$$K_j(R[t]) \to K_j(R) \quad \text{induced by} \quad t \mapsto 0$$

as corresponding to "algebraic homotopy," and to expect this map to be an isomorphism for suitable rings R. This turns out to be the case for R left regular (Grothendieck's Theorem), though homotopy invariance fails in general.

The case of the Laurent polynomial ring $R[t, t^{-1}]$ is more complicated. When $R = C^{\mathbb{C}}(X)$, there is a map $R[t, t^{-1}] \hookrightarrow C^{\mathbb{C}}(X \times S^1)$ defined via specialization of t to a complex number of absolute value 1, and the image is dense by the Stone-Weierstrass Theorem. On the other hand, in topological K-theory, one has the formula

$$KU^{-j}(X \times S^1) \cong KU^{-j}(X) \oplus KU^{-j-1}(X).$$

In **complex** K-theory, Bott periodicity holds and K^{-j} only depends on j modulo 2. Thus if we specialize to $j = 1$, we have

$$KU^{-1}(X \times S^1) \cong KU^{-1}(X) \oplus KU^0(X) \cong KU^{-1}(X) \oplus K_0(R).$$

Since, by Exercise 3.1.23, $KU^{-1}(X \times S^1)$ and $KU^{-1}(X)$ are closely related to $K_1(C^{\mathbb{C}}(X \times S^1))$ and to $K_1(R)$, respectively, and since $R[t, t^{-1}]$ is dense in $C^{\mathbb{C}}(X \times S^1)$, this suggests that perhaps one can expect to have $K_1(R[t, t^{-1}]) \cong K_1(R) \oplus K_0(R)$ when R is a nice enough ring. In other words, taking Laurent polynomials should correspond to "algebraic desuspension." Again, this will turn out to be the case for R left regular (the Bass-Heller-Swan Theorem). We will also be able to study the extent to which this and algebraic homotopy invariance fail for rings which are **not** left regular. Finally, further study of these ideas will also lead to a definition of K-groups extending the exact sequence of Theorem 2.5.4 arbitrarily far to the right.

We begin with a review of two famous theorems of Hilbert, the Basis Theorem and the Syzygy Theorem, which together imply that if R is left regular, so are $R[t]$ and $R[t, t^{-1}]$. The reader who is already familiar with these classical theorems can skip to formulas 3.2.5 and 3.2.6 and to the discussion surrounding them.

3.2.1. Theorem ("Hilbert Basis Theorem"). *Let R be a left Noetherian ring. Then the polynomial ring $R[t]$ is left Noetherian.*

Proof. Let J be a left ideal of $R[t]$, and consider the sets I, I_j of all leading coefficients of the polynomials in J (respectively, of the polynomials in J of degree $\leq j$). Since J is closed under addition and left multiplication

by elements of R, so are I and the I_j, hence I and the I_j are left ideals of R. Since R is left Noetherian, we may choose a finite set of polynomials

$$\left.\begin{array}{c} f_1(t) = a_1^{r_1}t^{r_1} + \cdots + a_1^0 \\ \cdots \\ f_n(t) = a_n^{r_n}t^{r_n} + \cdots + a_n^0 \end{array}\right\} \in J$$

whose leading coefficients $a_1^{r_1}, \ldots, a_n^{r_n}$ are generators of I. Let $m = \max_j \deg f_j = \max_j r_j$. Then if $f(t) = b_k t^k + \cdots + b_0 \in J$, the leading coefficient b_k of f may be written in the form $\sum_{j=1}^n c_j a_j^{r_j}$, $c_j \in R$, and if $k \geq m$, then $f(t) - \sum_{j=1}^n c_j t^{k-r_j} f_j(t)$ lies in J and has smaller degree than f. A simple induction thus shows that J is generated as a left ideal of $R[t]$ by $f_1(t), \ldots, f_n(t)$ and by the polynomials in J of degree $< m$. If we similarly choose successively, for $j = 0, \ldots, m-1$, finitely many polynomials $g_1^j, \ldots g_{s_j}^j \in J$ of degree $\leq j$ whose leading coefficients generate I_j, then it is evident that the $f_1(t), \ldots, f_n(t)$ together with the $g_l^j(t)$ generate J as a left ideal of $R[t]$. □

3.2.2. Corollary. *If R is a left Noetherian ring, then so is the Laurent polynomial ring $R[t, t^{-1}]$.*

Proof. $R[t, t^{-1}]$ is a localization of $R[t]$, and a localization of a left Noetherian ring is left Noetherian. For a more explicit proof, let J be a left ideal of $R[t, t^{-1}]$, and let $J_0 = J \cap R[t]$, which is a left ideal of $R[t]$. (Here we think of $R[t]$ as a subring of $R[t, t^{-1}]$.) Using Theorem 3.2.1, choose finitely many generators for J_0. Then these also generate J, since for $f(t) \in J$, $f(t) = t^{-n}t^n f(t)$, and $t^n f(t) \in J_0$ for $n \geq 0$ sufficiently large. □

3.2.3. Theorem ("Hilbert Syzygy Theorem"). *If R is a left regular ring, then so is $R[t]$. Furthermore, if R has (left) global dimension $\leq n$, then $R[t]$ has (left) global dimension $\leq n+1$.*

Proof. By the Basis Theorem, $R[t]$ is left Noetherian. Let M be a finitely generated left $R[t]$-module. By Lemma 3.1.15, to show that M has a resolution of finite type by projective $R[t]$-modules, it will be enough to show that there exists a positive integer K such that $\operatorname{Ext}^K_{R[t]}(M, N) = 0$ for all $R[t]$-modules N, and to prove the final statement about global dimension, we only need to show that if R has (left) global dimension $\leq n$, then K can be taken to be $n+2$.

By restriction, any $R[t]$-module can be considered to be an R-module, which comes naturally with an R-module endomorphism φ defined by left multiplication by t. So we can form the short exact sequence

$$0 \to R[t] \otimes_R M \xrightarrow{t \otimes 1_M - 1 \otimes \varphi} R[t] \otimes_R M \to M \to 0.$$

Suppose that for some k, $\operatorname{Ext}^k_R(M, N) = 0$ for all R-modules N. In fact, if R has (left) global dimension $\leq n$, k can be taken to be $n+1$. We have an exact sequence

$$\operatorname{Ext}^k_{R[t]}(R[t] \otimes_R M, N) \to \operatorname{Ext}^{k+1}_{R[t]}(M, N) \to \operatorname{Ext}^{k+1}_{R[t]}(R[t] \otimes_R M, N).$$

Since $R[t]$ is free as an R-module, $\operatorname{Ext}^j_{R[t]}(R[t]\otimes_R M,\, N) \cong \operatorname{Ext}^j_R(M,\, N) = 0$ for $j \geq k$, so $\operatorname{Ext}^K_{R[t]}(M,\, N) = 0$ for $K = k+1$, as desired.

To complete the proof, we only need to get around one technical point: M is assumed to be finitely generated as an $R[t]$-module, but may not be finitely generated as an R-module, so that in the case where R is not assumed to have finite global dimension, the definition of left regular ring doesn't immediately tell us that M is of finite homological dimension as an R-module. The following trick for getting around this may be found in [Bass, pp. 634–635], though part of the idea is older. Let M_0 be a finitely generated R-submodule of M which generates M as an $R[t]$-module, and let $M_n = t^n M_0 + \cdots + M_0$. Then M_n is an increasing sequence of finitely generated R-submodules of M and $M = \varinjlim M_n$. Let

$$Q_n = \{x \in M_0 : t^n x \in M_{n-1}\}$$

and observe that this is an increasing sequence of R-submodules of M_0. Since M_0 is finitely generated and R is left Noetherian, there is some n_0 such that $Q_n = Q_{n_0}$ for all $n \geq n_0$. We claim that for $n \geq n_0$, the homological dimension of M_n is $\leq d$, where d is the larger of the homological dimensions of M_{n_0} and M_{n_0}/M_{n_0-1}. Indeed, this is true for $n = n_0$, and if $n > n_0$ and it's true for $n-1$, we can apply the exact sequence

$$0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0.$$

By choice of n_0 and the assumption that $n > n_0$, the map $M_{n_0}/M_{n_0-1} \xrightarrow{t^{n-n_0}} M_n/M_{n-1}$ has a trivial kernel and thus is an isomorphism. So for any R-module N, we have an exact sequence

$$\operatorname{Ext}^j_R(M_{n_0}/M_{n_0-1},\, N) \to \operatorname{Ext}^j_R(M_n,\, N) \to \operatorname{Ext}^j_R(M_{n-1},\, N),$$

and since $\operatorname{Ext}^d_R(M_{n_0}/M_{n_0-1},\, N) = 0$ and $\operatorname{Ext}^d_R(M_{n-1},\, N) = 0$, we get that $\operatorname{Ext}^d_R(M_n,\, N) = 0$. This proves the claim by induction.

To complete the proof, one needs to see that the homological dimension of M is bounded by

$$\limsup \text{hom. dim.}\, M_n \leq d.$$

This follows from the "$\varprojlim^1$-sequence"

$$0 \to \varprojlim{}^1 \operatorname{Ext}^{d-1}_R(M_n,\, N) \to \operatorname{Ext}^d_R(\varinjlim M_n,\, N) \to \varprojlim \operatorname{Ext}^d_R(M_n,\, N) \to 0$$

which comes from the long exact sequence in Ext associated to the short exact sequence

$$0 \to \bigoplus_n M_n \xrightarrow{(x_n)\mapsto(x_0,x_1-x_0,\dots,x_n-x_{n-1},\dots)} \bigoplus_n M_n \xrightarrow{(x_n)\mapsto\Sigma_n x_n} \varinjlim M_n \to 0. \quad \square$$

3.2.4. Corollary. *If R is a left regular ring, then so is $R[t,\, t^{-1}]$. Furthermore, if R has (left) global dimension $\le n$, then $R[t,\, t^{-1}]$ has (left) global dimension $\le n+1$.*

Proof. Again this follows from the fact that $R[t,\, t^{-1}]$ is a localization of $R[t]$. More explicitly, if M is a finitely generated $R[t,\, t^{-1}]$-module, choose generators $x_1, \dots, x_n$ for M and let M_1 be the finitely generated $R[t]$-module they generate. Then $M = R[t,\, t^{-1}] \otimes_{R[t]} M_1$, and since $R[t,\, t^{-1}]$ is flat over $R[t]$,

$$\mathrm{Ext}^j_{R[t,\, t^{-1}]}(M,\, N) = \mathrm{Ext}^j_{R[t,\, t^{-1}]}(R[t,\, t^{-1}] \otimes_{R[t]} M_1,\, N) \cong \mathrm{Ext}^j_{R[t]}(M_1,\, N).$$

Hence the homological dimension of M over $R[t,\, t^{-1}]$ is the same as that of M_1 over $R[t]$. □

Now we're ready to proceed with the study of the K-theory of $R[t]$ and $R[t,\, t^{-1}]$. Note first of all that there are split short exact sequences

$$0 \to tR[t] \to R[t] \leftrightarrows R \to 0, \tag{3.2.5}$$

$$0 \to (t-1)R[t,\, t^{-1}] \to R[t,\, t^{-1}] \leftrightarrows R \to 0, \tag{3.2.6}$$

so that the K-theory of $R[t]$ or of $R[t,\, t^{-1}]$ contains that of R as a direct summand. The basic problem is to study the other summands, if any. It turns out that in this context, G-theory behaves better than K-theory, at least for rings which are left Noetherian. Hence it is worth saying something about the functoriality of G-theory under change of rings. In general, if $\varphi : R \to S$ is a ring homomorphism, though it induces an exact functor from $\mathbf{Proj}\, R \to \mathbf{Proj}\, S$, φ_* is usually **not** exact as a functor from R-$\mathbf{Mod_{fg}}$ to S-$\mathbf{Mod_{fg}}$, hence does not induce a homomorphism $G_j(R) \to G_j(S)$. However, if S is **flat** over R (which is another way of saying φ_* is an exact functor), in particular if S is projective as an R-module (via φ), then $\varphi_* : G_j(R) \to G_j(S)$ is defined. This will be the case, for instance, when φ is the obvious injection of R into $S = R[t]$ or $R[t,\, t^{-1}]$.

We would like, however, to have maps $G_j(R[t]) \to G_j(R)$ and

$$G_j(R[t,\, t^{-1}]) \to G_j(R)$$

in spite of the fact that the obvious maps $R[t] \xrightarrow{t\mapsto 0} R$ and $R[t,\, t^{-1}] \xrightarrow{t\mapsto 1} R$ are **not** flat. The device for constructing such maps, due to Grothendieck, is based on the ideas that went into the proof of the Resolution Theorem (Theorem 3.1.13). We use the fact that R has finite homological dimension over $R[t]$ or $R[t,\, t^{-1}]$. In fact, from the resolutions

$$\begin{cases} 0 \to R[t] \xrightarrow{t} R[t] \xrightarrow{t\mapsto 0} R \to 0, \\ 0 \to R[t,\, t^{-1}] \xrightarrow{(t-1)} R[t,\, t^{-1}] \xrightarrow{t\mapsto 1} R \to 0, \end{cases} \tag{3.2.7}$$

we see that R has homological dimension 1 over $R[t]$ and $R[t, t^{-1}]$, and thus that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of $R[t]$-modules, there is a corresponding exact sequence of R-modules

$$\begin{aligned}(3.2.8)\quad 0 \to \operatorname{Tor}_1^{R[t]}(R, M_1) \to \operatorname{Tor}_1^{R[t]}(R, M_2) \to \operatorname{Tor}_1^{R[t]}(R, M_3)\\ \to R \otimes_{R[t]} M_1 \to R \otimes_{R[t]} M_2 \to R \otimes_{R[t]} M_3 \to 0,\end{aligned}$$

and similarly with $R[t, t^{-1}]$ in place of $R[t]$.

3.2.9. Proposition. *Let R be a left Noetherian ring. There are well-defined homomorphisms $G_0(R[t]) \to G_0(R)$ and $G_0(R[t, t^{-1}]) \to G_0(R)$ defined by*

$$[M] \mapsto [R \otimes_{R[t]} M] - [\operatorname{Tor}_1^{R[t]}(R, M)]$$

(or the same formula with $R[t]$ replaced by $R[t, t^{-1}]$). When R is left regular, these agree with the usual homomorphisms $K_0(R[t]) \to K_0(R)$ and $K_0(R[t, t^{-1}]) \to K_0(R)$.

Similarly, there are well-defined homomorphisms $G_1(R[t]) \to G_1(R)$ and $G_1(R[t, t^{-1}]) \to G_1(R)$ defined by

$$[M, \alpha] \mapsto [R \otimes_{R[t]} M, 1 \otimes \alpha] - [\operatorname{Tor}_1^{R[t]}(R, M), \operatorname{Tor}(1, \alpha)]$$

(or the same formula with $R[t]$ replaced by $R[t, t^{-1}]$) which agree with the usual homomorphisms $K_1(R[t]) \to K_1(R)$ and $K_1(R[t, t^{-1}]) \to K_1(R)$ when R is left regular.

Proof. First consider the case of G_0 and $R[t]$. The indicated formula gives a well-defined homomorphism for two reasons:

(i) If M is finitely generated as an $R[t]$-module, then $R \otimes_{R[t]} M$ is finitely generated, and also $\operatorname{Tor}_1^{R[t]}(R, M)$ is finitely generated since it may be computed from (3.2.7) to be the kernel of multiplication by t on M. This is a submodule of M, so it is finitely generated if M is, since we are assuming R is left Noetherian, hence $R[t]$ is also left Noetherian by the Hilbert Basis Theorem (Theorem 3.2.1).

(ii) We need to show that the relations in G_0 are preserved by the map. But this follows immediately from the exact sequence (3.2.8) together with Lemma 3.1.10. (The hypothesis on the category R-$\mathbf{Mod_{fg}}$ needed for the Lemma follows from the assumption that M is left Noetherian.)

If R is left regular, then so is $R[t]$ by the Syzygy Theorem (Theorem 3.2.3). Hence by Corollary 3.1.16, the natural maps $K_0(R) \to G_0(R)$ and $K_0(R[t]) \to G_0(R[t])$ are isomorphisms. The diagram

$$\begin{array}{ccc} K_0(R[t]) & \xrightarrow{\cong} & G_0(R[t]) \\ \downarrow & & \downarrow \\ K_0(R) & \xrightarrow{\cong} & G_0(R) \end{array}$$

commutes since if M is a finitely generated projective module over $R[t]$, then $\operatorname{Tor}_1^{R[t]}(R, M) = 0$, hence $[M] \mapsto [R \otimes_{R[t]} M]$ under both of the vertical maps in the diagram.

Exactly the same reasoning works with $R[t]$ replaced by $R[t, t^{-1}]$, except that now Tor_1 is computed to be the kernel of multiplication by $t-1$. The proof for G_1 is also almost exactly the same. □

3.2.10. Corollary. *Let R be a left Noetherian ring. Then for $j = 0, 1$, $G_j(R)$ sits naturally as a direct summand in $G_j(R[t])$ and in $G_j(R[t, t^{-1}])$.*

Proof. If M is a finitely generated R-module, then

$$\operatorname{Tor}_*^{R[t]}(R, R[t] \otimes_R M) \text{ and } \operatorname{Tor}_*^{R[t, t^{-1}]}(R, R[t, t^{-1}] \otimes_R M)$$

are by (3.2.7) computed as the homology of the complexes

$$R[t] \otimes_R M \xrightarrow{t} R[t] \otimes_R M, \qquad R[t, t^{-1}] \otimes_R M \xrightarrow{t-1} R[t, t^{-1}] \otimes_R M.$$

So Tor_1 can be seen to vanish and Tor_0 gives back M. Hence the composites

$$G_j(R) \to G_j(R[t]) \xrightarrow{\text{map of Proposition 3.2.9}} G_j(R),$$

$$G_j(R) \to G_j(R[t, t^{-1}]) \xrightarrow{\text{map of Proposition 3.2.9}} G_j(R)$$

are the identity. □

Now we're almost ready for the first major result of this section, which is Grothendieck's Theorem comparing G_0 for a ring R and for the ring of polynomials $R[t]$ or $R[t, t^{-1}]$. It is convenient to begin by first proving the version of the theorem for **graded** modules. Then we will use a trick to go back to the ungraded case.

3.2.11. Theorem. *Let R be a left Noetherian ring, viewed as a graded ring with trivial grading concentrating everything in degree 0. Give the polynomial ring $R[t_1, \ldots, t_r]$ its usual grading in which the elements $t_1, \ldots, t_r$ have degree 1. For a (non-negatively) graded left Noetherian ring, let G_0^{graded} denote K_0 of the category of finitely generated graded modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$. Note that because of the finite generation hypothesis and the fact that the ring is non-negatively graded, these modules are automatically bounded below (i.e., given the module M, there is some $n_0 \in \mathbb{Z}$ such that $M_n = 0$ for $n < n_0$). Morphisms in this category are required to preserve the grading. Then the exact functor $M \mapsto R[t_1, \ldots, t_r] \otimes_R M$ induces an isomorphism $G_0(R) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \cong G_0^{\text{graded}}(R) \to G_0^{\text{graded}}(R[t_1, \ldots, t_r])$.*

Proof. First of all, it is obvious that if R is trivially graded, then

$$G_0(R) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \cong G_0^{\text{graded}}(R),$$

since a finitely generated graded R-module is just a finite direct sum of finitely generated graded modules M_n, each concentrated in a single degree, and we identify $[M_n] \in G_0^{\text{graded}}(R)$ with $[M_n] \otimes t^n$ in $G_0(R) \otimes_{\mathbb{Z}} \mathbb{Z}[t]$.

Next, observe that everything we have just done works with graded modules as well. In other words, there is a map $G_0^{\text{graded}}(R[t]) \to G_0^{\text{graded}}(R)$ defined by sending $[M]$ to $[R \otimes_{R[t]} M] - [\text{Tor}_1^{R[t]}(R, M)[-1]]$, where for N a graded module, the symbol $N[r]$ denotes N shifted in degree by r:

$$N[r]_n = N_{n+r}.$$

The degree shift comes from the fact that in the category of **graded** modules, the resolution (3.2.7) is not as it stands a resolution by graded modules, since multiplication by t increases degree by 1, but we can replace it by

$$0 \to R[t][-1] \to R[t] \to R \to 0.$$

Corollary 3.2.10 holds in the graded context, and tells us that the map $G_0^{\text{graded}}(R[t]) \to G_0^{\text{graded}}(R)$ is a split surjection, with right inverse the map $[M] \mapsto [R[t] \otimes_R M]$. Iterating all of this r times, we see that $R[t_1, \ldots, t_r]$ has finite homological dimension over R, and that there is a split surjection

$$G_0^{\text{graded}}(R[t_1, \ldots, t_r]) \to G_0^{\text{graded}}(R)$$

defined using higher Tor's.

For simplicity of notation, let $S = R[t_1, \ldots, t_r]$. So it suffices to show that the map $G_0^{\text{graded}}(R) \to G_0^{\text{graded}}(S)$ is surjective. Let $\mathcal{F}$ be the full subcategory of finitely generated "R-flat" graded S-modules, whose objects are graded modules M satisfying $\text{Tor}_j^S(R, M) = 0$ for $j > 0$. (For instance, when $r = 1$, these are modules which are t_1-torsion-free.) If M lies in this subcategory, the map $G_0^{\text{graded}}(S) \to G_0^{\text{graded}}(R)$ takes the simpler form $[M] \mapsto [R \otimes_S M]$. Because of the long exact Tor sequence, $\mathcal{F}$ is a category with exact sequences and contains the kernel of each of its surjective morphisms. So the hypotheses of the Resolution Theorem (Theorem 3.1.13) are satisfied, and the inclusion of $\mathcal{F}$ induces an isomorphism $K_0(\mathcal{F}) \xrightarrow{\cong} G_0^{\text{graded}}(S)$. Now if M is a graded R-module, $S \otimes_R M$ is R-flat, so the map $G_0^{\text{graded}}(R) \to G_0^{\text{graded}}(S)$ naturally factors through $K_0(\mathcal{F})$, and it's enough to show that the map $G_0^{\text{graded}}(R) \to K_0(\mathcal{F})$ is surjective.

Let M be an object of $\mathcal{F}$, thus a finitely generated graded S-module, and recall that S is left Noetherian by Theorem 3.2.1. For each integer i, let

$$F_i(M) = \text{the } S\text{-submodule of } M \text{ generated by } M_j,\ j \le i,$$

$$Q_i = M_i / M_i \cap F_{i-1}(M).$$

Note that Q_i is just the component of $R \otimes_S M$ in degree i, since (using multi-index notation)

$$M_i \cap F_{i-1}(M) = \sum_{j=1}^{\infty} \sum_{|I|=j} t^I M_{i-j}.$$

(The sum is really finite, since M is bounded below.) Similarly, $R \otimes_S F_i(M)$ vanishes in degrees $> i$ and coincides with $R \otimes_S M$ in degrees $\le i$.

If $M_i = 0$ for $i < n_0$, then

$$0 = F_{n_0-1}(M) \subseteq F_{n_0}(M) \subseteq \cdots \subseteq F_{+\infty}(M) = M,$$

and the filtration must terminate at some finite stage, *i.e.*, $F_{n_1}(M) = M$ for some n_1, since M is finitely generated and Noetherian. Note that there is a map of graded S-modules from $S \otimes_R M_i$ to $F_i(M)$, which induces by passage to the quotient a **surjective** map of graded S-modules

$$S \otimes_R Q_i \to F_i(M)/F_{i-1}(M).$$

Here we are viewing M_i and Q_i as graded modules concentrated in the single degree i. We will show this map is an isomorphism for each i.

For $i < n_0$ or $i > n_1$, this is obvious since both sides are zero. Suppose we know that $\operatorname{Tor}_1^S(R, F_i(M)) = 0$, which is at least the case for $i = n_1$ since $M \in \operatorname{Obj} \mathcal{F}$. From the short exact sequence of graded modules

$$0 \to F_{i-1}(M) \to F_i(M) \to F_i(M)/F_{i-1}(M) \to 0$$

and the fact that the natural map

$$R \otimes_S F_{i-1}(M) \to R \otimes_S F_i(M)$$

is injective with cokernel Q_i, we see first that $\operatorname{Tor}_1^S(R, F_i(M)) = 0$ implies also $\operatorname{Tor}_1^S(R, F_i(M)/F_{i-1}(M)) = 0$ and

$$\operatorname{Tor}_1^S(R, F_{i-1}(M)) \cong \operatorname{Tor}_2^S(R, F_i(M)/F_{i-1}(M)).$$

Then if K_i denotes the kernel of

$$S \otimes_R Q_i \to F_i(M)/F_{i-1}(M),$$

tensoring with R gives the exact sequence

$$0 = \operatorname{Tor}_1^S(R, F_i(M)/F_{i-1}(M)) \to R \otimes_S K_i$$

$$\begin{array}{ccc} \longrightarrow R \otimes_S (S \otimes_R Q_i) & \longrightarrow & R \otimes_S (F_i(M)/F_{i-1}(M)) \\ \| & & \| \\ Q_i & = & Q_i. \end{array}$$

This shows $R \otimes_S K_i = 0$, which since K_i is a finitely generated graded module forces $K_i = 0$. Hence $F_i(M)/F_{i-1}(M) \cong S \otimes_R Q_i$, which shows that $F_i(M)/F_{i-1}(M) \in \operatorname{Obj} \mathcal{F}$. Substituting back the fact that Tor_2 vanishes, we get $\operatorname{Tor}_1^S(R, F_{i-1}(M)) = 0$. So by descending induction on i, $K_i = 0$ and $\operatorname{Tor}_1^S(R, F_i(M)) = 0$ for all i, as desired.

Now to conclude the argument, note that $[M] = \sum_i [F_i(M)/F_{i-1}(M)]$ in $G_0^{\text{graded}}(S)$. But we have seen that $F_i(M)/F_{i-1}(M) \cong R[t] \otimes_R Q_i$, hence $[F_i(M)/F_{i-1}(M)]$ lies in the image of $G_0^{\text{graded}} R$ for each i, as required. □

As we have noted, the next theorem is really due to Grothendieck, though this version of it first appeared in [BassHellerSwan].

3.2.12. Theorem (Grothendieck). *Let R be a left Noetherian ring. Then the natural maps $G_0(R) \to G_0(R[t])$ and $G_0(R) \to G_0(R[t, t^{-1}])$ are isomorphisms, with inverses given by the maps of Proposition 3.2.9.*

Proof. We begin with the case of $R[t]$; the case of $R[t, t^{-1}]$ will follow. We need to show the map $G_0(R) \to G_0(R[t])$ is surjective. The trick is to observe that if M is a finitely generated $R[t]$-module, then $M = R[t] \otimes_\psi N$, where N is a finitely generated **graded** $R[t, s]$-module (we give $R[t, s]$ the grading by the total degree of a polynomial) and where $\psi : R[t, s] \to R[t]$ is the surjective homomorphism sending $t \mapsto t$, $s \mapsto 1$. To see this, note that $M = R[t]^n/Q$ for some module of relations $Q \subseteq R[t]^n$, and since M is finitely generated and R is left Noetherian, Q is finitely generated because of the Hilbert Basis Theorem (Theorem 3.2.1). Choose a finite set

$$f_j = (f_{j,1}(t), \dots, f_{j,n}(t)), \quad 1 \le j \le m,$$

of generators of Q and let $d = \max \deg f_{j,k}$. Define

$$g_j = (g_{j,1}(t, s), \dots, g_{j,n}(t, s)) \in R[t, s]^n, \quad 1 \le j \le m,$$

by replacing each monomial at^l in the $f_{j,k}$'s by $at^l s^{d-l}$. Then each $g_{j,k}$ is homogeneous of degree d, and $g_{j,k} \mapsto f_{j,k}$ under the map $\psi : R[t, s] \to R[t]$. Hence if Q' is the submodule of $R[t, s]^n$ generated by the $g_{j,k}$'s, $N = R[t, s]^n/Q'$ is a finitely generated graded $R[t, s]$-module and $\psi_*(N) = M$.

Observe in addition that the functor ψ_* from graded $R[t, s]$-modules to $R[t]$-modules is exact. Indeed, the tensor product functor is always right exact. On the other hand, $\psi_*(N)$ may also be written as $N/(s-1)N$, and we have left exactness because if N is a graded $R[t, s]$-module and N' is a graded submodule, and if $x = \sum_{j=n_0}^{\infty} x_j \in N$, $(s-1)x \in N'$, then

$$-x_{n_0} + \sum_{j=n_0+1}^{\infty} (s x_{j-1} - x_j) \in N',$$

so that $x_{n_0} \in N'$, $x_{n_0+1} \in N', \dots$, and $x \in N'$. Thus ψ_* induces by Proposition 3.1.9 a homomorphism $G_0^{\text{graded}}(R[t, s]) \to G_0(R[t])$.

Now consider the diagram

$$\begin{array}{ccc} G_0^{\text{graded}}(R) & \longrightarrow & G_0^{\text{graded}}(R[t,\, s]) \\ {\scriptstyle \text{forget grading}}\downarrow & & \downarrow{\scriptstyle \psi_*} \\ G_0(R) & \longrightarrow & G_0(R[t]), \end{array}$$

where the horizontal arrows are induced by the inclusions of R into $R[t]$ and $R[t,\, s]$. This diagram obviously commutes. By Theorem 3.2.11, the top horizontal arrow is an isomorphism. From the fact that every finitely generated $R[t]$-module is $\psi_*(N)$ for some finitely generated graded $R[t,\, s]$-module, the right-hand vertical arrow is surjective. Thus the bottom horizontal arrow is surjective and we are done.

Now consider the case of $G_0(R) \to G_0(R[t,\, t^{-1}])$. We must show that this map is also surjective. Since $R[t,\, t^{-1}]$ is flat over $R[t]$ (as an $R[t]$-module, $R[t,\, t^{-1}] = \varinjlim_{n\to\infty} t^{-n}R[t]$, and $t^{-n}R[t]$ is free over $R[t]$), the inclusion $R[t] \hookrightarrow R[t,\, t^{-1}]$ induces a homomorphism

$$G_0(R[t]) \to G_0(R[t,\, t^{-1}])$$

by $[M] \mapsto [R[t,\, t^{-1}] \otimes_{R[t]} M]$. The map $G_0(R) \to G_0(R[t,\, t^{-1}])$ obviously factors through this map. Since we have seen that $G_0(R) \to G_0(R[t])$ is an isomorphism, we only need to show that $G_0(R[t]) \to G_0(R[t,\, t^{-1}])$ is surjective. But if M is a finitely generated $R[t,\, t^{-1}]$-module, $M = R[t,\, t^{-1}]^n/S$ for some finitely generated module of relations S. (We are using Corollary 3.2.2.) Multiplication by t^k induces an automorphism of $R[t,\, t^{-1}]^n$, and for large enough k, it will kill off all negative powers of t in a finite set of generators for S. Thus for large enough k, $t^kS \subseteq R[t]^n$, and $M \cong t^kR[t,\, t^{-1}]^n/t^kS$ is extended from a finitely generated $R[t]$-module. This shows $G_0(R[t]) \to G_0(R[t,\, t^{-1}])$ is surjective, completing the proof. □

3.2.13. Corollary. *Let R be a left regular ring. Then the natural map $K_0(R[t]) \to K_0(R)$ induced by (3.2.5) and the natural map $K_0(R[t,\, t^{-1}]) \to K_0(R)$ induced by (3.2.6) are isomorphisms. Alternatively, $K_0(tR[t])$ and $K_0((t-1)R[t,\, t^{-1}])$ (computed in the sense of K_0 for rings without unit) vanish.*

Proof. This follows from combining Theorem 3.2.12, Corollary 3.1.16, and Proposition 3.2.9. □

Remark. For rings which are **not** left regular, the maps $K_0(R[t]) \to K_0(R)$ and $K_0(R[t,\, t^{-1}]) \to K_0(R)$ can have a non-zero kernel. For an example of the former phenomenon, see Exercise 3.2.24. The kernel of the map $K_0(R[t,\, t^{-1}]) \to K_0(R)$ actually consists of two different parts, both of which can be non-zero. The first is related to the kernel of $K_0(R[t]) \to K_0(R)$; the second is the functor $K_{-1}(R)$ which will be studied in the next section.

3.2.14. Definition. If R is any ring with unit, we define $NK_j(R)$, $j = 0$ or 1, to be the relative K-group $K_j(R[t], tR[t])$. By the split short exact sequence (3.2.5), this is the same as the kernel of the map on K-theory induced by $R[t] \xrightarrow{t \mapsto 0} R$. (Recall Exercises 1.5.11 and 2.5.19.) Corollary 3.2.13 states that $NK_0(R)$ vanishes if R is left regular.

Next we come to the study of K_1 and G_1. We would like to show as in the case of K_0 that $G_1(R[t]) \cong G_1(R)$ for R left Noetherian, so that $NK_1(R)$ vanishes if R is left regular. The case of Laurent polynomials will now be a bit different, since as we remarked at the beginning of this section, there is reason to believe $K_1(R[t, t^{-1}])$ should be related to $K_1(R) \oplus K_0(R)$, not just to $K_1(R)$.

First we have the analogue of Theorems 3.2.11 and 3.2.12 for G_1.

3.2.15. Theorem. *Let R be a left Noetherian ring, viewed as a graded ring with trivial grading, and let $R[t_1, \dots, t_r]$ be given its usual grading. For a (non-negatively) graded left Noetherian ring, let G_1^{graded} denote K_1 of the category of finitely generated graded modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$. (Morphisms in this category are required to preserve the grading.) Then the exact functor $M \mapsto R[t_1, \dots, t_r] \otimes_R M$ induces an isomorphism $G_1(R) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \cong G_1^{\text{graded}}(R) \to G_1^{\text{graded}}(R[t_1, \dots, t_r])$.*

Proof. For simplicity of notation, let $S = R[t_1, \dots, t_r]$. As in the proof of Theorem 3.2.11, it suffices to show that the map $G_1^{\text{graded}}(R) \to G_1^{\text{graded}}(S)$ is surjective. As in the proof of Theorem 3.2.11, let $\mathcal{F}$ be the full subcategory of the category of finitely generated R-flat graded S-modules, whose objects are graded modules M satisfying $\text{Tor}_j^S(R, M) = 0$ for $j > 0$. These include the finitely generated free graded modules, and any graded morphism lifts to a morphism of a free graded module. As in the proof of Theorem 3.2.11, the hypotheses of the Resolution Theorem (Theorem 3.1.14) are satisfied, and the inclusion of $\mathcal{F}$ induces an isomorphism $K_1(\mathcal{F}) \xrightarrow{\cong} G_1^{\text{graded}}(S)$. Also the map $G_1^{\text{graded}}(R) \to G_1^{\text{graded}}(S)$ naturally factors through $K_1(\mathcal{F})$, and it's enough to show that the map $G_1^{\text{graded}}(R) \to K_1(\mathcal{F})$ is surjective.

Furthermore, by the method of the proof of Theorem 3.2.11, it is enough to consider classes in $K_1(\mathcal{F})$ defined by an automorphism α of $S \otimes_R M$, where M is a graded R-module. Then since α is required to be grading-preserving, α induces an automorphism of $(S \otimes_R M)_n = M$, and since M generates $S \otimes_R M$ as an S-module, α is determined by its restriction to M, *i.e.*, $\alpha = 1 \otimes \alpha|_M$. Thus the map $G_1^{\text{graded}}(R) \to G_1^{\text{graded}}(S)$ is surjective. □

3.2.16. Theorem (Grothendieck Theorem for G_1). *Let R be a left Noetherian ring. Then the natural map $G_1(R) \to G_1(R[t])$ is an isomorphism, with inverse given by the map of Proposition 3.2.9.*

Proof. As in the proof of Theorem 3.2.12, we consider the diagram

$$\begin{array}{ccc} G_1^{\text{graded}}(R) & \longrightarrow & G_1^{\text{graded}}(R[t,\, s]) \\ {\scriptstyle \text{forget grading}}\downarrow & & \downarrow{\scriptstyle \psi_*} \\ G_1(R) & \longrightarrow & G_1(R[t]), \end{array}$$

where the horizontal arrows are induced by the inclusions of R into $R[t]$ and $R[t, s]$. This diagram obviously commutes. By Theorem 3.2.15, the top horizontal arrow is an isomorphism. So it will be enough to show that the vertical arrow on the right is surjective. This is a bit more delicate than in the case of G_0 since we need to consider not just modules but also their automorphisms. But $\psi : R[t, s] \to R[t]$ factors through the inclusion $R[t, s] \hookrightarrow R[t, s, s^{-1}]$, so closer examination shows that the right-hand side of the diagram above can be factored as

$$\begin{array}{ccc} G_1^{\text{graded}}(R[t,\, s]) & & \\ & \searrow & \\ {\scriptstyle \psi_*}\downarrow & & G_1^{\text{graded}}(R[t,\, s,\, s^{-1}]) \\ & \swarrow & \\ G_1(R[t]), & & \end{array}$$

where $G_1^{\text{graded}}(R[t, s, s^{-1}])$ is defined using $\mathbb{Z}$-graded modules that are not necessarily bounded below. (If N is a graded $R[t, s, s^{-1}]$-module, then multiplication by s induces isomorphisms $N_j \to N_{j-1}$ for all j, so N can't be bounded below unless it's the zero module.) Furthermore, if M is an $R[t]$-module, then $R[s, s^{-1}] \otimes_R M$ can be given the structure of a **graded** $R[t, s, s^{-1}]$-module $F(M)$ (in which t acts by the original action of t composed with multiplication by s), and the functors ψ_* and F are inverses to one another, defining an equivalence of the category of finitely generated $R[t]$-modules with the category of finitely generated graded $R[t, s, s^{-1}]$-modules. So

$$G_1^{\text{graded}}(R[t,\, s,\, s^{-1}]) \to G_1(R[t])$$

is an isomorphism and we need only see that

$$G_1^{\text{graded}}(R[t,\, s]) \to G_1^{\text{graded}}(R[t,\, s,\, s^{-1}])$$

is surjective.

Thus let N be a finitely generated graded $R[t, s, s^{-1}]$-module and let α be a grading-preserving automorphism of N. Let $P = \bigoplus_{n=0}^{\infty} N_n$ be the $R[t, s]$-module generated by N_0. Then P is finitely generated and α maps P into itself, and $N = R[t, s, s^{-1}] \otimes_{R[t,\, s]} P$. Since obviously $\alpha = 1 \otimes (\alpha|_P)$, this shows $[N, \alpha]$ is in the image of $G_1^{\text{graded}}(R[t, s])$. So

$$G_1^{\text{graded}}(R[t,\, s]) \to G_1^{\text{graded}}(R[t,\, s,\, s^{-1}])$$

is surjective. □

3.2.17. Corollary. *If R is a left regular ring, then the natural map $K_1(R[t]) \to K_1(R)$ is an isomorphism. In other words, $NK_1(R) = 0$.*

Proof. This follows from combining Theorem 3.2.16, Corollary 3.1.16, and Proposition 3.2.9. □

We come now to the case of the Laurent polynomial ring $R[t, t^{-1}]$. We begin with G_1 and then go on to K_1.

3.2.18. Proposition. *Let R be a left Noetherian ring. Then there is a natural embedding of $G_1(R) \oplus G_0(R)$ as a direct summand in $G_1(R[t, t^{-1}])$ via*

$$\Phi : ([M, \alpha], [M']) \mapsto \left[R[t, t^{-1}] \otimes_R M,\ 1 \otimes \alpha\right] + \left[R[t, t^{-1}] \otimes_R M',\ t \otimes 1\right].$$

The left inverse Ψ to Φ is given by $G_1(R[t, t^{-1}]) \to G_1(R)$ as defined in Proposition 3.2.9, together with the following map $G_1(R[t, t^{-1}]) \to G_0(R)$: Let N be a finitely generated $R[t, t^{-1}]$-module, $\beta \in \operatorname{Aut} N$, and let N' be a finitely generated $R[t]$-submodule of N that generates N as an $R[t, t^{-1}]$-module. Then for suitably large k, $t^k\beta$ maps N' into itself, and $\operatorname{coker}((t^k\beta)|_{N'})$ is a finitely generated R-module. Map $[N, \beta] \in G_1(R[t, t^{-1}])$ to

$$[\operatorname{coker}((t^k\beta)|_{N'})] - k[\operatorname{coker}(t|_{N'})] \in G_0(R).$$

Proof. It is clear that Φ defines a homomorphism, and we already verified in Corollary 3.2.10 that it embeds $G_1(R)$ in $G_1(R[t, t^{-1}])$ as a direct summand. It therefore suffices to check that the indicated formula gives a well-defined homomorphism $G_1(R[t, t^{-1}]) \to G_0(R)$, and that the composite $\Psi \circ \Phi$ is the identity.

The first problem is to show that, given a finitely generated $R[t, t^{-1}]$-module N and $\beta \in \operatorname{Aut} N$, $[\operatorname{coker}((t^k\beta)|_{N'})] - k[\operatorname{coker}(t|_{N'})] \in G_0(R)$ is independent of the choice of N' and of k. First of all, suppose N' is fixed and $t^k\beta$ maps N' into itself. Note that $(t^k\beta|_{N'})$ is an injective $R[t]$-module homomorphism since it is the restriction of an automorphism of N. If we replace k by $k+j$, $j > 0$, then $t^{k+j}\beta(N') \subseteq t^k\beta(N') \subseteq N'$, and $t^k\beta$ induces an isomorphism

$$N'/t^jN' \to t^k\beta(N')/t^{k+j}\beta(N').$$

Hence, in $G_0(R)$, we have

$$[\operatorname{coker}((t^k\beta)|_{N'})] + [\operatorname{coker}(t^j|_{N'})] = [\operatorname{coker}((t^{k+j}\beta)|_{N'})].$$

In particular, iterating this with $\beta = 1$ shows that

$$[\operatorname{coker}(t^j|_{N'})] = j[\operatorname{coker}(t|_{N'})],$$

and so

$$[\operatorname{coker}((t^{k+j}\beta)|_{N'})] - (k+j)[\operatorname{coker}(t|_{N'})] = [\operatorname{coker}((t^k\beta)|_{N'})] - k[\operatorname{coker}(t|_{N'})].$$

Thus for fixed N' we have independence of k.

Now suppose N'' is another finitely generated $R[t]$-submodule of N that generates N as an $R[t, t^{-1}]$-module. Then

$$N = \bigcup_{j=0}^{\infty} t^{-j} N' = \bigcup_{j=0}^{\infty} t^{-j} N'',$$

so for suitably large j, $t^j N'' \subseteq N'$. If we choose k large enough so that $t^k\beta$ maps N' and N'' into themselves, then $(t^{j+k}\beta)N'' \subseteq t^j N'' \cap (t^k\beta)N' \subseteq N'$ and in $G_0(R)$ we have

$$[N'/(t^k\beta)N'] + [(t^k\beta)N'/(t^{j+k}\beta)N'')] = [N'/t^j N''] + [t^j N''/(t^{j+k}\beta)N'']$$

or

$$[N'/(t^k\beta)N'] + [N'/t^j N''] = [N'/t^j N''] + [N''/(t^k\beta)N''],$$

so $[N'/(t^k\beta)N'] = [N''/(t^k\beta)N'']$. This proves independence of the choice of N' and shows Ψ is well defined.

Next we have to show that Ψ is a homomorphism. If β and γ are two automorphisms of N, and if we choose k large enough so that $t^k\beta$ and $t^k\gamma$ both map N' into itself, then $(t^{2k}\beta\gamma)N' \subseteq (t^k\beta)N' \cap (t^k\gamma)N' \subseteq N'$, so in $G_0(R)$ we have

$$\begin{aligned}[N'/(t^{2k}\beta\gamma)N'] &= [N'/(t^k\gamma)N'] + [(t^k\gamma)N'/(t^{2k}\beta\gamma)N'] \\ &= [N'/(t^k\gamma)N'] + [N'/(t^k\beta)N'],\end{aligned}$$

showing that $\Psi([N, \beta\gamma]) = \Psi([N, \beta]) + \Psi([N, \gamma])$. It remains to show that Ψ is additive on short exact sequences. Suppose

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

is a short exact sequence of finitely generated $R[t, t^{-1}]$-modules, and β is an automorphism of N_2 that maps N_1 onto itself and that induces an automorphism γ of N_3. Choose a finitely generated $R[t]$-submodule N_2' of N_2 that generates N_2 as an $R[t, t^{-1}]$-module. Let $N_1' = N_2' \cap N_1$, and let N_3' be the image of N_2' in N_3. Then N_1' and N_3' are finitely generated and generate N_1 and N_3, respectively. If we choose k so that $t^k\beta$ maps N_2' and N_1' into themselves, then if $x \in N_1' \cap (t^k\beta)N_2'$, we have $(t^k\beta)^{-1}x \in N_1 \cap N_2'$ (since N_1 is stable under $(t^k\beta)^{-1}$), which is N_1'. Hence $N_1' \cap (t^k\beta)N_2' = (t^k\beta)N_1'$ and so in $G_0(R)$,

$$\begin{aligned}[N_2'/(t^k\beta)N_2'] &= [N_2'/\left(N_1' + (t^k\beta)N_2'\right)] + [\left(N_1' + (t^k\beta)N_2'\right)/(t^k\beta)N_2'] \\ &= [N_3'/(t^k\gamma)N_3'] + [N_1'/\left(N_1' \cap (t^k\beta)N_2'\right)] \\ &= [N_3'/(t^k\gamma)N_3'] + [N_1'/(t^k\beta)N_1'].\end{aligned}$$

It follows that $\Psi([N_3, \gamma]) + \Psi([N_1, \beta|_{N_1}]) = \Psi([N_2, \beta])$, so Ψ is additive on short exact sequences.

Finally, we show that the composite $\Psi \circ \Phi$ is the identity. If M and M' are finitely generated R-modules and $\alpha \in \operatorname{Aut} M$, then

$$\begin{aligned}\Psi \circ \Phi\left([M, \alpha], [M']\right) &= \Psi\left(\left[R[t, t^{-1}] \otimes_R M, 1 \otimes \alpha\right]\right. \\ &\qquad \left.+ \left[R[t, t^{-1}] \otimes_R M', t \otimes 1\right]\right) \\ &= \left([M, \alpha] + [M', 1],\right. \\ &\qquad \left.\left[\operatorname{coker}(1 \otimes \alpha)|_{R[t] \otimes_R M}\right] + \left[\operatorname{coker}(t \otimes 1)|_{R[t] \otimes_R M'}\right]\right) \\ &= \left([M, \alpha] + 0, 0 + [M']\right) = \left([M, \alpha], [M']\right),\end{aligned}$$

as required. $\square$

3.2.19. Theorem. *Let R be a left Noetherian ring. Then the embedding Φ of $G_1(R) \oplus G_0(R)$ into $G_1(R[t, t^{-1}])$, defined in Proposition 3.2.18, is an isomorphism.*

Proof. We need to show that Φ is surjective. The intuitive idea is easy to explain. Suppose $[N, \beta] \in G_1(R[t, t^{-1}])$ is defined by a finitely generated $R[t, t^{-1}]$-module and $\beta \in \operatorname{Aut} N$. We need to show that if $[N, \beta] \mapsto 0$ in $G_0(R)$, then $[N, \beta]$ comes from a class in $G_1(R)$. Let N' be a finitely generated $R[t]$-submodule of N that generates N as an $R[t, t^{-1}]$-module, and suppose for simplicity that β maps N' into itself. The statement that $[N, \beta] \mapsto 0$ in $G_0(R)$ then means that $[N'/\beta(N')] = 0$ in $G_0(R)$. If $N'/\beta(N')$ were literally the zero-module, this would mean that β restricts to an automorphism of N'. But then $N = R[t, t^{-1}] \otimes_{R[t]} N'$ and $\beta = 1 \otimes \beta|_{N'}$, which shows that $[N, \beta]$ lies in the image of the map $G_1(R[t]) \to G_1(R[t, t^{-1}])$. Since (using Theorem 3.2.16) we have a commutative diagram

$$\begin{array}{ccc} G_1(R) & & \\ \cong\big\downarrow & \searrow & \\ G_1(R[t]) & \longrightarrow & G_1(R[t, t^{-1}]), \end{array}$$

this shows $[N, \beta]$ is in the image of $G_1(R)$.

To make this argument rigorous takes a bit of work, and can be done in a number of ways. The easiest is probably to appeal to the method of proof of Theorem 3.2.16, which shows that we can take $N' = R[t] \otimes_{\psi} P$, with P a finitely generated graded $R[t, s, s^{-1}]$-module, and that β extends to a graded automorphism of $R[t, t^{-1}, s, s^{-1}] \otimes_{R[t, s, s^{-1}]} P$. Instead of assuming that $[N, \beta] \mapsto 0$ in $G_0(R)$, we'll make no assumption on β and show how to write $[N, \beta]$ in terms of elements in the image of Φ. First multiply β by a suitably high power of t so that β maps N' into itself. Then there will be an induced grading-preserving endomorphism $\tilde{\beta}$ of P. Let $P' = \bigoplus_{n=0}^{\infty} P_n$ be the graded $R[t, s]$-submodule of P generated by the elements of degree 0; $\tilde{\beta}$ maps P' into itself.

By the method of proof of Theorem 3.2.15, we may reduce to the case where $P' = R[t, s] \otimes_R M$, M a finitely generated graded R-module, and $\tilde{\beta}$ is determined by a graded endomorphism α of M. But then $N = R[t, t^{-1}] \otimes_R$

M and β induces α on the quotient $M \cong N/(t-1)N$. Since β is invertible, α is in fact an automorphism of M. We may identify N with $\bigoplus_{n\in\mathbb{Z}} t^n M$ and write $\beta(t^n x) = \sum_{j\in\mathbb{Z}} \beta_j(x) t^{n+j}$ for $x \in M$, where almost all of the $\beta_j(x)$ are 0, and $\sum_j \beta_j(x) = \alpha(x)$. From this one can see M is an iterated extension of R-modules M_j on which β takes the simple form $t^j \alpha_j$, and we have

$$\begin{aligned}[N, \beta] &= \sum_{j\in\mathbb{Z}} [R[t, t^{-1}] \otimes_R M_j, t^j \otimes \alpha_j] \\ &= \sum_{j\in\mathbb{Z}} \left([R[t, t^{-1}] \otimes_R M_j, t^j \otimes 1] + [R[t, t^{-1}] \otimes_R M_j, 1 \otimes \alpha_j] \right) \\ &= \sum_{j\in\mathbb{Z}} \Phi\left([M_j, \alpha_j], j[M_j]\right),\end{aligned}$$

which shows Φ is surjective. □

3.2.20. Corollary (Bass-Heller-Swan). *Let R be a left regular ring. Then there is a canonical isomorphism $K_1(R[t, t^{-1}]) \cong K_1(R) \oplus K_0(R)$.*

Proof. This follows from combining Theorem 3.2.19, Corollary 3.1.16, and Proposition 3.2.9. □

Since we are also interested in rings which are not left regular or even left Noetherian, it will be convenient to try to analyze $K_1(R[t, t^{-1}])$ directly, without going through the intermediary of G-theory. This will lead to another proof of the Bass-Heller-Swan Theorem (Corollary 3.2.20) as well as a motivation for the definition of negative K-theory in the next section.

3.2.21. Lemma ([BassHellerSwan]). *Let R be a ring. Then*

(*a*) *Any matrix $B \in GL(R[t])$ can be reduced, modulo $GL(R)$ and $E(R[t])$, to a matrix of the form $1 + At$, where A is a* **nilpotent** *matrix with entries in R.*

(*b*) *Any matrix $B \in GL(R[t, t^{-1}])$ can be reduced, modulo $GL(R)$ and $E(R[t, t^{-1}])$, to a matrix of the form $\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}(1 + A(t-1))$, where A is a matrix with entries in R and $A(1 - A)$ is nilpotent.*

Proof. (a) Write $B = B_0 + tB_1 + \cdots + t^d B_d$, where the B_j are matrices with entries in R. We will first show by induction that B can be reduced to something with $d \leq 1$. So assume $d > 1$. Then if $\sim$ stands for "is equal to, modulo $GL(R)$ and $E(R[t])$," we have

$$\begin{aligned}B &\sim \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} B & t^{d-1}B_d \\ 0 & 1 \end{pmatrix}\end{aligned}$$

(since we can add a multiple of the bottom row to the top row)

$$\sim \begin{pmatrix} B - t^d B_d & t^{d-1} B_d \\ -t & 1 \end{pmatrix},$$

(since we can subtract $t\times$ (the last column) from the first column).

Now we have something of degree $\leq d-1$. Continuing by induction, B can be reduced to something with $d \leq 1$.

If we can reduce to $d = 0$, the assertion of the Lemma is obvious. Otherwise, we can reduce to the case $d = 1$, and assume $B = B_0 + tB_1$. Since B is invertible as a matrix over $R[t]$, B_0 must be invertible. Factoring out B_0, we reduce B to the form $1 + At$. This must be invertible as a matrix over $R[t]$, so we have $B^{-1} = C_0 + tC_1 + \cdots + t^r C_r$ for some C_j's and some integer r. Multiplying out the equation

$$1 = (1 + At)(C_0 + tC_1 + \cdots + t^r C_r) = (C_0 + tC_1 + \cdots + t^r C_r)(1 + At),$$

we obtain the equations

$$1 = C_0, \quad 0 = AC_0 + C_1 = C_0 A + C_1, \quad \ldots,$$

$$0 = AC_{r-1} + C_r = C_{r-1}A + C_r, \quad 0 = AC_r = C_r A.$$

Solving inductively, we obtain $C_0 = 1$, $C_1 = -A$, ... , $C_j = (-A)^j$, and A is nilpotent since $A^{r+1} = 0$.

(b) Since we are allowed to multiply B by a power of t, we may suppose B involves only non-negative powers of t. Then we may repeat the same trick and come down to the case where $B = B_0 + tB_1 = (B_0 + B_1) + (t-1)B_1$. Since B is invertible as a matrix over $R[t, t^{-1}]$, $B_0 + B_1$ must be invertible. Factoring out $B_0 + B_1$, we reduce B to the form $1 + A(t-1) = 1 - A + At$. This must be invertible as a matrix over $R[t, t^{-1}]$, so after multiplying by a power of t it has an inverse which is a matrix over $R[t]$. By the same reasoning as in (a), $(1-A)A$ is nilpotent. □

3.2.22. Theorem (Bass-Heller-Swan). *Let R be a ring. Let* **Nil** *R be the category whose objects are pairs (P, A), where P is a finitely generated free R-module and A is a nilpotent endomorphism of P. The morphisms $(P, A) \to (P', A')$ are R-module homomorphisms $T : P \to P'$ such that $A'T = TP$. Note that* **Nil** *R is a category with exact sequences, and K_0 of this category contains an obvious homomorphic image of $\mathbb{Z}$ coming from the full subcategory of objects with $A = 0$. Then*

(*a*) *$K_1(R[t]) = K_1(R) \oplus NK_1(R)$, where $NK_1(R)$ is canonically isomorphic to $\tilde{K}_0($***Nil** *$R)$. (The notation $\tilde{K}_0$ means K_0 divided out by the canonical image of $\mathbb{Z}$.)*

(*b*) *There is a natural splitting of $K_1(R[t, t^{-1}])$ as $K_1(R) \oplus K_0(R) \oplus NK_1(R) \oplus NK_1(R)$. The two copies of $NK_1(R)$ come from the embeddings $R[t] \hookrightarrow R[t, t^{-1}]$ and $R[t^{-1}] \hookrightarrow R[t, t^{-1}]$.*

Proof. (a) We already know there must be a splitting

$$K_1(R[t]) = K_1(R) \oplus NK_1(R),$$

and (a) of Lemma 3.2.21 shows that $NK_1(R)$ is the image in $K_1(R[t])$ of matrices of the form $1 + At$, A nilpotent. We define a map $NK_1(R) \to \tilde{K}_0(\mathbf{Nil}\ R)$ by (when A is a nilpotent $n \times n$ matrix)

$$NK_1(R) \ni [1 + At] \mapsto [(R^n, A)] \in \tilde{K}_0(\mathbf{Nil}\ R).$$

To check that this is well defined, note that if $1 + At$ is conjugate to $1 + A't$ under $GL(n, R[t])$, then sending $t \mapsto 1$, we see $1 + A$ is conjugate to $1 + A'$ under $GL(n, R)$, hence A is conjugate to A' under $GL(n, R)$ and $[(R^n, A)] = [(R^n, A')]$ in $K_0(\mathbf{Nil}\ R)$. Furthermore, if we replace $1_n + At$ by the $(n+k) \times (n+k)$ matrix $(1_n + At) \oplus (1_k)$, this corresponds to replacing A by $A \oplus 0_k$ and $[(R^n, A)]$ by $[(R^n, A)] + [(R^k, 0)]$. This element is different in K_0, but the same in $\tilde{K}_0$, so the map $NK_1(R) \to \tilde{K}_0(\mathbf{Nil}\ R)$ is well defined. Finally, the map is a homomorphism since

$$\begin{aligned}[1 + At] + [1 + A't] = [(1 + At) \oplus (1 + A't)] &= [1 + (A \oplus A')t] \\ &\mapsto [(R^n, A)] + [(R^n, A')].\end{aligned}$$

To show the map is an isomorphism, we construct its inverse by the obvious formula $[(R^n, A)] \mapsto [1 + At]$. Note that $[1 + At]$ indeed defines a class in $NK_1(R)$, since for any nilpotent A, $1 + At \in GL(R[t])$ and maps to 1 in $GL(R)$ under the homomorphism defined by $t \mapsto 0$. Since we know all classes in $NK_1(R)$ can be represented in the form $[1 + At]$, we will be done if we can show that the map is indeed well defined. Clearly the class of $1 + At$ doesn't change if we replace A by $A \oplus 0_k$ (which corresponds to changing our class in $K_0(\mathbf{Nil}\ R)$ by something in the canonical image of $\mathbb{Z}$). So we only need to check additivity on exact sequences. Suppose

$$0 \to P_1 \xrightarrow{\alpha_1} P_2 \xrightarrow{\alpha_2} P_3 \to 0$$

is a short exact sequence of finitely generated free R-modules and we have nilpotent endomorphisms $A_j \in \mathrm{End}(P_j)$ such that $A_2\alpha_1 = \alpha_1 A_1$, $A_3\alpha_2 = \alpha_2 A_2$. This means that also $(1 + A_2 t)\alpha_1 = \alpha_1(1 + A_1 t)$ and $(1 + A_3 t)\alpha_2 = \alpha_2(1 + A_2 t)$, so

$$[1 + A_2 t] = [1 + A_1 t] + [1 + A_3 t] \quad \text{in } K_1(R[t]),$$

and we're done.

(b) The maps Φ and Ψ from Proposition 3.2.18 can be defined in K-theory instead of in G-theory, by exactly the same formulas. The only point that needs checking is that the second component of Ψ indeed sends $K_1(R[t, t^{-1}])$ into $K_0(R)$ and not just into $G_0(R)$. To check this, we use the fact that the cokernel of the map $K_1(R) \to K_1(R[t, t^{-1}])$ is described by

Lemma 3.2.21(b) as being generated by matrices of the form $1+A(t-1)$ with A an $n\times n$ matrix over R and $(1-A)A$ nilpotent. Equivalently, $A = P+N$, where P is idempotent, N is nilpotent, and P and N commute with each other. To see this, suppose $A^r(1-A)^r = 0$. Then since the polynomials x^r and $(1-x)^r$ are relatively prime in $\mathbb{Z}[x]$, there are polynomials $p(x)$ and $q(x)$ with integer coefficients such that $p(x)x^r + q(x)(1-x)^r = 1$. Let $P = p(A)A^r$. Then $1 - P = q(A)(1-A)^r$ and since $A^r(1-A)^r = 0$, $P(1-P) = 0$. This shows P is idempotent, and P is a polynomial in A. If $N = A - P$, then N is also a polynomial in A, so P and N commute with one another. Furthermore, $N = A\left(1-p(A)A^{r-1}\right)$ and $N = (A-1)+(1-P) = (1-A)\left(-1+q(A)(1-A)^{r-1}\right)$, so N is divisible by both A and $1-A$, hence divisible by $A(1-A)$, hence nilpotent. But then $R[t]^n/\left(1+A(t-1)\right)R[t]^n \cong \operatorname{im} P$ is projective as an R-module. So $K_1(R) \oplus K_0(R)$ naturally embeds as a direct summand in $K_1(R[t])$.

The cokernel of this embedding is once again described by Lemma 3.2.21 (b) as being generated by matrices of the form $1+(P+N)(t-1)$, where P is idempotent, N is nilpotent, and P and N commute with one another. Thus $1+(P+N)(t-1)$ corresponds to a pair of nilpotent matrices, PN and $(1-P)N$. These correspond to the two copies of $NK_1(R)$. The rest of the proof is just as in part (a). □

Remark. This explains a commonly used notation: the group $NK_1(R)$ is often called $\operatorname{Nil} R$ because of (a) of the theorem.

3.2.23. Exercise (Non-triviality of NK_1). Let k be a commutative field, and let $R = k[t]/(t^2)$.

(a) Show that R is a local ring and thus compute $K_0(R)$ and $K_1(R)$.

(b) Let s be another indeterminate and compute the group of units $R[s]^\times$ in $R[s]$.

(c) From the exact sequence (split on the right)

$$NK_1(R) \to K_1(R[s]) \leftrightarrows K_1(R),$$

deduce that $NK_1(R)$ is not finitely generated (as an abelian group). (Recall that since the ring $R[s]$ is commutative, $R[s]^\times \hookrightarrow K_1(R[s])$.)

3.2.24. Exercise (Non-triviality of NK_0). Let k be a commutative field, and let $S = k[t^2, t^3]$.

(a) From the split exact sequence

$$0 \to t^2k[t] \to S \leftrightarrows k \to 0$$

and the long exact K-theory sequence coming from the short exact sequence

$$0 \to t^2k[t] \to k[t] \to k[t]/(t^2) \to 0,$$

compute $K_0(S)$. (Hint: use Exercise 3.2.23(a) and the fact that $k[t]$ is a Euclidean ring.)

(b) Let s be another indeterminate and similarly use the short exact sequences

$$0 \to t^2k[t,s] \to S[s] \leftrightarrows k[s] \to 0$$

and

$$0 \to t^2k[t,s] \to k[t,s] \to k[t,s]/(t^2) \to 0$$

to relate $K_0(S[s])$ and thus $NK_0(S)$ to $NK_1(R)$ in Exercise 3.2.24(c). (Recall that $K_0(S[s]) \cong K_0(S) \oplus NK_0(S)$.) Use Grothendieck's Theorem which implies that $\tilde{K}_0(k[t,s]) = 0$. Deduce that $NK_0(S)$ is not finitely generated (as an abelian group).

3.2.25. Exercise. Give another proof of Corollaries 3.2.17 and 3.2.20 from Theorem 3.2.22, by showing that the group NK_1 described in that Theorem has to vanish if R is left regular. (Hint: if A is an $n \times n$ matrix over R and $A^r = 0$, it gives a filtration of the free module R^n by

$$0 = \operatorname{im} A^r \subseteq \operatorname{im} A^{r-1} \subseteq \cdots \subseteq \operatorname{im} A \subseteq R^n.$$

If R is left regular, the subquotients can be resolved by finitely generated projective modules.)

3.2.26. Exercise. Let R be any ring.

(1) Show that $NK_1(R[t,\, t^{-1}])$ contains $NK_1(R) \oplus NK_0(R)$ as a direct summand. You can do this by computing $K_1(R[t,\, t^{-1},\, s])$ two ways.

(2) [Vorst] A ring R is called **j-regular** if $K_j(R[t_1, \ldots, t_r]) \cong K_j(R)$ for any r. By Corollary 3.2.17, a left regular ring is 1-regular.

Show that $NK_1(R[t]) = 0$ implies $NK_1(R[t,\, t^{-1}]) = 0$, by noting that a nilpotent matrix over $R[t,\, t^{-1}]$ is of the form $t^{-n} \times$ (a nilpotent matrix over $R[t]$). Then deduce from (1) that if R is 1-regular, $NK_0(R) = 0$.

(3) Iterating the result of (2), prove the multivariable version of this, that 1-regularity implies 0-regularity. (However, there are 0-regular rings which are not 1-regular.)

3.2.27. Exercise.

(1) Show from the Bass-Heller-Swan Theorem that if π is any group, then $\mathrm{Wh}(\pi \times \mathbb{Z}) \cong \mathrm{Wh}(\pi) \oplus \tilde{K}_0(\mathbb{Z}\pi) \oplus (NK_1(\mathbb{Z}\pi))^2$. (We are mixing additive and multiplicative notation here.)

(2) Deduce that the Whitehead group of any free abelian group vanishes. (Hint: $\mathbb{Z}[t_1,\, t_1^{-1}, \ldots,\, t_r,\, t_r^{-1}]$ is regular. Why?)

(3) See if you can find a topological interpretation of the formula in (1), and in particular a relationship between Wall finiteness obstructions for spaces with fundamental group π and Whitehead torsion obstructions for spaces with fundamental group $\pi \times \mathbb{Z}$.

3.2.28. Exercise. Let k be a (commutative) field. Show that Corollary 3.2.20 applied with $R = k$ amounts to the assertion that SK_1 vanishes for the PID $k[t,\, t^{-1}]$. Can you prove this directly using the results of Chapter 2?

3.2.29. Exercise [Farrell]. Let R be a ring, and view $NK_1(R)$ as being represented by classes of nilpotent matrices N over R, as in Theorem 3.2.22.

(1) Fix an integer n, and let $\iota_n : R[t^n] \hookrightarrow R[t]$ be the inclusion. Note that $R[t]$ is a free $R[t^n]$-module of rank n, and thus that an $r \times r$ matrix over $R[t]$ gives rise to an $rn \times rn$ matrix over $R[t^n]$. In this way a **transfer map** $\iota_n^* : K_1(R[t]) \to K_1(R[t^n])$ is defined. Show that $\iota_n^* \circ (\iota_n)_*$ is multiplication by n on $K_1(R[t^n])$ (if we use additive notation).

(2) Suppose N is a nilpotent $r \times r$ matrix over R, so that $1 + Nt$ represents a typical element of $NK_1(R)$. Show that $\iota_n^*(1 + Nt)$ is given by the block matrix

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & Nt^n \\ N & 1 & 0 & & 0 \\ 0 & N & \ddots & & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & N & 1 \end{pmatrix}.$$

(3) Let A be the strictly lower-triangular block matrix

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ N & 1 & 0 & & 0 \\ 0 & N & \ddots & & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & N & 1 \end{pmatrix}.$$

Show that if $N^n = 0$, then $A^{-1}M$ is strictly upper-triangular and hence elementary, and thus that $\iota_n^*([1 + Nt]) = 0$ in $K_1(R[t^n])$. Conclude that if $NK_1(R)$ is finitely generated, then there is some integer n_0 such that $\iota_n^* = 0$ on $NK_1(R)$ for all $n \geq n_0$.

(4) Using (1), deduce that if there is a prime p such that multiplication by p is injective on $NK_1(R)$, then $\iota_n^* \circ (\iota_n)_*([1+Nt^n]) \neq 0$ for $n = p^i$ a power of p and for all nilpotent matrices N.

(5) Conclude from (3) and (4) that if $NK_1(R)$ is finitely generated and non-zero, there can be no prime p such that multiplication by p is injective on $NK_1(R)$, which contradicts the structure theorem for finitely generated abelian groups. Therefore if $NK_1(R) \neq 0$, then $NK_1(R)$ is **not finitely generated.**

3. Negative K-theory

One immediate consequence of the Bass-Heller-Swan Theorem (Theorem 3.2.22) is that for any ring R,

$$K_0(R) = \operatorname{coker}\left(K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t, t^{-1}])\right).$$

This motivates the following inductive definition:

3.3.1. Definition. For any ring R, $K_{-1}(R)$ is defined to be the cokernel of the natural map $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t, t^{-1}])$. (Since we have defined K_0 even for rings without unit, it is not necessary here to assume that R has a unit.) By Corollary 3.2.20, $K_{-1}(R)$ vanishes if R is left regular. Then for any $n \geq 1$, we define $K_{-n}(R)$ to be the cokernel of the natural map $K_{-(n-1)}(R[t]) \oplus K_{-(n-1)}(R[t^{-1}]) \to K_{-(n-1)}(R[t, t^{-1}])$. Note that this is functorial in R, since $K_{-n}(R)$ is a natural direct summand in $K_0(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$. We also define $NK_{-n}(R)$ to be the cokernel of the natural map $K_{-n}(R) \to K_{-n}(R[t])$. (Because of the splitting map $R[t] \to R$ sending $t \mapsto 0$, $K_{-n}(R[t])$ splits as $K_{-n}(R) \oplus NK_{-n}(R)$.) By iterated use of the Syzygy Theorem (Theorem 3.2.3) and Corollary 3.2.20, $K_{-n}(R)$ and $NK_{-n}(R)$ vanish for all n if R is left regular.

The following theorem shows that Theorem 3.2.22 has an exact analogue for K_0, using the new functor K_{-1}.

3.3.2. Theorem. *For any ring R, there is a natural splitting*

$$K_0(R[t, t^{-1}]) \cong K_0(R) \oplus K_{-1}(R) \oplus NK_0(R) \oplus NK_0(R),$$

where the two copies of $NK_0(R)$ come from the embeddings

$$R[t] \hookrightarrow R[t, t^{-1}] \text{ and } R[t^{-1}] \hookrightarrow R[t, t^{-1}].$$

Proof. Theorem 3.2.22 says that for any ring, $K_0(R)$ naturally sits as a direct summand in $K_1(R[t_1, t_1^{-1}])$. It also says that for any ring S, there is a natural exact sequence

$$0 \to K_1(S) \to K_1(S[t]) \oplus K_1(S[t^{-1}]) \to K_1(S[t, t^{-1}]),$$

and the cokernel of the map on the right splits. Let us put these two statements together, but taking $S = R[t_1, t_1^{-1}]$. Then $K_0(R)$ naturally sits as a direct summand in $K_1(S)$, and similarly $K_0(R[t])$, $K_0(R[t^{-1}])$, and $K_0(R[t, t^{-1}])$ naturally sit as direct summands in $K_1(S[t])$, in $K_1(S[t^{-1}])$, and in $K_1(S[t, t^{-1}])$, respectively. Furthermore, the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_0(R) & \longrightarrow & K_0(R[t]) \oplus K_0(R[t^{-1}]) & \longrightarrow & \\
\| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1(S) & \longrightarrow & K_1(S[t]) \oplus K_1(S[t^{-1}]) & \longrightarrow &
\end{array}$$

$$\begin{array}{cccccc}
\longrightarrow & K_0(R[t, t^{-1}]) & \longrightarrow & K_{-1}(R) & \longrightarrow & 0 \\
& \downarrow & & \| & & \\
\longrightarrow & K_1(S[t, t^{-1}]) & \rightleftarrows & K_0(S) & \longrightarrow & 0
\end{array}$$

clearly commutes. The bottom row is exact, and the top row is exact on the right by definition of $K_{-1}(R)$ and on the left since $K_0(R) \to K_0(R[t])$ is split injective. Let us show that the top row is also exact at $K_0(R[t]) \oplus K_0(R[t^{-1}])$, and that the top exact sequence splits on the right. To prove the first of these statements, note that by commutativity of the diagram, the kernel of $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t, t^{-1}])$ may be identified with the intersection in $K_1(S[t]) \oplus K_1(S[t^{-1}])$ of the images of $K_0(R[t]) \oplus K_0(R[t^{-1}])$ and of $K_1(S)$. This is obviously $K_0(R)$. To construct a splitting map $K_{-1}(R) \to K_0(R[t, t^{-1}])$, note that the projection of $K_1(S[t, t^{-1}])$ onto $K_1(S) \oplus NK_1(S) \oplus NK_1(S)$ killing $K_0(S)$ restricts to a projection of $K_0(R[t, t^{-1}])$ onto $K_0(R) \oplus NK_0(R) \oplus NK_0(R)$ killing $K_{-1}(R)$, hence the latter must split off as a direct summand. □

Note that iteration of the same argument clearly proves the following.

3.3.3. Theorem ("Fundamental Theorem of Algebraic K-Theory"). *For any ring R and any $n \geq 1$, there is a natural splitting*

$$K_{-(n-1)}(R[t, t^{-1}]) \cong K_{-(n-1)}(R) \oplus K_{-n}(R) \oplus NK_{-(n-1)}(R) \oplus NK_{-(n-1)}(R),$$

where the two copies of $NK_{-(n-1)}(R)$ come from the embeddings $R[t] \hookrightarrow R[t, t^{-1}]$ and $R[t^{-1}] \hookrightarrow R[t, t^{-1}]$.

The advantage of the construction of the functors $K_{-n}(R)$ is that it now gives us a way of extending the exact sequence of an ideal arbitrarily far to the right, and thus a way of computing $K_0(R/I)$ from information about R and the ideal I.

3.3.4. Theorem. *Let R be a ring with unit, and let I be a two-sided ideal in R, viewed as a ring without unit. Then the exact sequence of Theorem 2.5.4 extends to an exact sequence*

$$\cdots \to K_0(R) \xrightarrow{q_*} K_0(R/I) \xrightarrow{\partial} K_{-1}(I) \xrightarrow{\iota_*} K_{-1}(R) \xrightarrow{q_*} K_{-1}(R/I) \xrightarrow{\partial} K_{-2}(I) \xrightarrow{\iota_*} \cdots,$$

where ι_* and q_* are the maps induced by the inclusion $\iota : I \hookrightarrow R$ and by the quotient map $q : R \twoheadrightarrow R/I$.

Proof. Take $S = R[t, t^{-1}]$ and $J = I[t, t^{-1}]$. To avoid unnecessary extra notation, again denote the inclusion $J \hookrightarrow S$ by ι and the quotient map $S \to S/J \cong (R/I)[t, t^{-1}]$ by q. Then $J \trianglelefteq S$, and by Theorem 2.5.4, there is a natural exact sequence

$$K_1(S) \xrightarrow{q_*} K_1(S/J) \xrightarrow{\partial} K_0(J) \xrightarrow{\iota_*} K_0(S) \xrightarrow{q_*} K_0(S/J).$$

On the other hand, we have natural embeddings of $K_0(R)$ in $K_1(S)$, $K_0(R/I)$ in $K_1(S/J)$, $K_{-1}(I)$ in $K_0(J)$, etc., as direct summands, and

$$\begin{array}{ccc} K_1(S) & \xrightarrow{q_*} & K_1(S/J) \\ \uparrow & & \uparrow \\ K_0(R) & \xrightarrow{q_*} & K_0(R/I), \end{array} \qquad \begin{array}{ccccc} K_0(J) & \xrightarrow{\iota_*} & K_0(S) & \xrightarrow{q_*} & K_0(S/J) \\ \uparrow & & \uparrow & & \uparrow \\ K_{-1}(I) & \xrightarrow{\iota_*} & K_{-1}(R) & \xrightarrow{q_*} & K_{-1}(R/I) \end{array}$$

commute. We also have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1(R) & \longrightarrow & K_1(R[t]) \oplus K_1(R[t^{-1}]) & \longrightarrow \\
\| & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & \\
0 & \longrightarrow & K_1(R/I) & \longrightarrow & K_1((R/I)[t]) \oplus K_1((R/I)[t^{-1}]) & \longrightarrow \\
\| & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & \\
0 & \longrightarrow & K_0(I) & \longrightarrow & K_0(I[t]) \oplus K_0(I[t^{-1}]) & \longrightarrow \\
\| & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & \\
0 & \longrightarrow & K_0(R) & \longrightarrow & K_0(R[t]) \oplus K_0(R[t^{-1}]) & \longrightarrow \\
\| & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & \\
0 & \longrightarrow & K_0(R/I) & \longrightarrow & K_0((R/I)[t]) \oplus K_0((R/I)[t^{-1}]) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & K_1(R[t, t^{-1}]) & \rightleftarrows & K_0(R) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle q_*} & & & & \| \\
\cdots & \longrightarrow & K_1((R/I)[t, t^{-1}]) & \rightleftarrows & K_0(R/I) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \partial} & & & & \| \\
\cdots & \longrightarrow & K_0(I[t, t^{-1}]) & \rightleftarrows & K_{-1}(I) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \iota_*} & & & & \| \\
\cdots & \longrightarrow & K_0(R[t, t^{-1}]) & \rightleftarrows & K_{-1}(R) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle q_*} & & & & \| \\
\cdots & \longrightarrow & K_0((R/I)[t, t^{-1}]) & \rightleftarrows & K_{-1}(R/I) & \longrightarrow & 0.
\end{array}$$

A diagram chase now gives the desired exact sequence as far as the $K_{-1}(R/I)$, but we can iterate the construction to include K_{-2} terms and eventually K_{-n} terms for all n. □

3.3.5. Examples. (Cf. Examples 1.5.10 and 2.5.6.)

(a) Suppose $R = \mathbb{Z}$ and $I = (m)$, where $m > 0$. Then R/I is a product of k local rings, where k is the number of distinct prime factors of m, and we determined earlier that $K_0(R/I) \cong \mathbb{Z}^k$ and that the map $K_0(R) \to K_0(R/I)$ may be identified with the diagonal map $\mathbb{Z} \to \mathbb{Z}^k$. The cokernel of this map is free abelian of rank $k - 1$. Since R is a PID, it is certainly a regular ring, hence $K_{-1}(R) = 0$. Therefore the exact sequence of the ideal terminates with $K_0(R) \to K_0(R/I) \to K_{-1}(I) \to 0$, and $K_{-1}(I)$ is free abelian of rank $k-1$.

Looking at the rest of the exact sequence (and using the fact that all negative K-groups of R must vanish) also shows that

$K_{-n}(R/I) \cong K_{-n-1}(I)$ for $n \geq 1$. We can use this fact to compute the rest of the negative K-groups of I. For instance, suppose m is square-free, *i.e.*, a product of distinct primes. Then R/I is a product of fields, hence is left regular, so all the negative K-groups of R/I must also vanish. Hence $K_{-n}(I) = 0$ for $n \geq 2$, though $K_{-1}(I)$ will be non-zero if $k > 1$.

If, say, $m = p^2$ with p prime, then R/I is local but **not** regular (see Exercise 3.1.24). Thus $K_{-1}(I) = 0$ from the exact sequence, but, at least a priori, R/I could have plenty of negative K-groups. However we have an exact sequence

$$0 \to \mathfrak{m} \to \mathbb{Z}/(p^2) \to \mathbb{F}_p \to 0,$$

where $\mathfrak{m} = p\mathbb{Z}/(p^2)$ is the maximal ideal of $\mathbb{Z}/(p^2)$. Since $\mathfrak{m}^2 = 0$, it is easy to see that for any n,

$$\mathfrak{m}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}] = \operatorname{rad} \mathbb{Z}/(p^2)[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}],$$

with quotient $\mathbb{F}_p[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. By the argument of Theorem 1.3.11 (see Exercise 3.3.6 below), the map

$$K_0\left(\mathbb{Z}/(p^2)[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]\right) \to K_0\left(\mathbb{F}_p[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]\right) \cong \mathbb{Z}$$

(by iterated use of Corollary 3.2.13)

is injective, hence all negative K-groups of R/I vanish. Hence $K_{-n}(I) = 0$ for all $n \geq 1$.

(b) Suppose G is a cyclic group of prime order p, say with generator t, and $R = \mathbb{Z}G$ is its integral group ring, which may be identified with $\mathbb{Z}[t]/(t^p - 1)$. If $\xi = e^{2\pi i/p}$, a primitive p-th root of unity, and if $S = \mathbb{Z}[\xi]$, then S is the ring of integers in the cyclotomic field $\mathbb{Q}(\xi)$, hence is a Dedekind domain by Theorem 1.4.18. There is a surjective homomorphism $R \twoheadrightarrow S$ defined by sending $t \mapsto \xi$. We have seen that the kernel I of the map $R \to S$ is, as a ring without unit, the same as in the last example if we specialize to the case $m = p$. Thus from the calculation in (a) above, $K_{-n}(I) = 0$ for all $n \geq 1$. From the exact sequence

$$K_0(R) \to K_0(S) \to K_{-1}(I) = 0,$$

we conclude that the map $K_0(R) \to K_0(S)$ is surjective. On the other hand, by Corollary 2.5.9, the map $K_0(R) \to K_0(S)$ is injective, so we conclude that $\tilde{K}_0(\mathbb{Z}G) \cong \tilde{K}_0(\mathbb{Z}[\xi])$. Thus the "Wall obstruction group" for G is trivial if and only if the cyclotomic field $\mathbb{Q}(\xi)$ has class number 1. It is known that his happens if and only if $p \leq 19$ [Washington, Ch. 11]. From the fact that S is a Dedekind domain, hence a regular ring, and from the exact sequence

$$0 = K_{-n}(I) \to K_{-n}(R) \to K_{-n}(S) = 0,$$

$K_{-n}(\mathbb{Z}G) = 0$ for $n > 0$.

It is perhaps worth mentioning a geometric application of negative K-theory. This involves the concept, which has been increasingly important in geometric topology during the last several years, of **topology with control**. For simplicity, we consider one of the simplest illustrations of this idea, as developed in [Pedersen]. Namely, we consider h-cobordisms W between two manifolds M and M' as in Theorem 2.4.4, but this time with a **control map** $p : W \to \mathbb{R}^k$. The control map p is required to be proper, and its restriction to either M or M' is required to be surjective. Of course, none of the manifolds W, M, or M' will be compact. We use the map p to measure "distances," that is, we define

$$\text{"dist"}\ (x, y) = |p(x) - p(y)|.$$

Then we require that W have **bounded fundamental group**, *i.e.*, that there be a fixed constant C such that for every $x, y \in W$, and for every homotopy class of paths from x to y, there be a representative for the class of length $< |p(x) - p(y)| + C$, and similarly that null-homotopic loops be contractible within a set of diameter $< C +$ the diameter of the loop. The result of [Pedersen] then gives a necessary and sufficient condition for a "bounded" h-cobordism W to have a bounded product structure, in terms of an invariant in $\tilde{K}_{-k+1}(\mathbb{Z}\pi_1(W))$, provided that $\dim W > 5$. (Here $\tilde{K}_{-k+1}$ refers to Wh if $k = 0$ and to K_{-k+1} if $k > 1$.) If $k = 0$, this reduces to the usual s-cobordism theorem (Theorem 2.4.4).

Of course, one way a bounded h-cobordism can arise is from a compact h-cobordism W' with fundamental group $\pi \times \mathbb{Z}^k$. The projection of the fundamental group onto $\mathbb{Z}^k$ induces a map $p' : W' \to T^k$, and taking coverings, we get a map

$$p = \widetilde{p'} : \widetilde{W'} = W \to \tilde{T}^k = \mathbb{R}^k.$$

Theorem 1.7 of [Pedersen] identifies the associated invariant as the image of the original Whitehead torsion in $\mathrm{Wh}(\pi \times \mathbb{Z}^k)$. But there are controlled non-compact problems that do not arise so simply from compact situations.

3.3.6. Exercise.

(1) Fill in the details of the argument copied from Theorem 1.3.11, that if S is a ring and if J is an ideal of R contained in $\operatorname{rad} R$, then the map $K_0(S) \to K_0(S/J)$ induced by the quotient map $S \to S/J$ is injective.

(2) Let R be a local ring, not necessarily commutative, and let $I = \operatorname{rad} R$. Assume $I^k = 0$ for some k. Let

$$S = R[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}], \quad J = I[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}].$$

Show that S/J is left regular and deduce that $NK_n(S/J) = 0$ for all $n \leq 1$ and that $K_{-n}(S/J) = 0$ for all $n > 0$.

(3) Conclude from (1) that the map $K_0(S) \to K_0(S/J)$ is injective. Deduce that $NK_{-n}(R) = 0$ for $n \geq 0$ and that $K_{-n}(R) = 0$ for $n > 0$.

3.3.7. Exercise. Use the results of the last exercise to show that for a finite product of local rings all of whose radicals are nilpotent, all negative K-groups must vanish. Apply this to the ring $\mathbb{Z}/(m)$ to complete the calculation of the negative K-groups of $(m) \subseteq \mathbb{Z}$ for an arbitary positive integer m.

3.3.8. Exercise [KaroubiAlgOp]. Let R be a complex Banach algebra (with unit), and observe that $C(S^1, R)$ is also a Banach algebra with pointwise multiplication of functions and with norm

$$\|f\| = \sup_{t \in S^1} \|f(t)\|.$$

We have an isometric inclusion $R \hookrightarrow C(S^1, R)$ as constant functions.

(1) For all $t \in S^1$, the evaluation map at t induces a retraction

$$C(S^1, R) \to R.$$

Show that the induced map on K_0 is independent of t, hence that $K_0(R)$ sits as a direct summand in $K_0(C(S^1, R))$ in a canonical way. (Use Corollary 1.6.11.) Define

$$K_{-1}^{\text{top}}(R) = \ker\left(K_0(C(S^1, R)) \to K_0(R)\right).$$

(2) There is a map $R[t, t^{-1}] \hookrightarrow C(S^1, R)$ obtained by viewing a Laurent polynomial as a function of $t \in S^1$ (identified with the unit circle in the complex plane). This induces a map $K_{-1}(R) \to K_{-1}^{\text{top}}(R)$.

Now if R is a Banach algebra, so is $M_n(R)$ for all n, so $GL(n, R) = (M_n(R))^\times$ is an open subset of $M_n(R)$ by Lemma 1.6.6. It therefore has a natural topology making it a locally contractible topological group. Let $GL(n, R)^0$ denote the connected component of the identity in $GL(n, R)$. This contains $E(n, R)$ since each elementary matrix $e_{ij}(a)$ is path-connected to the identity via the path $e_{ij}(ta)$, $0 \le t \le 1$. Let $GL(R)^0 = \varinjlim GL(n, R)^0$. Then this is a normal subgroup of $GL(R)$ and the quotient is abelian since $GL(R)^0 \supseteq E(R)$. It is customary to define $K_1^{\text{top}}(R) = GL(R)/GL(R)^0$.

Show (see [Blackadar], Theorem 8.2.2) that there is a functorial isomorphism $\theta : K_1^{\text{top}}(R) \xrightarrow{\cong} K_{-1}^{\text{top}}(R)$. This is constructed as follows.

(a) If $u \in GL(n, R)$, then $u \oplus u^{-1} \in E(2n, R) \subseteq GL(2n, R)^0$ (Corollary 2.1.3). Choose $z \in C\left([0, 2\pi], GL(2n, R)^0\right)$ with $z(0) = 1_{2n}$, $z(2\pi) = u \oplus u^{-1}$. Then let an idempotent $p \in C\left(S^1, M_{2n}(R)\right) = M_{2n}\left(C(S^1, R)\right)$ be defined by $p(e^{it}) = z(t)(1_n \oplus 0_n)z(t)^{-1}$. (This is indeed a continuous function of e^{it}, not just a function of t, since $z(0)$ and $z(2\pi)$ both commute with $1_n \oplus 0_n$.) Define

$$\theta([u]) = [p] - [(1_n \oplus 0_n)].$$

Since $p(1) = (1_n \oplus 0_n)$,

$$\theta([u]) \in \ker\left(K_0\left(C(S^1,\, R)\right) \to K_0\left(R\right)\right) = K_{-1}^{\mathrm{top}}(R).$$

First show that this is independent of all choices and gives a homomorphism with respect to the block sum operations $\oplus$ on K_1^{top} and on K_{-1}^{top}.

(b) Next, to prove injectivity of θ, suppose $\theta([u]) = 0$. Stabilizing u if necessary, reduce to the case where p is conjugate to $1_n \oplus 0_n$ in $GL(2n)$, say

$$h(e^{it})p(t)h(e^{it})^{-1} = \begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix}$$

for some $h \in C\left(S^1,\, GL(2n,\, R)\right)$. Then show $h(e^{it})z(t) = \begin{pmatrix} z_1(t) & 0 \\ 0 & z_2(t) \end{pmatrix}$ for some $z_1,\, z_2 \in C\left([0,\, 2\pi],\, GL(n,\, R)\right)$ and deduce that $u \in GL(n,\, R)^0$, so that $[u] = 0$ in $K_1^{\mathrm{top}}(R)$.

(c) Finally, to prove surjectivity of θ, show that every element of K_{-1}^{top} may be represented in the form $[p] - [(1_n \oplus 0_n)]$.

(3) Let $A = \mathbb{C}\langle s,\, s^{-1}\rangle$, the commutative Banach algebra of absolutely convergent Laurent series (with norm coming from $\ell^1(\mathbb{Z})$). There is a norm-decreasing homomorphism $A \to C(S^1)$ with dense image obtained by viewing a Laurent series as a function of $u \in S^1$ (identified with the unit circle in the complex plane). Show that $K_{-1}^{\mathrm{top}}(A) \cong \tilde{K}^0(S^1 \times S^1) \cong \mathbb{Z}$ and that the map $K_{-1}(A) \to K_{-1}^{\mathrm{top}}(A)$ is surjective.

(4) Let R be a complex C*-algebra, that is, a norm-closed subalgebra of the bounded operators $\mathcal{B}(\mathcal{H})$ on some complex Hilbert space $\mathcal{H}$ which is invariant under the involution $*$ sending an operator to its adjoint. Then if $b \in R^\times$, $b^*b \in R^\times$ and is strictly positive, so $|b| = (b^*b)^{\frac{1}{2}} \in R^\times$ (argue as in Lemma 1.6.6) and we have a polar decomposition $b = |b|u$ in R with u unitary. In particular, u and u^{-1} each have norm 1.

Now if R is a C*-algebra, so is $M_n(R)$ for each n. (If R acts on a Hilbert space $\mathcal{H}$, $M_n(R)$ acts on $\mathbb{C}^n \otimes \mathcal{H}$.) Show that if $b \in GL(n,\, R)$, there is a path joining $|b|$ to the identity in $GL(n,\, R)$, and hence the class of b in $K_1^{\mathrm{top}}(R)$ may be represented by the unitary $u = |b|^{-1}b$. Then show that with A as above, $s \mapsto u$ defines a continuous (in fact norm-decreasing) homomorphism $\varphi : A \to M_n(R)$ sending $[s]$, which generates $K_1^{\mathrm{top}}(A)$, to $[b] \in K_1^{\mathrm{top}}(R)$.

(5) From (2) and (3) above, from commutativity of the diagram

$$\begin{array}{ccccc} K_{-1}(A) & \longrightarrow & K_{-1}^{\mathrm{top}}(A) & \underset{\cong}{\overset{\theta}{\longleftarrow}} & K_1^{\mathrm{top}}(A) \\ \varphi_* \big\downarrow & & \varphi_* \big\downarrow & & \varphi_* \big\downarrow \\ K_{-1}(R) & \longrightarrow & K_{-1}^{\mathrm{top}}(R) & \underset{\cong}{\overset{\theta}{\longleftarrow}} & K_1^{\mathrm{top}}(R), \end{array}$$

and from the fact that φ can be chosen to have any desired class in $K_1^{\text{top}}(R)$ in its image, deduce that the map $K_{-1}(R) \to K_{-1}^{\text{top}}(R)$ is surjective.

(6) If X is a compact Hausdorff space, the algebra $R = C^{\mathbb{C}}(X)$ is a C*-algebra. (It may be represented, for instance, on a Hilbert space of the form $L^2(X, \mu)$, with μ a measure on X of full support.) Deduce that $K_{-1}(R) \twoheadrightarrow KU^{-1}(X)$. This provides many examples of commutative rings with complicated K_{-1}.

3.3.9. Exercise. (Cf. Exercise 3.2.27.)

(1) Show from Theorem 3.3.2 that if π is any group, then

$$\tilde{K}_0\left(\mathbb{Z}\left(\pi \times \mathbb{Z}\right)\right) \cong \tilde{K}_0(\mathbb{Z}\pi) \oplus K_{-1}(\mathbb{Z}\pi) \oplus \left(NK_0(\mathbb{Z}\pi)\right)^2.$$

(2) Deduce that the Wall obstruction group $\tilde{K}_0$ vanishes for free abelian groups but is non-zero for $\pi \times \mathbb{Z}$ if $K_{-1}(\mathbb{Z}\pi) \neq 0$. (An example of a finite abelian group with this property is given in the next Exercise.)

3.3.10. Exercise. Let G and H be finite cyclic groups of orders 2 and 3, respectively, so that $G \times H$ is cyclic of order 6. Note that $\mathbb{Z}(G \times H) = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}H$. From the exact sequences

$$0 \to 2\mathbb{Z} \to \mathbb{Z}G \leftrightarrows \mathbb{Z} \to 0,$$
$$0 \to 3\mathbb{Z} \to \mathbb{Z}H \to \mathbb{Z}[\omega] \to 0,$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$, deduce the exact sequences

$$0 \to \mathbb{Z}G \otimes_{\mathbb{Z}} 3\mathbb{Z} \to \mathbb{Z}(G \times H) \to \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}[\omega] \to 0,$$
$$0 \to 6\mathbb{Z} \to \mathbb{Z}G \otimes_{\mathbb{Z}} 3\mathbb{Z} \leftrightarrows 3\mathbb{Z} \to 0,$$
$$0 \to 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\omega] \to \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}[\omega] \leftrightarrows \mathbb{Z}[\omega] \to 0.$$

Note also that there is an exact sequence

$$0 \to 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\omega] \to \mathbb{Z}[\omega] \to \mathbb{F}_4 \to 0.$$

Compute from these and from the fact that $\mathbb{Z}[\omega]$ is a PID that $K_{-1}(\mathbb{Z}(G \times H)) \cong K_{-1}(6\mathbb{Z})$ is infinite cyclic and that $K_{-n}(\mathbb{Z}(G \times H)) = 0$ for $n \geq 2$.

The groups $K_{-n}(\mathbb{Z}\pi)$ have been computed for arbitrary finite groups π by Carter [Carter], and it turns out that $K_{-n}(\mathbb{Z}\pi)$ is always finitely generated for $n = 1$ and vanishes for $n \geq 2$. Furthermore, torsion can occur in K_{-1}, but it is always of exponent 2.

4

Milnor's K_2

1. Universal central extensions and H_2

For the reader who might have been alarmed by the category-theoretic approach of the last chapter, this chapter, which discusses Milnor's K_2 functor, will seem a comforting retreat to more familiar territory. However, we will need to refer to the homology of a group, at least in order to speak of H_2. Since group homology will be needed in a more serious way in the next chapter anyway, we provide a brief introduction to the subject later in this section. The reader who wants a more serious approach to the homology theory of groups and its applications should consult a source such as [Brown] or [CartanEilenberg].

First, though, we begin with the theory of universal central extensions, as developed in [Kervaire2] and [Milnor, §5]. This a cute and fairly self-contained topic in group theory, but it's hard to see at first what it has to do with K-theory. Roughly speaking, the idea here is that K-theory for rings is supposed to measure "abelian" invariants of the highly non-commutative group $GL(R)$. For example, $K_1(R)$ is defined by taking the abelianization of $GL(R)$, in other words, the quotient of this group by its commutator subgroup $E(R)$. Since $E(R)$ is its own commutator subgroup (Proposition 2.1.4), repeating this process with $E(R)$ doesn't yield anything. However, the deep structure of linear algebra over R should be connected with the deep structure of the group $E(R)$, in other words the relations satisfied by its generators $e_{ij}(a)$. One way of measuring these is by looking at extensions of $E(R)$ by abelian groups. There turns out to be a universal such extension $\mathrm{St}(R)$, and the (abelian) kernel of the map $\mathrm{St}(R) \twoheadrightarrow E(R)$ is Milnor's $K_2(R)$. Even when R is a field, this turns out to be an interesting invariant with lots of number-theoretic significance. But since the number-theoretic applications of K_2 are described quite nicely in [Milnor], we have only touched on the most important of these and have chosen to emphasize some other applications instead.

Universal Central Extensions.

4.1.1. Definition. Let G be any group, and let A be an abelian group. A **central extension** of G by A is a pair (E, φ), where E is a group containing A as a central subgroup, and $\varphi : E \twoheadrightarrow G$ is a surjective homomorphism whose kernel is exactly A. Alternatively, in the language of exact sequences, a central extension of G by A is a short exact sequence

$$1 \to A \to E \xrightarrow{\varphi} G \to 1$$

with A central in E.

Remark. There are still those who call the above a central extension of A by G, but the terminology above is more in keeping with the formalism of group cohomology, since it turns out that the central extensions of G by A are classified by $H^2(G, A)$ (and everyone agrees that here one should put the G before the A!).

4.1.2. Definition. Next we note that central extensions of G (by arbitrary abelian groups) form a category. If (E, φ) and (E', φ') are central extensions of (the same group) G, a **morphism** of central extensions $(E, \varphi) \to (E', \varphi')$ is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & G \\ {\scriptstyle\psi}\downarrow & & \| \\ E' & \xrightarrow{\varphi'} & G. \end{array}$$

A central extension (E, φ) of G by A is called **trivial** if it is isomorphic in the category of central extensions of G to $G \times A \xrightarrow{p_1} G$, where p_1 is projection on the first factor. A central extension (E, φ) of G is called **universal** if, for any other central extension (E', φ') of G, there is a **unique** morphism $(E, \varphi) \to (E', \varphi')$. Not every group has a universal central extension, but if it has one, then it is clear from the definition that any two universal central extensions must be isomorphic (in the category of central extensions of G).

4.1.3. Theorem. *A group G has a universal central extension if and only if it is* **perfect**, *that is, $G = [G, G]$. When this is the case, a central extension (E, φ) of G is universal if and only if the following two conditions hold:*

(*i*) *E is perfect, that is, $E = [E, E]$, and*
(*ii*) *all central extensions of E are trivial.*

(Roughly speaking, condition (i) says that E is not too big, and condition (ii) says that it is not too small.)

If G is perfect and

$$1 \to R \to F \to G \to 1$$

is a presentation of G (i.e., a short exact sequence with F a free group), then the universal central extension (E, φ) may be constructed as $E = [F, F]/[F, R]$, with φ the quotient map

$$[F, F]/[F, R] \twoheadrightarrow [F, F]/R = [F/R, F/R] = [G, G] = G.$$

Proof. If G is not perfect, it means it has a non-trivial abelian quotient, say A. Let $\psi : G \to A$ be the quotient map. Now if (E, φ) is a central extension of G, we can construct two distinct morphisms from (E, φ) to the trivial extension $G \times A \xrightarrow{p_1} G$, namely $(\varphi, 1)$ and $(\varphi, \psi \circ \varphi)$ (here $1 : E \to A$ is the trivial map sending everything in E to the identity of A). This shows that (E, φ) cannot be universal. Hence, for G to have a universal central extension, G must be perfect.

Now assume G is perfect with presentation

$$1 \to R \to F \to G \to 1.$$

We will show first that any central extension of G satisfying (i) and (ii) is universal. Then to complete the proof, we will show that

$$[F, F]/[F, R] \twoheadrightarrow [F, F]/R = G$$

is a central extension satisfying (i) and (ii). This will show in particular that every perfect group has a universal central extension, and since universal central extensions are unique and non-perfect groups do not have universal central extensions, every universal central extension must satisfy (i) and (ii).

So assume (E, φ) is a central extension of the perfect group G satisfying (i) and (ii), and let (E', φ') be any other central extension of G. Suppose $\psi, \psi' : (E, \varphi) \to (E', \varphi')$ are two morphisms of central extensions. For $x \in E$, $\varphi' \circ \psi(x) = \varphi(x) = \varphi' \circ \psi'(x)$, hence $\psi(x) = c_x\psi'(x)$ for some element c_x of the kernel A' of $\varphi' : E \to G$. Similarly, if $y \in E$, then $\psi(y) = c_y\psi'(y)$ for some $c_y \in A'$. So

$$\psi([x, y]) = [\psi(x), \psi(y)] = [c_x\psi'(x), c_y\psi'(y)] = [\psi'(x), \psi'(y)] = \psi'([x, y]).$$

(In this calculation we have used the fact that A' is central in E'.) Hence ψ and ψ' coincide on commutators. Since $E = [E, E]$ by (i), ψ and ψ' coincide on all of E. This shows there can be at most one morphism from E to E'.

We still need to construct a morphism from E to E'. Let $E'' = E \times_G E'$, that is,

$$E'' = \{(x, y) \in E \times E' : \varphi(x) = \varphi'(y)\}.$$

Since φ and φ' are surjective, projection p_1 on the first factor is a surjective homomorphism $E'' \twoheadrightarrow E$. The kernel of p_1 is obviously isomorphic to the

kernel A' of φ', and so is central. So $p_1 : E'' \twoheadrightarrow E$ is a central extension of E. By (ii), this central extension is trivial, which says that there is a homomorphism $E \to E'$ commuting with the projections onto G. This means there is a morphism of central extensions from E to E'. Since E' was arbitrary and we already showed that morphisms from E to E' are unique, thus E is a universal central extension of G.

Finally, let $E = [F, F]/[F, R]$, with φ the quotient map

$$[F, F]/[F, R] \twoheadrightarrow [F, F]/R = [F/R, F/R] = [G, G] = G.$$

To begin with, note that $E \subseteq E_1 = F/[F, R]$, which also projects onto G via the quotient map $\varphi_1 : F/[F, R] \twoheadrightarrow F/R = G$, and φ is the restriction of φ_1 to E. (Since $R \trianglelefteq F$, $[F, R] \subseteq R$, and $[F, R]$ is also normal in F.) Note that the kernel of φ_1 is contained in $R/[F, R]$, hence commutators of elements of the kernel with elements of E_1 lie in $[F, R]/[F, R]$, which is trivial. Thus the kernel of φ_1 is central in E_1 and (E, φ), (E_1, φ_1) are central extensions of G. We need to verify properties (i) and (ii) for E.

Directly from the definition of E and E_1, we see that E_1 has E as its commutator subgroup. On the other hand, since φ and φ_1 are both surjective onto G, E_1 is generated by E together with the kernel of φ_1, which is central. So

$$E = [E_1, E_1] = [E \cdot Z(E_1), E \cdot Z(E_1)] = [E, E]$$

and E is perfect. (Here, as usual, $Z(E_1)$ denotes the center of E_1; the letter Z comes from the German *Zentrum.*) This proves (i).

As far as (ii) is concerned, let

$$1 \to A \to E_2 \xrightarrow{\psi} E \to 1$$

be any central extension of E. This induces an extension $E_3 = E_1 \times_G E_2 \xrightarrow{p_1} E_1$ of E_1, where

$$E_1 \times_G E_2 = \{(x, y) \in E_1 \times E_2 : \varphi_1(x) = \varphi \circ \psi(y)\}.$$

This is actually a central extension. Indeed, the kernel of $p_1 : E_3 \to E_1$ is clearly isomorphic to the kernel of $\varphi \circ \psi : E_2 \to G$. But since $E = [E, E]$, $\psi([E_2, E_2]) = [\psi(E_2), \psi(E_2)] = [E, E] = E$, and thus $E_2 = [E_2, E_2] \cdot A$. Also, $\psi([E_2, \ker \varphi \circ \psi]) \subseteq [E, \ker \varphi] = 1$, so $[E_2, \ker \varphi \circ \psi] \subseteq A$. Thus for $x \in \ker \varphi \circ \psi$ and $s, t \in E_2$, $xsx^{-1} = sz_1$ and $xtx^{-1} = tz_2$ for some central z_1 and z_2, and $x[s, t]x^{-1} = [xsx^{-1}, xtx^{-1}] = [sz_1, tz_2] = [s, t]$. Thus x commutes with $[E_2, E_2]$. Since x also commutes with A (A is central), it commutes with all of E_2, and E_3 is a central extension of E_1.

Since F is free, we can fill in the diagram

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ E_3 = E_1 \times_G E_2 & \xrightarrow{p_1} & E_1 \end{array}$$

to get a homomorphism $F \to E_3$ lifting the quotient map $F \twoheadrightarrow E_1$. This amounts to a homomorphism $\theta : F \to E_2$ such that for $x \in F$, $\varphi \circ \psi(\theta(x))$ coincides with the image of x in $G \cong F/R$. So $\theta(R) \subseteq \ker \varphi \circ \psi \subseteq Z(E_2)$, and

$$\theta([F, R]) \subseteq [\theta(F), \theta(R)] \subseteq [E_2, Z(E_2)] = 1.$$

Hence θ descends to a map $\tilde{\theta} : F/[F, R] = E_1 \to E_2$ which, together with the identity map on E_1, gives a splitting $E_1 \to E_3 = E_1 \times_G E_2$ of p_1. Restricting to E then gives a trivialization of $\psi : E_2 \to E$, verifying (ii). Thus we have constructed a universal central extension of G. □

4.1.4. Remark. Of the conditions in Theorem 4.1.3 for checking when one has a universal central extension, (i) is fairly straightforward, but (ii) is rather difficult to check (without using machinery from group homology theory, which will make it possible to restate the condition in the form $H_2(E, \mathbb{Z}) = 0$). However, the proof of Theorem 4.1.3 gives the following additional piece of information which is sometimes useful. Suppose G is a perfect group and (E', φ') is a central extension of G satisfying (i). Then if (E, φ) denotes the universal central extension of G, there is a unique morphism ψ of central extensions from E to E', and ψ must map E **onto** E' and the abelian group $\ker \varphi$ **onto** the abelian group $\ker \varphi'$. Thus condition (i) (without condition (ii)) at least guarantees that one has a quotient of the universal central extension.

To see this, note that since $E' = [E', E']$, to prove surjectivity of ψ, it's enough to show that every commutator is in the image. Let $x', y' \in E'$. Then we can choose $x, y \in E$ such that $\varphi(x) = \varphi'(x')$, $\varphi(y) = \varphi'(y')$, and it follows from the relation $\varphi = \varphi' \circ \psi$ that $x' = \psi(x)z_1$, $y' = \psi(y)z_2$ for some $z_1, z_2 \in \ker \varphi'$. Since z_1 and z_2 are central,

$$[x', y'] = [\psi(x)z_1, \psi(y)z_2] = [\psi(x), \psi(y)] = \psi([x, y]),$$

and thus ψ is surjective. Furthermore, if $z' \in \ker \varphi'$ and $\psi(z) = z'$, then $\varphi' \circ \psi(z) = 1$, hence $\varphi(z) = 1$, which shows that $\psi^{-1}(\ker \varphi') \subseteq \ker \varphi$. The other inclusion is trivial. □

4.1.5. Examples. Let G and $\tilde{G}$ be connected Hausdorff topological groups and $p : \tilde{G} \to G$ a continuous surjective homomorphism with discrete kernel $D \subseteq \tilde{G}$. Since D is normal in $\tilde{G}$, for each $d \in D$, we have a continuous map $g \mapsto gdg^{-1}$ from $\tilde{G}$ to D. Since $\tilde{G}$ is connected and D is discrete, this map must be constant, and thus $gdg^{-1} = d$ for all $d \in D$, $g \in \tilde{G}$, *i.e.*, D is central. So $\tilde{G}$ is a central extension of G. If $\tilde{G}$ is also perfect, which is easy to check in many examples, then it is a quotient of the universal central extension.

In particular, let's take $\tilde{G} = SU(2)$, the group of matrices $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. G will be $SO(3)$, the group of rotations (orthogonal linear transformations of determinant 1) of Euclidean 3-space. $\tilde{G}$ and G are clearly connected and Hausdorff; in fact $\tilde{G}$ is clearly homeomorphic to the unit sphere in $\mathbb{C}^2$, or to S^3, which is **simply** connected.

The **Lie algebra** of $\tilde{G}$ is defined to be the 3-dimensional real vector space

$$\begin{aligned}\tilde{\mathfrak{g}} &= \{X \in M_2(\mathbb{C}) : \bar{X}^t = -X,\ \operatorname{Tr} X = 0\}\\ &= \left\{\begin{pmatrix} ix & y+iz \\ -y+iz & -ix\end{pmatrix} : x,\ y,\ z \in \mathbb{R}\right\}.\end{aligned}$$

We can make this into a Euclidean space via the inner product

$$\langle X,\ Y\rangle = -\operatorname{Tr}(XY),$$

since this pairing is clearly symmetric and bilinear, and it's positive-definite since for $X \in \tilde{\mathfrak{g}}$,

$$\langle X,\ X\rangle = -\operatorname{Tr}(XX) = \operatorname{Tr}\left(X \cdot (\bar{X}^t)\right) \geq 0,$$

with equality only if $X = 0$. Note that $\tilde{G}$ acts on $\tilde{\mathfrak{g}}$ by conjugation, since if $g \in \tilde{G}$ and $X \in \tilde{\mathfrak{g}}$,

$$\overline{(gXg^{-1})}^t = (\bar{g}\bar{X}\bar{g}^{-1})^t = (\bar{g}^t)^{-1}(\bar{X}^t)(\bar{g}^t) = g \cdot (-X) \cdot g^{-1} = -(gXg^{-1}).$$

Furthermore, $\tilde{G}$ preserves the inner product on $\tilde{\mathfrak{g}}$ since

$$\begin{aligned}\langle gXg^{-1},\ gYg^{-1}\rangle &= -\operatorname{Tr}(gXg^{-1}gYg^{-1}) = -\operatorname{Tr}(gXYg^{-1})\\ &= -\operatorname{Tr}(XY) = \langle X,\ Y\rangle.\end{aligned}$$

So we obtain a homomorphism p from $\tilde{G}$ to the connected component of the identity in the orthogonal group of $\tilde{\mathfrak{g}}$, in other words, if we identify $\tilde{\mathfrak{g}}$ with $\mathbb{R}^3$, to G. This homomorphism is easily seen to be surjective. Its kernel D consists of matrices g which commute with everything in $\tilde{\mathfrak{g}}$, and thus with everything in

$$\tilde{\mathfrak{g}} + i\tilde{\mathfrak{g}} = \{X \in M_2(\mathbb{C}) : \operatorname{Tr} X = 0\},$$

and thus with all of $M_2(\mathbb{C})$. (Any matrix differs by a scalar multiple of the identity from a traceless matrix.) So

$$D = \tilde{G} \cap \{\text{scalar matrices in } M_2(\mathbb{C})\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix},\ \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\right\} \cong \mathbb{Z}/2,$$

and thus $\tilde{G}$ is a central extension of $SO(3)$ by $\mathbb{Z}/2$. However, $\tilde{G}$ is perfect, since the Implicit Function Theorem shows that the image of the commutator map $\tilde{G} \times \tilde{G} \to \tilde{G}$, $(g,\ h) \mapsto ghg^{-1}h^{-1}$, contains an open neighborhood of the identity, and thus generates $\tilde{G}$. So $\tilde{G}$ is a quotient of the universal central extension of G.

There are many other "naturally occurring" pairs $(\tilde{G},\ G)$ giving quotients of the universal central extensions of matrix groups. For instance (this example will be important later), let $G = SL(n,\ \mathbb{R}) = E(n,\ \mathbb{R})$, which

is a perfect connected matrix group for any $n \geq 2$. Then because of polar decomposition, $G = SO(n) \cdot \exp \mathfrak{s}_n$, where $\mathfrak{s}_n$ is the vector space of $n \times n$ symmetric matrices with trace zero, and so G has the rotation group $SO(n)$ as a deformation retract. In particular,

$$\pi_1(G) \cong \pi_1(SO(n)) \cong \begin{cases} \mathbb{Z} & \text{for } n = 2 \\ \mathbb{Z}/2 & \text{for } n \geq 3. \end{cases}$$

Thus for $n \geq 3$, the universal covering group $\tilde{G}$ of G is a non-trivial double cover, and is again perfect (since the commutator subgroup contains an open neighborhood of the identity, and thus is all of $\tilde{G}$). Incidentally, one can show that the topological group $\tilde{G}$ **cannot** itself be realized as a group of real or complex matrices. Thus the universal central extension of $SL(\mathbb{R})$ is non-trivial, and the kernel of this universal central extension has $\mathbb{Z}/2$ as a quotient group. Later we will come back to this from the point of view of $K_2(\mathbb{R})$.

Homology of Groups. Next we give a quick introduction to the homology of groups, which is an essential tool both for translating the theory of universal central extensions into something computable and for defining and understanding the higher K-groups which will appear in Chapter 5.

4.1.6. Definition. Let G be any group. A **(left) G-module** M is an abelian group equipped with a (left) action of G by automorphisms, satisfying the usual conditions $g \cdot (h \cdot m) = (gh) \cdot m$, $1 \cdot m = m$ for $g, h \in G$, $m \in M$, or equivalently, a left R-module, where $R = \mathbb{Z}G$ is the group ring of G. Note that any such M may also be made into a **right G-module** if we define $m \tilde{\cdot} g =_{\text{def}} g^{-1} \cdot m$, though there is one source of possible confusion: if G is abelian, then R is a commutative ring, and we customarily make any left module for such a ring into a right module by defining $m \cdot r =_{\text{def}} r \cdot m$, and this convention disagrees with the previous one. When it makes a difference, we will specify which right action we are using. We will denote by $\mathbb{Z}$ the trivial G-module which is $\mathbb{Z}$ as an abelian group and with $g \cdot n = n$ for all $g \in G$, $n \in \mathbb{Z}$. (In this case, when G is abelian, the two right actions agree.)

4.1.7. Definition. Let G be any group, M a G-module. We define the **homology and cohomology groups of G with coefficients in M** as follows. First note that we can construct a free resolution of the trivial G-module $\mathbb{Z}$ by letting $P_j = \mathbb{Z}G^{j+1}$, the free abelian group on the $(j+1)$-st Cartesian product of G with itself, for $j \geq 0$, and defining boundary maps $d_j : P_j \to P_{j-1}$, $j \geq 1$, by

$$d_j(g_0, g_1, \ldots, g_j) = \sum_{k=0}^{j} (-1)^k (g_0, g_1, \ldots, \widehat{g_k}, \cdots, g_j),$$

where $\widehat{g_k}$ indicates that g_k is omitted. The augmentation $\varepsilon : P_0 \to \mathbb{Z}$ is defined by $g \mapsto 1$, and it is a trivial calculation to check that

$$\cdots \xrightarrow{d_{j+1}} P_j = \mathbb{Z}G^{j+1} \xrightarrow{d_j} P_{j-1} = \mathbb{Z}G^j \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} P_0 = \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \tag{4.1.8}$$

is exact. Furthermore, the boundary maps d_j are clearly both left and right G-module maps for the diagonal actions

$$g \cdot (g_0, g_1, \ldots, g_j) = (gg_0, gg_1, \ldots, gg_j),$$

$$(g_0, g_1, \ldots, g_j) \cdot g = (g_0g, g_1g, \ldots, g_jg).$$

(Note that the right action used here is the same as the left action if G is abelian, and is in general **not** the same as the "flipped left action" $m \tilde{\cdot} g = g^{-1} \cdot m$.)

The **homology groups of G with coefficients in M** are the homology groups of the complex

$$C_\bullet(G, M) = P_\bullet \otimes_{\mathbb{Z}G} M \cong \mathbb{Z}G^\bullet \otimes_{\mathbb{Z}} M,$$

and the **cohomology groups of G with coefficients in M** are the homology groups of the dual complex

$$C^\bullet(G, M) = \mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G^\bullet, M).$$

(Here we are always using the **left** module structure on M. To identify $P_j \otimes_{\mathbb{Z}G} M$ with $\mathbb{Z}G^j \otimes_{\mathbb{Z}} M$, we view P_j as the free right $\mathbb{Z}G$-module on generators $(1, g_1, \ldots, g_j)$. To identify $\mathrm{Hom}_{\mathbb{Z}G}(P_j, M)$ with $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G^j, M)$, we let $F \in \mathrm{Hom}_{\mathbb{Z}G}(P_j, M)$ correspond to $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G^j, M)$, where

$$f(g_1, \ldots, g_j) = F(1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n),$$
$$F(g_0, \ldots, g_j) = g_0 \cdot f(g_0^{-1}g_1, g_1^{-1}g_2, \ldots, g_{j-1}^{-1}g_j)\,.)$$

The lowest-dimensional parts of these complexes are of particular interest, so we write them out explicitly:

$$C_\bullet(G, M): \qquad \cdots \xrightarrow{d_3} \mathbb{Z}G^2 \otimes_{\mathbb{Z}} M \xrightarrow{d_2} \mathbb{Z}G \otimes_{\mathbb{Z}} M \xrightarrow{d_1: g\otimes m \mapsto g\cdot m - m} M,$$

$$d_n((g_1, \ldots, g_n) \otimes m) = (g_2 g_1^{-1}, \ldots, g_n g_1^{-1}) \otimes g_1 \cdot m + \sum_{j=1}^{n} (-1)^j (g_1, \ldots, \widehat{g_j} \ldots, g_n) \otimes m,$$

$$C^\bullet(G, M): \qquad M \xrightarrow{d^0 : d^0 m(g) = g\cdot m - m} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M) \xrightarrow{d^1} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G^2, M) \xrightarrow{d^2} \cdots,$$

$$d^n f(g_0, \ldots, g_n) = g_0 \cdot f(g_1, \ldots, g_n) + \sum_{j=1}^{n} (-1)^j f(g_0, \ldots, g_{j-1}g_j, \ldots, g_n) + (-1)^{n+1} f(g_0, \ldots, g_{n-1}).$$

Note that when the G-action on M is trivial, $d_1 \equiv 0$ and $d^0 \equiv 0$, so $H_0(G, M) = H^0(G, M) = M$. Also in this case, we have

$$d_2((g_1, g_2) \otimes m) = \left(g_2 g_1^{-1} - g_2 + g_1\right) \otimes m,$$

so $H_1(G, M) = \mathbb{Z}G \otimes_{\mathbb{Z}} M / \operatorname{im} d_2 = G_{\text{ab}} \otimes_{\mathbb{Z}} M$, while $H^1(G, M)$ is simply $\operatorname{Hom}(G, M)$.

More generally, when the action of G on M can be arbitrary, we see that $H_0(G, M) = M_G$, the quotient of M by the submodule generated by the elements $g \cdot m - m$, $g \in G$, $m \in M$. Similarly, $H^0(G, M) = M^G$, the elements of M left fixed by all elements of G. The 1-cocycles $f \in Z^1(G, M)$ are functions $f : G \to M$ such that $g_0 \cdot f(g_1) - f(g_0 g_1) + f(g_0) = 0$ for all $g_0, g_1 \in G$, or such that $f(g_0 g_1) = f(g_0) + g_0 \cdot f(g_1)$. The 1-coboundaries are those of the special sort $f(g) = g\cdot m - m$ for some $m \in M$; $H^1(G, M)$ is the quotient of $Z^1(G, M)$ by this subgroup. Similarly, 2-cocycles $f \in Z^2(G, M)$ are functions $f : G \times G \to M$ such that $g_0 \cdot f(g_1, g_2) - f(g_0 g_1, g_2) + f(g_0, g_1 g_2) - f(g_0, g_1) = 0$ for all $g_0, g_1, g_2 \in G$.

While we will not develop that much of the homology theory of groups, we present at least a few tools for computing homology and cohomology and develop the relationship between H_2 and H^2 and the theory of central extensions. The first basic facts are contained in the following proposition.

4.1.9. Proposition. *Let G be a group. For each $k \geq 0$, $M \rightsquigarrow H_k(G, M)$ and $M \rightsquigarrow H^k(G, M)$ are (covariant) functors on the category of G-modules. If M is a projective G-module, then $H_k(G, M) = 0$ for all $k > 0$. Similarly, if M is an injective G-module, then $H^k(G, M) = 0$ for all $k > 0$. If*

$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$$

is a short exact sequence of G modules, there are associated long exact sequences

$$\ldots \xrightarrow{\beta_*} H_{k+1}(G, M_3) \xrightarrow{\partial} H_k(G, M_1) \xrightarrow{\alpha_*} H_k(G, M_2) \xrightarrow{\beta_*} H_k(G, M_3) \xrightarrow{\partial} H_{k-1}(G, M_1) \xrightarrow{\alpha_*} \ldots$$

and

$$\cdots \xrightarrow{\beta_*} H^{k-1}(G, M_3) \xrightarrow{\partial} H^k(G, M_1) \xrightarrow{\alpha_*} H^k(G, M_2)$$
$$\xrightarrow{\beta_*} H^k(G, M_3) \xrightarrow{\partial} H^{k^1}(G, M_1) \xrightarrow{\alpha_*} \cdots .$$

Proof. It is obvious from the definition that homology and cohomology are functorial. If M is a free $\mathbb{Z}G$-module, with a free basis indexed by a set I, then for any right $\mathbb{Z}G$-module N, $N \otimes_{\mathbb{Z}G} M$ is naturally isomorphic to N^I, and so tensoring with M preserves exactness. A similar argument applies if M is a direct summand in a free module. Thus, if M is projective, since the complex $P_\bullet$ is exact, so is $P_\bullet \otimes_{\mathbb{Z}G} M$, and so $H_k(G, M) = 0$ for all $k > 0$. Similarly, if M is injective, then $\mathrm{Hom}_{\mathbb{Z}G}(\cdot, M)$ preserves exactness and so $\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, M)$ is exact, so that $H^k(G, M) = 0$ for all $k > 0$. The statement about long exact sequences follows immediately from Theorem 1.7.6 (the Fundamental Theorem of Homological Algebra), since a short exact sequence of G-modules yields short exact sequences

$$0 \to P_\bullet \otimes_{\mathbb{Z}G} M_1 \xrightarrow{\alpha_*} P_\bullet \otimes_{\mathbb{Z}G} M_2 \xrightarrow{\beta_*} P_\bullet \otimes_{\mathbb{Z}G} M_3 \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, M_1) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, M_2) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, M_3) \to 0$$

of chain complexes. □

4.1.10. Corollary. *If G is a group and M is a G-module, then homology of M can be computed from a projective resolution of M, while cohomology can be computed from an injective resolution. More precisely, if*

$$\cdots \xrightarrow{d_3} N_2 \xrightarrow{d_2} N_1 \xrightarrow{d_1} N_0 \to M \to 0$$

is exact and each N_j is G-projective, then $H_\bullet(G, M)$ is the homology of the complex $\mathbb{Z} \otimes_{\mathbb{Z}G} N_\bullet$. Similarly, if

$$0 \to M \to N_0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \cdots$$

is exact and each N_j is G-injective, then $H^\bullet(G, M)$ is the homology of the complex $N_\bullet^G$.

Proof. This follows by iteration, splitting the resolution into a series of short exact sequences and using Proposition 4.1.9 over and over again. For instance, consider the case of a projective resolution $N_\bullet$ of M. First consider the short exact sequence

$$0 \to N_1/(\mathrm{im}\, d_2) \xrightarrow{d_1} N_0 \to M \to 0.$$

From this we obtain a long exact sequence of homology groups, but since N_0 is projective, $H_k(G, N_0) = 0$ for $k > 0$. Thus

$$H_k(G, M) \cong H_{k-1}(G, N_1/(\operatorname{im} d_2))$$

for $k \geq 2$, and similarly there is an exact sequence

$$0 \to H_1(G, M) \to H_0(G, N_1/(\operatorname{im} d_2)) \xrightarrow{(d_1)_*} H_0(G, N_0) \to H_0(G, M) \to 0.$$

On the other hand, we have an exact sequence

$$0 \to N_2/(\operatorname{im} d_3) \xrightarrow{d_2} N_1 \to N_1/(\operatorname{im} d_2) \to 0, \tag{4.1.11}$$

and N_1 is projective. Repeating the argument,

$$H_k(G, M) \cong H_{k-1}(G, N_1/(\operatorname{im} d_2)) \cong H_{k-2}(G, N_2/(\operatorname{im} d_3))$$

for $k \geq 3$, and we obtain an exact sequence

$$0 \to H_2(G, M) \cong H_1(G, N_1/(\operatorname{im} d_2)) \to H_0(G, N_2/(\operatorname{im} d_3)) \xrightarrow{(d_2)_*} H_0(G, N_1) \to H_0(G, N_1/(\operatorname{im} d_2)) \to 0.$$

Putting this together with (4.1.11), we see that $H_1(G, M)$ and $H_0(G, M)$ are the lowest-degree homology groups of the complex $H_0(G, N_\bullet) \cong \mathbb{Z} \otimes_{\mathbb{Z}G} N_\bullet$. Then we continue inductively to compute $H_2(G, M)$, and so on. □

For future applications, the following easy consequence of Corollary 4.1.10 is often useful.

4.1.12. Corollary ("Shapiro's Lemma"). *Let G be a group and let $H \subseteq G$ be a subgroup of G. Let M be an H-module. Then there are natural isomorphisms*

$$H_j(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong H_j(H, M)$$

for all j.

Proof. Choose a $\mathbb{Z}H$-projective resolution $N_\bullet$ of M. By Corollary 4.1.10, $H_\bullet(H, M)$ is the homology of the complex $\mathbb{Z} \otimes_{\mathbb{Z}H} N_\bullet$. However, if g_i is a set of representatives for the right H-cosets in G, then $\mathbb{Z}G$ is a free right $\mathbb{Z}H$-module with basis g_i, so $\mathbb{Z}G \otimes_{\mathbb{Z}H} N_\bullet = \bigoplus_i g_i \otimes N_\bullet$ is a $\mathbb{Z}G$-projective resolution of $\mathbb{Z}G \otimes_{\mathbb{Z}H} M$. Thus $H_\bullet(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M)$ is the homology of the complex

$$\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}H} N_\bullet \cong \mathbb{Z} \otimes_{\mathbb{Z}H} N_\bullet,$$

which proves the result. □

For purposes of studying central extensions, we will be particularly interested in homology and cohomology of G-modules with trivial G-action. These are related by the following.

4.1.13. Theorem ("Universal Coefficient Theorem"). *Let G be a group and let M be an abelian group, viewed as a G-module with trivial G-action. Then there are short exact sequences*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}\left(H_{k-1}\left(G,\,\mathbb{Z}\right),\,M\right) \to H^k(G,\,M) \to \mathrm{Hom}_{\mathbb{Z}}\left(H_k\left(G,\,\mathbb{Z}\right),\,M\right) \to 0$$

for all k, which split (though not in a natural way). In particular,

$$H^2\left(G,\,M\right) = 0 \text{ for all } G\text{-modules } M \text{ with trivial } G\text{-action}$$

if and only if the abelianization of G is free (abelian) and $H_2\left(G,\,\mathbb{Z}\right) = 0$.

Proof. Recall that $H_\bullet = H_\bullet\left(G,\,\mathbb{Z}\right)$ was defined to be the homology of $P_\bullet \otimes_{\mathbb{Z}G} \mathbb{Z}$. However, since $P_\bullet$ is a G-projective resolution of $\mathbb{Z}$, by Corollary 4.1.10 we can also compute it as the homology of the complex $C_\bullet = \mathbb{Z} \otimes_{\mathbb{Z}G} P_\bullet \cong \mathbb{Z}G^\bullet$. On the other hand, $H^\bullet(G,\,M)$ is the homology of $\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet,\,M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G^\bullet,\,M) \cong \mathrm{Hom}_{\mathbb{Z}}(C_\bullet,\,M)$ with the dual differential. Let $Z_k = \ker(C_k \xrightarrow{d_k} C_{k-1})$ and $B_{k-1} = \mathrm{im}(C_k \xrightarrow{d_k} C_{k-1})$, so that $H_k(G,\,\mathbb{Z}) = Z_k/B_k$. Note that C_k is a free abelian group; hence its subgroups Z_k and B_k are also free. We have short exact sequences

$$0 \to Z_k \to C_k \xrightarrow{d_k} B_{k-1} \to 0$$

and

$$0 \to B_k \to Z_k \to H_k \to 0.$$

Since B_{k-1} is free abelian, the first of these splits, and so goes under the functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot,\,M)$ to a (split) short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(B_{k-1},\,M) \xrightarrow{d^*_{k-1}} \mathrm{Hom}_{\mathbb{Z}}(C_k,\,M) \to \mathrm{Hom}_{\mathbb{Z}}(Z_k,\,M) \to 0. \tag{4.1.14}$$

Exactness of the second is not preserved in general, but from the definition of Ext there is an exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_{\mathbb{Z}}(H_k,\,M) &\to \mathrm{Hom}_{\mathbb{Z}}(Z_k,\,M)\\ &\to \mathrm{Hom}_{\mathbb{Z}}(B_k,\,M) \to \mathrm{Ext}^1_{\mathbb{Z}}(H_k,\,M) \to 0.\end{aligned} \tag{4.1.15}$$

Now note that (4.1.14) may be viewed as giving a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(B_{\bullet-1},\,M) \to \mathrm{Hom}_{\mathbb{Z}}(C_\bullet,\,M) \to \mathrm{Hom}_{\mathbb{Z}}(Z_\bullet,\,M) \to 0,$$

where the outside chain complexes have vanishing differentials. So from the Fundamental Theorem of Homological Algebra, we obtain a long exact homology sequence

$$\begin{aligned}\cdots \to \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1},\,M) &\to \mathrm{Hom}_{\mathbb{Z}}(B_{k-1},\,M)\\ &\to H^k(G,\,M) \to \mathrm{Hom}_{\mathbb{Z}}(Z_k,\,M) \to \mathrm{Hom}_{\mathbb{Z}}(B_k,\,M) \to \cdots.\end{aligned}$$

Substituting from (4.1.15) we obtain the desired short exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}\left(H_{k-1}\left(G,\, \mathbb{Z}\right),\, M\right) \to H^k(G,\, M) \to \operatorname{Hom}_{\mathbb{Z}}\left(H_k\left(G,\, \mathbb{Z}\right),\, M\right) \to 0.$$

The sequence splits since (4.1.14) splits (though not naturally).

To prove the last statement, recall that $H_1(G,\, \mathbb{Z})$ is the abelianization of G. Hence $\operatorname{Ext}^1_{\mathbb{Z}}\left(H_1\left(G,\, \mathbb{Z}\right),\, M\right)$ vanishes for all M if and only if $H_1(G,\, \mathbb{Z})$ is $\mathbb{Z}$-projective, *i.e.*, free abelian. Similarly, $\operatorname{Hom}_{\mathbb{Z}}\left(H_2\left(G,\, \mathbb{Z}\right),\, M\right)$ vanishes for all M if and only if $H_2\left(G,\, \mathbb{Z}\right)$ vanishes. □

Now it is time to make the connection between group homology and the theory of extensions. This follows from the following basic classification theorem, due originally to Eilenberg and Mac Lane. The theorem has a version for non-central extension extensions and even a version for extensions by a non-abelian normal subgroup, but in the interests of simplicity we stick with the simplest case, which is all we will need for applications to K-theory.

4.1.16. Theorem. *Let G be a group and let A be an abelian group. Then the isomorphism classes of parameterized central extensions of G by A, that is, triples $(E,\, \varphi,\, \iota)$, where $(E,\, \varphi)$ is a central extension of G and $\iota : A \to E$ is an isomorphism of A with $\ker\varphi$, naturally form an abelian group* $\operatorname{Ext}(G,\, A)$*, in which the trivial extension gives the* 0*-element. This group is naturally isomorphic to $H^2(G,\, A)$, where we view A as a trivial G-module. Hence every central extension of G by A is trivial if and only if $H^2(G,\, A) = 0$.*

Proof. We should make precise what we mean by isomorphism; two triples $(E,\, \varphi,\, \iota)$ and $(E',\, \varphi',\, \iota')$ are isomorphic if and only if there is an isomorphism of central extensions from E to E' compatible with ι and ι', *i.e.*, a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\varphi} & G & \longrightarrow & 1 \\
\| & & \| & & \cong\downarrow & & \| & & \| \\
1 & \longrightarrow & A & \xrightarrow{\iota'} & E' & \xrightarrow{\varphi'} & G & \longrightarrow & 1.
\end{array}$$

Next, we explain the group structure on $\operatorname{Ext}(G,\, A)$. If $(E_1,\, \varphi_1,\, \iota_1)$ and $(E_2,\, \varphi_2,\, \iota_2)$ are central extensions of G by A, we define their **Baer sum** as follows. First let $E = E_1 \times_G E_2$, *i.e.*, $\{(x,\, y) \in E_1 \times E_2 : \varphi_1(x) = \varphi_2(y)\}$. Note that this group comes with a natural surjection $\varphi : E \twoheadrightarrow G$ given by $\varphi(x,\, y) = \varphi_1(x) = \varphi_2(y)$. The kernel of φ is central, but it's too big; it's isomorphic to $A \times A$. Therefore we define

$$E_3 = E / \left\{\left(\iota_1\left(a\right),\, -\iota_2\left(a\right)\right) : a \in A\right\}. \tag{4.1.17}$$

Note that φ factors through E_3 and gives a map $\varphi_3 : E_3 \twoheadrightarrow G$. The kernel of φ_3 is central and is given by

$$\left\{\left(\iota_1\left(a_1\right),\, \iota_2\left(a_2\right)\right) : a_1,\, a_2 \in A\right\} / \left\{\left(\iota_1\left(a\right),\, -\iota_2\left(a\right)\right) : a \in A\right\},$$

so we can define an isomorphism $\iota_3 : A \to \ker \varphi_3$ by

$$\iota_3(a) = [(\iota_1(a), \iota_2(0))] = [(\iota_1(0), \iota_2(a))].$$

(Here we use the relation that $[(\iota_1(-a), \iota_2(a))] = 0$.) We define

$$[(E_1, \varphi_1, \iota_1)] + [(E_2, \varphi_2, \iota_2)] = [(E_3, \varphi_3, \iota_3)].$$

This addition operation is actually commutative, since if we define $(E_4, \varphi_4, \iota_4)$ similarly but with E_1 and E_2 interchanged, then we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\iota_3} & E_3 & \xrightarrow{\varphi_3} & G & \longrightarrow & 1 \\ \| & & \| & & \cong\downarrow\psi & & \| & & \| \\ 1 & \longrightarrow & A & \xrightarrow{\iota_4} & E_4 & \xrightarrow{\varphi_4} & G & \longrightarrow & 1 \end{array}$$

with ψ defined by the "flip." It is easy to verify that the Baer sum is associative on isomorphism classes of central extensions, that the isomorphism class of the trivial extension

$$(G \times A, p_1, i_2),$$

where p_1 is projection on the first factor and i_2 is injection into the second factor, acts as an identity with respect to the Baer sum, and that the class of (E, φ, ι) has as its inverse $(E, \varphi, -\iota)$. Thus we obtain an abelian group $\operatorname{Ext}(G, A)$ of parameterized central extensions of G by A, in which the trivial extension gives the 0-element.

Now we want to set up an isomorphism between $\operatorname{Ext}(G, A)$ and $H^2(G, A)$. First we define a map $\Phi : \operatorname{Ext}(G, A) \to H^2(G, A)$, then we'll define the inverse map $\Psi : H^2(G, A) \to \operatorname{Ext}(G, A)$. Start with a class in $\operatorname{Ext}(G, A)$ represented by (E, φ, ι). Choose a (set-theoretic) section $s : G \to E$, that is, a map such that $\varphi \circ s = id_G$. We may suppose that $s(1_G) = 1_E$. For $g, h \in G$, $\varphi(s(gh)) = \varphi(s(g))\varphi(s(h))$, so we can define a map $f : G\times G \to A$ by

$$s(gh) = \iota \circ f(g, h)s(g)s(h).$$

Since $1_E = s(1_G)$, $f(g, 1_G) = f(1_G, g) = 0$ and $f(g, g^{-1}) = f(g^{-1}, g)$ for all $g \in G$. If $g, h, k \in G$, then $s(gh)s(k) = \iota \circ f(g, h)s(g)s(h)s(k)$, while $s(g)s(hk) = \iota \circ f(h, k)s(g)s(h)s(k)$, so the associative rule gives

$$\begin{aligned}\iota \circ f(g, hk)\iota \circ f(h, k)s(g)s(h)s(k) &= \iota \circ f(g, hk)s(g)s(hk) = s(ghk) \\ &= \iota \circ f(gh, k)s(gh)s(k) = \iota \circ f(gh, k)\iota \circ f(g, h)s(g)s(h)s(k),\end{aligned}$$

or $f(g, hk) + f(h, k) = f(gh, k) + f(g, h)$. (We write products multiplicatively in E but additively in A.) Comparison with the formulas in Definition 4.1.7 shows that this is precisely the condition for f to define a 2-cocycle in $Z^2(G, A)$. So we let $\Phi([(E, \varphi, \iota)]) = [f] \in H^2(G, A)$. Of course, it is not immediately obviously that this is independent of the choice of s.

But if s' is some other choice for s, then we must have $s'(g) = s(g)\iota \circ u(g)$ for some map $u : G \to A$. Then if f' is the 2-cocycle defined by s, we have

$$\begin{aligned} s(gh)\iota \circ u(gh) = s'(gh) &= \iota \circ f'(g, h)s'(g)s'(h) \\ &= \iota \circ f'(g, h)\iota \circ u(g)\iota \circ u(h)s(g)s(h), \end{aligned}$$

and comparison with the definition of f gives

$$f(g, h) + u(gh) = f'(g, h) + u(g) + u(h),$$

which says that f' differs from f by the coboundary of u. Hence f' and f define the same cohomology class and Φ is well defined.

To show that Φ is a homomorphism, note first that Φ sends the 0-element of $\text{Ext}(G, A)$ to the 0-element of $H^2(G, A)$, since when $(E, \varphi, \iota) = (G \times A, p_1, i_2)$, we can take $s = i_1$, which gives $f = 0$. Next we show that Φ respects the Baer sum. Given $(E_1, \varphi_1, \iota_1)$ and $(E_2, \varphi_2, \iota_2)$, choose corresponding sections s_1 and s_2 giving cocycles f_1 and f_2, and let $E = E_1 \times_G E_2$ and E_3 be as in (4.1.17). Note that $s = (s_1, s_2)$ gives a section of (E, φ) which descends to a section s_3 for E_3. Then if $g, h \in G$, we have

$$\begin{aligned} s_3(gh) &= [(s_1(gh), s_2(gh))] \\ &= [(s_1(g)s_1(h)\iota_1 \circ f_1(g, h), s_2(gh)\iota_2 \circ f_2(g, h))] \\ &= s_3(g)s_3(h)[(\iota_1 \circ f_1(g, h), \iota_2 \circ f_2(g, h))] \\ &= s_3(g)s_3(h)\iota_3 \left(f_1(g, h) + f_2(g, h)\right), \end{aligned}$$

so that the cocycle f_3 defined by s_3 is just $f_1 + f_2$. This shows that Φ is additive.

To complete the proof, we show that Φ is bijective. First suppose $\Phi([(E, \varphi, \iota)]) = 0$. This means that if we choose a section f as above, the corresponding cocycle f is the coboundary of some $u : G \to A$, *i.e.*,

$$f(g, h) = u(gh) - u(g) - u(h), \qquad g, h \in G.$$

Replacing s by s', where $s'(g) = s(g)(\iota \circ u(g))^{-1}$, we have

$$\begin{aligned} s'(gh) &= s(gh)(\iota \circ u(gh))^{-1} \\ &= s(g)s(h)\iota \circ (f(g, h) - u(gh)) \\ &= s(g)s(h)\iota\left(u(gh) - u(g) - u(h) - u(gh)\right) \\ &= s'(g)s'(h), \end{aligned}$$

so s' is a homomorphism, which shows (E, φ, ι) is trivial. Thus Φ is injective.

Now we construct a right inverse Ψ for Φ, showing that Φ is surjective. Let $f \in Z^2(G, A)$, which we can take (by changing it within its cohomology class) to be normalized so that $f = 0$ on $\{1_G\} \times G$ and on $G \times \{1_G\}$ (this

implies $f(g, g^{-1})) = f(g^{-1}, g))$ and let $E = G \times A$ **as a set**, but with the following binary operation:

$$(g_1, a_1) \cdot (g_2, a_2) = (g_1 \cdot g_2, a_1 + a_2 - f(g_1, g_2)).$$

Since f is normalized, $(1_G, 0_A)$ acts as an identity element with respect to the operation $\cdot$. Furthermore, we have from the cocycle identity

$$\begin{aligned}((g_1, a_1) \cdot (g_2, a_2)) \cdot (g_3, a_3) &= (g_1 \cdot g_2, a_1 + a_2 - f(g_1, g_2)) \cdot (g_3, a_3) \\ &= (g_1 \cdot g_2 \cdot g_3, a_1 + a_2 + a_3 - f(g_1, g_2) - f(g_1 \cdot g_2, g_3)) \\ &= (g_1 \cdot g_2 \cdot g_3, a_1 + a_2 + a_3 - f(g_2, g_3) - f(g_1, g_2 \cdot g_3)) \\ &= (g_1, a_1) \cdot (g_2 \cdot g_3, a_2 + a_3 - f(g_2, g_3)) \\ &= (g_1, a_1) \cdot ((g_2, a_2) \cdot (g_3, a_3)).\end{aligned}$$

Also,

$$\begin{aligned}(g, a) \cdot (g^{-1}, -a + f(g, g^{-1})) &= (1_G, 0_A), \\ (g^{-1}, -a + f(g, g^{-1})) \cdot (g, a) &= (1_G, 0_A),\end{aligned}$$

so E is a group with respect to $\cdot$. Define $\varphi : E \to G$ to be projection on the first factor and define $\iota : A \to E$ by $\iota(a) = (1_G, a)$. Then it is clear that φ and ι are homomorphisms and that $\iota(A)$ is central in E and equal to the kernel of φ. Furthermore, $s : G \to E$ defined by $s(g) = (g, 0)$ is a section which gives rise to the cocycle f since

$$s(gh) = (gh, 0) = (g, 0) \cdot (h, f(g, h)) = s(g)s(h)\iota(f(g, h)),$$

so $\Psi : [f] \mapsto [(E, \varphi, \iota)]$ is a right inverse to Φ, showing that Φ is bijective. □

4.1.18. Corollary. *If G is a perfect group, then a central extension (E, φ) of G is universal if and only if $H_1(E, \mathbb{Z}) = 0$ and $H_2(E, \mathbb{Z}) = 0$.*

Proof. This follows immediately from Theorems 4.1.3, 4.1.13, and 4.1.16. (Note that a group is perfect if and only if its H_1 vanishes.) □

In fact, one can go a bit further.

4.1.19. Theorem. *Let G be a perfect group. Then the kernel of the universal central extension (E, φ) of G is naturally isomorphic to $A = H_2(G, \mathbb{Z})$, and under the isomorphisms*

$$\operatorname{Ext}(G, A) \cong H^2(G, A) \cong \operatorname{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), A)$$

defined by Theorems 4.1.16 and 4.1.13, the class of (E, φ) corresponds to the identity map $H_2(G, \mathbb{Z}) \to A$. (The Ext *term vanishes since H_1 vanishes.)*

The proof of this theorem requires developing some of the theory of how homology and cohomology behave under group extensions. To do this in the greatest generality requires the theory of spectral sequences and would take us a bit too far afield in homological algebra. However the following special case of the theory can be done directly.

4.1.20. Theorem ("Inflation-Restriction Sequence"). *Let G be a group, N a normal subgroup, and A a G-module. Then there is a natural exact sequence*

$$0 \to H^1(G/N, A^N) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(N, A)^{G/N} \xrightarrow{\partial} H^2(G/N, A^N) \xrightarrow{\text{inf}} H^2(G, A).$$

Here "res" comes from restriction of cocycles from G to N and "inf" denotes inflation, composition of cocycles on G/N with the quotient map $G \xrightarrow{q} G/N$. The action of G/N on $H^1(N, A)$ is induced from the conjugation action of G on N and from the action of G on A. If G acts trivially on A and N is central in G, the exact sequence simplifies to

$$0 \to H^1(G/N, A) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(N, A) \xrightarrow{\partial} H^2(G/N, A) \xrightarrow{\text{inf}} H^2(G, A),$$

where the map $H^1(N, A) \cong \text{Hom}(N, A) \xrightarrow{\partial} H^2(G/N, A)$ sends $u : N \to A$ to the class of $u \circ f$, where $f \in Z^2(G/N, A)$ is a cocycle defining the central extension of G/N by N as in Theorem 4.1.16.

Proof. Note that A^N is a G/N-module, since if $\dot{g} \in G/N$, $a \in A^N$ and $q(g) = \dot{g}$, then $\dot{g} \cdot a =_{\text{def}} g \cdot a$ will not change if we replace g by gn, $n \in N$. If $u : G/N \to A^N$ is a 1-cocycle, then $u \circ q : G \to A^N$ is a 1-cocycle, since for $g, h \in G$,

$$u \circ q(gh) = u(\dot{g}\dot{h}) = u(\dot{g}) + \dot{g} \cdot u(\dot{h}) = u \circ q(g) + g \cdot (u \circ q(h)).$$

Furthermore, if $u \circ q$ is the coboundary of some $a \in A$, then $u(\dot{g}) = g \cdot a - a$, hence taking $g \in N$ shows that $a \in A^N$ and u is the coboundary of a. Thus $H^1(G/N, A^N) \xrightarrow{\text{inf}} H^1(G, A)$ is injective.

It is clear that there is a homomorphism $H^1(G, A) \xrightarrow{\text{res}} H^1(N, A)$. To show that the image is fixed by G/N, consider a 1-cocycle $u : G \to A$ and let $g \in G$, $\dot{g} = q(g)$, $n \in N$. Then

$$\begin{aligned}(g \cdot (\text{res}\, u))(n) &=_{\text{def}} g \cdot u(g^{-1}ng) = g \cdot (u(g^{-1}) + g^{-1} \cdot u(ng)) \\ &= g \cdot (-g^{-1} \cdot u(g)) + g \cdot g^{-1} \cdot u(ng) \\ &= -u(g) + u(ng) = u(n) + n \cdot u(g) - u(g).\end{aligned}$$

Thus $g \cdot (\text{res}\, u)$ differs from $\text{res}\, u$ by the coboundary of $u(g)$, and so $\dot{g}$ fixes the cohomology class of $\text{res}\, u$.

That $\text{res} \circ \text{inf} : H^1(G/N, A^N) \to H^1(N, A)$ is 0 is trivial. We must show that if a 1-cocycle $u : G \to A$ restricts on N to the coboundary of some $a \in A$, then u is up to a coboundary the inflation of a cocycle on G/N with values in A^N. Replacing u by $u - \delta a$, we may suppose u vanishes on N. Then for $g \in G$ and $n \in N$, $u(gn) = u(g) + g \cdot u(n) = u(g) + 0 = u(g)$, so u is

constant on cosets of N. Furthermore, u takes values in A^N, since we then have (by normality of N) $u(g) = u(ng) = u(n) + n \cdot u(g) = 0 + n \cdot u(g) = n \cdot u(g)$, so u is the inflation of a cocycle $G/N \to A^N$.

We prove the last part of the theorem only in the case where A is a trivial G-module, which is the only case we'll need for applications. The general case works the same way but the calculations involved are much messier. We define the map $\partial : H^1(N, A)^{G/N} \to H^2(G/N, A)$. Let $u : N \to A$ be a 1-cocycle, *i.e.*, a homomorphism, which is invariant under conjugation by elements of G. (Since A is abelian, u is automatically fixed under conjugation by elements of N.) Fix a section $s : G/N \to G$ with $s(1_{G/N}) = 1_G$. We define $\psi : G/N \times G/N \to A$ by

$$\psi(\dot{g}, \dot{h}) = u(s(\dot{h})^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h})).$$

The same calculation as in Theorem 4.1.16 shows that ψ is a 2-cocycle with values in A, and that its cohomology class doesn't depend on the choice of s. We define $\partial([u])$ to be the class of ψ; it is obvious that ∂ is a homomorphism of abelian groups. Note that in the important special case when N is central, the condition that u be G-invariant is vacuous, and ψ is just $u \circ f$, where $f : G/N \times G/N \to N$ is the cocycle determined by s and the central extension of G/N by N, so in this case $\partial : H^1(N, A) \to H^2(G/N, A)$ is just composition with the class of f. It is also clear in general that ψ vanishes identically if s is a homomorphism. Let's show next that ∂ vanishes on the image of res. Suppose $u : G \to A$ is a homomorphism; we denote its restriction to N by the same letter. Define a map $v : G/N \to A$ by $v(\dot{g}) = u(s(\dot{g}))$. Then

$$\begin{aligned}\delta v(\dot{g}, \dot{h}) &=_{\text{def}} v(\dot{g}) + v(\dot{h}) - v(\dot{g}\dot{h}) \\ &= u(s(\dot{g})) + u(s(\dot{h})) - u(s(\dot{g}\dot{h})) = -\psi(\dot{g}, \dot{h}),\end{aligned}$$

so ψ is a coboundary and $\partial(\operatorname{res} u) = 0$.

In the other direction, if $u : N \to A$ is a homomorphism fixed under conjugation by elements of G and $\partial(u) = 0$, then ψ as defined above is a coboundary, say of some $-v : G/N \to A$. In other words,

$$u(s(\dot{h})^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h})) = -v(\dot{g}) - v(\dot{h}) + v(\dot{g}\dot{h}).$$

We may suppose $v(1_{G/N}) = 0$. Let $\tilde{u}(g) = v(\dot{g}) + u(s(\dot{g})^{-1}g)$. Then it is clear that $\tilde{u}$ agrees with u on N and that for $g \in G$ and $n \in N$, $\tilde{u}(gn) = \tilde{u}(g) + u(n)$. Finally we have

$$\begin{aligned}\tilde{u}(gh) &= v(\dot{g}\dot{h}) + u(s(\dot{g}\dot{h})^{-1}gh) \\ &= v(\dot{g}) + v(\dot{h}) + u(s(\dot{h})^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h})) + u(s(\dot{g}\dot{h})^{-1}gh) \\ &= v(\dot{g}) + v(\dot{h}) + u(s(\dot{h})^{-1}s(\dot{g})^{-1}gh) \\ &= \tilde{u}(h) + v(\dot{g}) + u(h^{-1}s(\dot{g})^{-1}gh) \\ &= \tilde{u}(h) + \tilde{u}(g),\end{aligned}$$

since u is invariant under conjugation by h, and thus $\tilde{u}$ is a homomorphism extending u and $u = \operatorname{res} \tilde{u}$.

It remains to prove exactness at $H^2(G/N, A)$. First of all, $\inf \circ \partial = 0$, for if $u : N \to A$ is a homomorphism fixed under conjugation by elements of G and ψ is defined as above, then

$$\begin{aligned}
\inf \psi(g, h) &=_{\text{def}} u(s(\dot{h})^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h})) \\
&= u(s(\dot{h})^{-1}h) + u(h^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h})) \\
&= u(s(\dot{h})^{-1}h) + u(s(\dot{g})^{-1}s(\dot{g}\dot{h})h^{-1}) \\
&= u(s(\dot{h})^{-1}h) + u(s(\dot{g})^{-1}g) + u(g^{-1}s(\dot{g}\dot{h})h^{-1}) \\
&= u(s(\dot{h})^{-1}h) + u(s(\dot{g})^{-1}g) + u(h^{-1}g^{-1}s(\dot{g}\dot{h})) \\
&= \delta v(g, h),
\end{aligned}$$

where $v(g) = u(s(\dot{g})^{-1}g)$. Finally, suppose $\psi \in Z^2(G/N, A)$, which we may suppose is normalized to vanish when one of its arguments is $1_{G/N}$, and $\inf \psi$ is a coboundary, say δv with $v : G \to A$. We need to show that the class of ψ is in the image of ∂. What we are given translates into the condition

$$\psi(\dot{g}, \dot{h}) = v(g) + v(h) - v(gh).$$

Since the left-hand side vanishes when g or h lies in N, this says in particular that v restricted to N is a homomorphism, and that $v(gn) = v(ng) = v(g) + v(n)$ for $n \in N$. Thus the restriction of v defines a class in $H^1(N, A)^{G/N}$. The class $\partial(v|_N)$ is defined by the cocycle

$$\psi'(\dot{g}, \dot{h}) = v(s(\dot{h})^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h})).$$

Then

$$(\psi - \psi')(\dot{g}, \dot{h}) = v(s(\dot{g})) + v(s(\dot{h})) - v(s(\dot{g})s(\dot{h})) - v(s(\dot{h})^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h})).$$

Since $n = s(\dot{h})^{-1}s(\dot{g})^{-1}s(\dot{g}\dot{h}) \in N$, the identity $v(s(\dot{g})s(\dot{h})) + v(n) = v(s(\dot{g})s(\dot{h})n)$ gives

$$(\psi - \psi')(\dot{g}, \dot{h}) = v(s(\dot{g})) + v(s(\dot{h})) - v(s(\dot{g}\dot{h})) = \delta(v \circ s)(\dot{g}, \dot{h}),$$

and $[\psi] = \partial(v|_N)$, as desired. $\square$

Proof of Theorem 4.1.19. Let G be a perfect group, let (E, φ) be a perfect central extension of G with kernel $\ker \varphi = K$, and let A be a trivial E-module. The exact sequence of Theorem 4.1.20 becomes

$$0 = H^1(E, A) \xrightarrow{\text{res}} H^1(K, A) \xrightarrow{\partial} H^2(G, A) \xrightarrow{\text{inf}} H^2(E, A).$$

Now for (E, φ) to be the universal central extension of G, E must be perfect and $H^2(E, A)$ must vanish for every A. So this can happen only

if $H^1(K, A) = \operatorname{Hom}_{\mathbb{Z}}(K, A) \xrightarrow{\partial} H^2(G, A)$ is an isomorphism for every A. But by Theorem 4.1.13, $H^2(G, A) \cong \operatorname{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), A)$, so we must have

$$\operatorname{Hom}_{\mathbb{Z}}(K, A) \cong \operatorname{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), A)$$

for every abelian group A, which is only possible if $K \cong H_2(G, \mathbb{Z})$. Furthermore, from the description of ∂ in Theorem 4.1.20, we see that the 2-cohomology class of the extension of G by K must correspond to an isomorphism $H_2(G, \mathbb{Z}) \to K$, which we can take to be the identity after reparameterizing K. $\square$

Remark. Milnor in [Milnor] gives a different proof of Theorem 4.1.19, by identifying the kernel of the map $[F, F]/[F, R] \twoheadrightarrow [F, F]/R$ in Theorem 4.1.3 directly with $H_2(G, \mathbb{Z})$. This comes from applying the analogue in homology of Theorem 4.1.20 to the group extension

$$1 \to R \to F \to G \to 1.$$

This basically concludes the discussion of the relationship between central extensions and homology. However, for future use in studying the homology of groups such as SL and GL, we mention a few more basic facts about group homology.

4.1.21. Definition. If $H \xrightarrow{\alpha} G$ is a homomorphism of groups and A is a G-module (viewed also as an H-module via α), there are induced maps $\alpha_* : H_\bullet(H, A) \to H_\bullet(G, A)$. (Merely take the complex defining $H_\bullet(H, A)$ and apply α to each copy of H.) When α is the inclusion of a subgroup, this map is commonly called **corestriction**, since it is the analogue in homology of $H^\bullet(G, A) \xrightarrow{\text{res}} H^\bullet(H, A)$. However, when H is of finite index $r = [G : H]$ in G, there is also a natural map in the other direction, called the **transfer**, sometimes denoted Tr_H^G or $\alpha^!$. This may be defined as follows. Note that $\mathbb{Z}G$ is naturally the free $\mathbb{Z}H$-module on the set $H\backslash G$, so that if A is a free $\mathbb{Z}G$-module, $H_0(H, A) \cong H_0(G, A)^{H\backslash G}$. (This uses the fact that $H\backslash G$ is a finite set, since in general we only get a direct **sum** of copies of $H_0(G, A)$, not a direct product.) Thus there is a diagonal map $H_0(G, A) \to H_0(H, A) \cong H_0(G, A)^{H\backslash G}$, given by $a \mapsto (\overbrace{a, \dots, a}^{r \text{ times}})$, called the transfer. In general, we can resolve A by free $\mathbb{Z}G$-modules, use Corollary 4.1.10, and do this to each step of the resolution.

There is another equivalent way of defining corestriction which is sometimes useful. Namely, we can use Shapiro's Lemma (Corollary 4.1.12), which sets up a natural isomorphism $H_j(H, A) \cong H_j(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} A)$. The map α_* is easily seen to be the composition of this isomorphism with the map $H_j(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} A) \to H_\bullet(G, A)$ induced by the map of G-modules $\mathbb{Z}G \otimes_{\mathbb{Z}H} A \twoheadrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}G} A \cong A$ coming from the fact that A is a G-module and not just an H-module.

4.1.22. Proposition. *If $\alpha : H \hookrightarrow G$ is the inclusion of a subgroup of finite index $r = [G : H]$ and A is a G-module, then $\alpha_* \circ \alpha^!$ is multiplication by r on $H_\bullet(G, A)$.*

Proof. If A is free, then $H_0(H, A) \cong H_0(G, A)^{H\backslash G}$ and

$$\alpha_* \circ \alpha^!(a) = \alpha_*(\overbrace{a, \dots, a}^{r \text{ times}}) = \overbrace{a + \cdots + a}^{r \text{ times}} = ra.$$

In general, we can resolve A by free $\mathbb{Z}G$-modules, use Corollary 4.1.10, and apply this at each step of the resolution. □

4.1.23. Theorem. *If G is a finite group of order r and A is a G-module, then $H_j(G, A)$ is a group of exponent r for each $j > 0$, and $H_j(G, \mathbb{Z})$ is a finite abelian group of exponent r for each $j > 0$.*

Proof. Let H be the trivial one-element subgroup of G. Then $H_0(H, A) = A$ and $H_j(H, A) = 0$ for $j > 0$ (this is obvious from Definition 4.1.7). Applying Proposition 4.1.22 to the inclusion α of H into G, we see that $\alpha_* \circ \alpha^!$ is multiplication by r on $H_j(G, A)$, while of course this composite is zero for $j > 0$. So multiplication by r on $H_j(G, A)$ acts by zero for $j > 0$. This proves the first statement.

Furthermore, the abelian groups P_j in (4.1.8) are finitely generated if G is finite. So each $H_j(G, A)$ is finitely generated if A is finitely generated, in particular if $A = \mathbb{Z}$. Since a finitely generated abelian group of finite exponent is finite, the last statement follows. □

4.1.24. Corollary. *Let G be a finite group, let p be a prime dividing the order of G, and let P_p be a Sylow p-subgroup of G. Then for each $j > 0$, the natural map $H_j(P_p, \mathbb{Z}) \to H_j(G, \mathbb{Z})$ is surjective onto the p-primary part. In particular, if $H_j(P_p, \mathbb{Z}) = 0$, then $H_j(G, \mathbb{Z})$ has no p-torsion, and if $H_j(P_p, \mathbb{Z}) = 0$ for each p dividing the order of G, then $H_j(G, \mathbb{Z}) = 0$.*

Proof. Apply Proposition 4.1.22 to the inclusion $\alpha : P_p \hookrightarrow G$. Thus $\alpha_* \circ \alpha^!$ is multiplication by $[G : P_p]$, which is relatively prime to p, and is thus an isomorphism on the p-primary part of $H_j(G, A)$ for $j > 0$. So α_* is surjective on the p-primary part. □

4.1.25. Exercise. Let G be a cyclic group of finite order r, and identify $\mathbb{Z}G$ with $\mathbb{Z}[t]/(t^r - 1)$, where t is a generator of G. Show that

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{(t-1)} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{(t-1)} \mathbb{Z}G \xrightarrow{t \mapsto 1} \mathbb{Z} \to 0,$$

where N is multiplication by $1 + t + \cdots + t^{r-1}$, gives a free resolution of the trivial G-module $\mathbb{Z}$. Deduce that $H_j(G, \mathbb{Z}) = 0$ for j positive and even, and that $H_j(G, \mathbb{Z}) \cong G$ for j positive and odd. Show also that $H^j(G, \mathbb{Z}) = 0$ for j positive and odd, and that $H^j(G, \mathbb{Z}) \cong G$ for j positive and even. The generator of $H^2(G, \mathbb{Z})$ corresponds to the non-trivial central extension

$$0 \to \mathbb{Z} \xrightarrow{r} \mathbb{Z} \to \mathbb{Z}/r \to 0.$$

4.1.26. Exercise. Let V be the Klein 4-group, the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

of $SO(3)$. (The "V" stands for *Viergruppe*, German for "4-group.")

(1) Show by direct calculation that $H_2(V, \mathbb{Z}) \cong \mathbb{Z}/2$, and deduce from Theorem 4.1.13 that

$$H^2(V, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3.$$

Determine which elements of this group correspond to the various groups of order 8 which are central extensions of V by $\mathbb{Z}/2$.

(2) Examples 4.1.5 may be interpreted as exhibiting $\mathbb{Z}/2$ as a quotient of $H_2(SO(3), \mathbb{Z})$. Show that the inclusion $V \hookrightarrow SO(3)$ induces a non-zero map $\mathbb{Z}/2 \cong H_2(V, \mathbb{Z}) \to H_2(SO(3), \mathbb{Z})$ which is a splitting for this $\mathbb{Z}/2$ factor, by noticing that the inverse image $\tilde{V}$ of V in $SU(2)$ is the quaternion group Q of order 8. (Since Q is non-abelian, this means the central extension of $SO(3)$ is non-trivial on the subgroup V.)

(3) Show that $H_1(Q, \mathbb{Z}) \cong V$ (this is almost trivial) and that $H_2(Q, \mathbb{Z}) = 0$ (it helps to construct a suitable resolution). Thus the quotient map $Q \to V$ induces an isomorphism on H_1 but is not surjective on H_2.

4.1.27. Exercise. This exercise will exhibit an interesting finite example of a universal central extension.

(1) Let G be the subgroup of $SO(3)$ consisting of rotations mapping a regular icosahedron to itself. Since G acts transitively on the 20 faces of the icosahedron, and each face is an equilateral triangle, and there are clearly 3 rotations stabilizing each face (the identity and rotations by $2\pi/3$ in either direction about an axis through the center of the face), G is a group of order 60. It is clear that the stabilizer S_f of a face is a Sylow 3-subgroup and that the stabilizer S_v of a vertex is a Sylow 5-subgroup.

(2) Show that one may position the icosahedron so that two of the edges are parallel to each of the three coordinate axes in $\mathbb{R}^3$. Deduce (by looking at the effect of rotations by π around these axes) that G contains the group $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ of Exercise 4.1.26 as a Sylow 2-subgroup.

(3) From Exercise 4.1.25, $H_2(S_f, \mathbb{Z}) = 0$ and $H_2(S_v, \mathbb{Z}) = 0$. From Exercise 4.1.26, $H_2(V, \mathbb{Z}) \cong \mathbb{Z}/2$. Deduce from Corollary 4.1.24 that $H_2(G, \mathbb{Z})$ has order at most 2.

(4) Show that G is perfect. In fact it is isomorphic to the simple group A_5, and is the smallest non-trivial perfect group. (One way to do this is to divide the 30 edges of the icosahedron into 5 equivalence

classes of 6 edges each, where each equivalence class consists of the edges pointing in directions parallel or perpendicular to the direction of a given edge. Then you just need to show that G acts faithfully, *i.e.*, without kernel, by permutations of the 5 equivalence classes. Since A_5 is the only subgroup of S_5 of order 60, this shows G is isomorphic to the simple group A_5.)

(5) From (2) and (3), deduce that $H^2(G, \mathbb{Z}/2)$ has order at most two, and that either $H_2(G, \mathbb{Z}) = 0$ or else G has exactly one non-trivial central extension by $\mathbb{Z}/2$. In this latter case, show that this must be the universal central extension of G.

(6) Let $\tilde{G}$ be the inverse image of G in $SU(2)$. This is a central extension of G by $\mathbb{Z}/2$, called the **binary icosahedral group**. It is a group of order 120. Show that $\tilde{G} \to G$ is not trivial, by using (2) of this Exercise and (2) of Exercise 4.1.26.

(7) Deduce from (4) and (5) that $\tilde{G}$ is the universal central extension of G, that $H_2(G, \mathbb{Z}) \cong \mathbb{Z}/2$, and that $\tilde{G}$ is perfect. (For another proof that $\tilde{G}$ is the universal central extension of G, you can show that $H_2(\tilde{G}, \mathbb{Z}) = 0$ using (3) of Exercise 4.1.26.) Since $SU(2)$ can be topologically identified with S^3, it follows that $SU(2)/\tilde{G}$ is a compact 3-manifold such that $\pi_1(SU(2)/\tilde{G})$ is perfect, hence with $H_1(SU(2)/\tilde{G}) = 0$. It follows from Poincaré duality that $H_2(SU(2)/\tilde{G}) = 0$ as well. Thus $SU(2)/\tilde{G}$ has the same homology groups as S^3, and is known as the **Poincaré homology 3-sphere**.

4.1.28. Exercise. This exercise will provide some more finite examples of universal central extensions.

(1) Show from the identity

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & (d^2-1)a \\ 0 & 1 \end{pmatrix}$$

and its transpose that if F is a field with more than 3 elements (so that there is an element $d \in F^\times$ with $d^2 \neq 1$), then $SL(2, F) = E(2, F)$ is a perfect group.

(2) Show that if $\mathbb{F}_q$ is a finite field with q elements, then $SL(2, \mathbb{F}_q)$ has order $q(q^2-1)$. Note that in fact the restriction in (1) on the cardinality of F is necessary, since $SL(2, \mathbb{F}_2) \cong S_3$ and $SL(2, \mathbb{F}_3)$ is a solvable group of order 24.

(3) From (1) and (2), $SL(2, \mathbb{F}_4)$ is a perfect group of order $4(15) = 60$. Show that it is isomorphic to the group $G \cong A_5$ of Exercise 4.1.27 by showing that G acts faithfully as a permutation group of the set of 5 elements

$$\mathbb{P}^1(\mathbb{F}_4) =_{\mathsf{def}} (\mathbb{F}_4^2 \smallsetminus \{(0, 0)\})/\mathbb{F}_4^\times \cong \mathbb{F}_4 \cup \{\infty\}.$$

Deduce from Exercise 4.1.27 that $H_2(SL(2, \mathbb{F}_4), \mathbb{Z})$ has order 2, so that its universal central extension is an extension by $\mathbb{Z}/2$.

(4) From (1) and (2), $SL(2, \mathbb{F}_5)$ is a perfect group of order $5(24) = 120 = 3 \cdot 5 \cdot 2^3$. Clearly the Sylow 3-subgroups and Sylow 5-subgroups of this group are cyclic. Show that the Sylow 2-subgroups are isomorphic to Q, which has vanishing H_2 by (3) of Exercise 4.1.26. Deduce that $H_2(SL(2, \mathbb{F}_5), \mathbb{Z}) = 0$ and that G is its own universal central extension. In fact $SL(2, \mathbb{F}_5)$ is isomorphic to the binary icosahedral group $\tilde{G}$ of Exercise 4.1.27.

(5) $GL(3, \mathbb{F}_2) = SL(3, \mathbb{F}_2)$ is a perfect group of order $(2^3 - 1)(2^3 - 2)(2^3 - 4) = 7 \cdot 6 \cdot 4 = 3 \cdot 7 \cdot 8$. Clearly the Sylow 3-subgroups and Sylow 7-subgroups of this group are cyclic. Show that the group of upper-triangular matrices with 1's on the diagonal is a Sylow 2-subgroup isomorphic to a dihedral group of order 8. Deduce that $H_2(SL(3, \mathbb{F}_2), \mathbb{Z})$ is a finite abelian 2-group, and see if you can compute it.

4.1.29. Exercise. Show that group homology commutes with direct limits (cf. Theorem 1.2.5). In other words, if $(G_\alpha)_{\alpha \in I}$, $(\theta_{\alpha\beta} : G_\alpha \to G_\beta)_{\alpha<\beta}$ is a direct system of groups, if $G = \varinjlim G_\alpha$ is the direct limit of the system, and if M is a G-module (hence also a G_α-module for each α, via the map $G_\alpha \to G$), then there are natural isomorphisms

$$H_j(G, M) \cong \varinjlim H_j(G_\alpha, M)$$

for each j. The key to this is the observation that each element of $P_j \otimes_{\mathbb{Z}G} M$ involves only finitely many group elements, and thus comes from some G_α.

The corresponding statement for cohomology is false, *i.e.*, in general $H^j(G, M) \neq \varprojlim H^j(G_\alpha, M)$.

4.1.30. Exercise. Let G be an abelian group (written additively for purposes of this exercise), and choose an exact sequence

$$0 \to F_1 \to F_0 \to G \to 0,$$

where F_0 and F_1 are free abelian groups. (In other words, choose a free resolution of G as a $\mathbb{Z}$-module.) Let A be a trivial G-module. The exact sequence of Theorem 4.1.20 becomes

$$\begin{aligned} 0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(F_0, A) &\to \mathrm{Hom}(F_1, A) \\ &\to H^2(G, A) \to H^2(F_0, A). \end{aligned}$$

Comparing this with the exact sequence coming from the definition of Ext,

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(F_0, A) \to \mathrm{Hom}(F_1, A) \to \mathrm{Ext}^1_{\mathbb{Z}}(G, A) \to 0,$$

deduce that

$$\ker(\inf : H^2(G, A) \to H^2(F_0, A)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(G, A).$$

Then using Theorem 4.1.13 and the fact that for an abelian group,

$$H_1(G, \mathbb{Z}) \cong G \quad \text{(in a natural way)},$$

deduce that inf gives an injection

$$\operatorname{Hom}(H_2(G, \mathbb{Z}), A) \hookrightarrow \operatorname{Hom}(H_2(F_0, \mathbb{Z}), A).$$

This being true for all A, deduce that the natural map

$$H_2(F_0, \mathbb{Z}) \to H_2(G, \mathbb{Z})$$

must be surjective.

4.1.31. Exercise. Let F be the free abelian group on generators $t_1, t_2, \ldots, t_n$, so that

$$R = \mathbb{Z}F \cong \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_n, t_n^{-1}].$$

Construct a free resolution for the trivial R-module $\mathbb{Z}$ of the form

$$0 \to R \to R^n \to \cdots \to R^{\binom{n}{k}} \to \cdots \to R^n \to R \to \mathbb{Z} \to 0.$$

(If you can't see how to do this in general, first try the special cases $n = 1$ and $n = 2$.) Use this resolution to compute $H_\bullet(F, \mathbb{Z})$. Deduce that there are natural isomorphisms

$$H_\bullet(F, \mathbb{Z}) \cong \bigwedge^\bullet F,$$

where the right-hand side denotes the exterior algebra on F (viewed as a free $\mathbb{Z}$-module).

4.1.32. Exercise.

(1) By Exercises 4.1.30 and 4.1.31, if G is a finitely generated abelian group, written in the form F_0/F_1, where F_0 is free on n generators, then there is a surjection

$$H_2(F_0, \mathbb{Z}) \cong \bigwedge^2(F_0) \twoheadrightarrow H_2(G, \mathbb{Z}).$$

Show that this map factors through $\bigwedge^2(G)$. If G is written with multiplicative instead of with additive ($\mathbb{Z}$-module) notation, then $\bigwedge^2(G)$ is the universal abelian group generated by elements $g \wedge h$, $g, h \in G$, with $g \wedge g = 1$, such that the map $(g, h) \mapsto g \wedge h$ is bilinear, *i.e.*,

$$(g_1 g_2) \wedge h = (g_1 \wedge h)(g_2 \wedge h).$$

Note that these relation imply $(g \wedge h)^{-1} = h \wedge g$, $g, h \in G$.

(2) Deduce from Exercises 4.1.29 and 4.1.31 that for **any torsion-free** abelian group G, there is a natural **isomorphism**

$$H_2(G,\,\mathbb{Z}) \cong \bigwedge^2 G.$$

(3) Suppose $G = G_1 \oplus (\mathbb{Z}/k)$, where $k > 1$ and G_1 is an abelian group written additively. Show that

$$H_2(G,\,\mathbb{Z}) \cong H_2(G_1,\,\mathbb{Z}) \oplus (G_1 \otimes_{\mathbb{Z}} (\mathbb{Z}/k)).$$

Deduce using the structure theorem for finitely generated abelian groups and induction on the number of finite cyclic summands that there is a natural isomorphism

$$H_2(G,\,\mathbb{Z}) \cong \bigwedge^2 G$$

for any finitely generated abelian group. Then use Exercise 4.1.29 to conclude that this is valid for **any** abelian group. Note that this calculation is consistent with the calculation that $H_2(V,\,\mathbb{Z}) \cong \mathbb{Z}/2$ in Exercise 4.1.26(1).

2. The Steinberg group

We're now ready to apply the theory of the previous section to the perfect group $E(R)$ of matrices over a ring R. Recall from Lemma 2.1.2 that this has generators $e_{ij}(a)$, $i \neq j$, $a \in R$, satisfying the relations

$$\begin{cases} e_{ij}(a)e_{ij}(b) = e_{ij}(a+b); & \text{(a)}\\ e_{ij}(a)e_{kl}(b) = e_{kl}(b)e_{ij}(a), \quad j \neq k \text{ and } i \neq l; & \text{(b)}\\ e_{ij}(a)e_{jk}(b)e_{ij}(a)^{-1}e_{jk}(b)^{-1} = e_{ik}(ab), \quad i,j,k \text{ distinct}; & \text{(c)}\\ e_{ij}(a)e_{ki}(b)e_{ij}(a)^{-1}e_{ki}(b)^{-1} = e_{kj}(-ba), \quad i,j,k \text{ distinct}. & \text{(d)} \end{cases}$$

4.2.1. Definition. Let R be a ring. For $n \geq 3$, we define $\mathrm{St}(n,\,R)$, the **Steinberg group** of order n over R, to be the free group on generators $x_{ij}(a)$, $i \neq j$, $1 \leq i,\, j \leq n$, $a \in R$, divided by the relations

$$\begin{cases} x_{ij}(a)x_{ij}(b) = x_{ij}(a+b); & \text{(a)}\\ x_{ij}(a)x_{kl}(b) = x_{kl}(b)x_{ij}(a), \quad j \neq k \text{ and } i \neq l; & \text{(b)}\\ x_{ij}(a)x_{jk}(b)x_{ij}(a)^{-1}x_{jk}(b)^{-1} = x_{ik}(ab), \quad i,j,k \text{ distinct}; & \text{(c)}\\ x_{ij}(a)x_{ki}(b)x_{ij}(a)^{-1}x_{ki}(b)^{-1} = x_{kj}(-ba), \quad i,j,k \text{ distinct}. & \text{(d)} \end{cases}$$

It is immediate that $\mathrm{St}(n,\,R)$ is a perfect group and that there is a unique surjective homomorphism $\varphi_n : \mathrm{St}(n,\,R) \to E(n,\,R)$ satisfying $x_{ij}(a) \mapsto$

$e_{ij}(a)$. Clearly there are natural maps $\mathrm{St}(n, R) \to \mathrm{St}(n+1, R)$. However, unlike the situation with the maps $GL(n, R) \to GL(n+1, R)$, it is not clear that these are injective (and in fact this is not always the case). We let $\mathrm{St}(R)$, called simply the Steinberg group of R, be the inductive limit. In other words, this is the universal group on generators $x_{ij}(a)$, $i \neq j$, $1 \leq i, j < \infty$, $a \in R$, satisfying the above relations. By construction, there is a canonical map $\varphi : \mathrm{St}(R) \twoheadrightarrow E(R)$ (the limit of the φ_n as $n \to \infty$).

The definition can be simplified a bit because relation (d) is redundant. By relation (a), $x_{ij}(a)^{-1} = x_{ij}(-a)$. Multiplying (c) on the left by $x_{ij}(a)^{-1}$ and on the right by $x_{ij}(a)$, we obtain

$$x_{jk}(b)x_{ij}(a)^{-1}x_{jk}(b)^{-1}x_{ij}(a) = x_{ij}(a)^{-1}x_{ik}(ab)x_{ij}(a) = x_{ik}(ab)$$

(using (b) and the fact that i, j, k were assumed distinct), which is the same as (d) if we renumber indices and replace b by a, a by $-b$. Thus it suffices to assume (a), (b), and (c).

Note also that the group $\mathrm{St}(R)$ is functorial in R, since if $\alpha : R \to S$ is a homomorphism of rings, there is a unique map from the free group on generators $x_{ij}(a)$, $a \in R$, to $\mathrm{St}(S)$ sending $x_{ij}(a)$ to $x_{ij}(\alpha(a))$, and since this is compatible with the relations in $\mathrm{St}(R)$, it factors through a map $\alpha_* : \mathrm{St}(R) \to \mathrm{St}(S)$.

4.2.2. Definition. Let R be a ring. We let $K_2(R) = \ker(\varphi : \mathrm{St}(R) \twoheadrightarrow E(R))$. This is functorial in R since the groups $E(R)$ and $\mathrm{St}(R)$ and the homomorphism φ are functorial.

The rationale for this definition is that $K_2(R)$ vanishes precisely when all relations among matrices in $E(R)$ follow from the "obvious" relations of Definition 4.2.1. Thus $K_2(R)$ measures the "non-obvious" relations among elementary matrices over R, just as $K_1(R)$ measures the failure of general invertible matrices to be expressible in terms of elementary matrices.

4.2.3. Lemma. *Let R be a ring and $3 \leq n < \infty$. The subgroup of* $\mathrm{St}(n, R)$ *generated by all $x_{ij}(a)$, $a \in R$, with $i < j$ is nilpotent, and φ_n restricted to this subgroup is an isomorphism onto the upper-triangular subgroup of $E(n, R)$. Thus $K_2(R)$ has trivial intersection with the subgroup of* $\mathrm{St}(n, R)$ *generated by all $x_{ij}(a)$, $a \in R$, with $i < j$.*

Proof. Let $N(n, R)$ be the subgroup of $\mathrm{St}(n, R)$ generated by all $x_{ij}(a)$, $a \in R$, with $i < j$. This contains the subgroup N_1 generated by all $x_{1j}(a)$, $a \in R$, $1 < j \leq n$. By relations (a) and (b), N_1 is abelian and R^{n-1} surjects onto N_1 via

$$(a_2, a_3, \ldots, a_n) \mapsto x_{12}(a_2)x_{13}(a_3)\cdots x_{1n}(a_n).$$

But under φ_n, N_1 maps to the upper-triangular matrices with 1's on the diagonal whose other non-zero entries are all in the first row, so the composition $R^{n-1} \twoheadrightarrow N_1 \xrightarrow{\varphi_n} E(n, R)$ is injective and φ_n must be injective on N_1. By relation (c), $N(n, R)$ normalizes N_1. Let N_2 be the subgroup of $\mathrm{St}(n, R)$ generated by all $x_{ij}(a)$, $a \in R$, with $i < j$ and $i = 1$ or 2. Then

N_2/N_1 is generated by the images of the $x_{2j}(a)$, $a \in R$, $2 < j \le n$. Arguing as before, the group generated by these is also abelian and an image of R^{n-2}, and maps to the upper-triangular matrices with 1's on the diagonal whose other non-zero entries are all in the second row. So φ_n is injective on this group as well and so on N_2. Continuing inductively, one sees that $N(n, R)$ is an iterated extension of abelian groups and maps isomorphically under φ_n to the group of upper-triangular $n \times n$ matrices with 1's on the diagonal. □

4.2.4. Theorem. *Let R be a ring. Then $K_2(R) = \ker(\varphi : \mathrm{St}(R) \to E(R))$ is an abelian group, and is precisely the center $Z(\mathrm{St}(R))$ of $\mathrm{St}(R)$. Thus $\mathrm{St}(R)$ is a central extension of $E(R)$.*

Proof. Let $x \in Z(\mathrm{St}(R))$. Then $\varphi(x)$ must commute with $\varphi(y)$ for all $y \in \mathrm{St}(R)$, and since φ is surjective, $\varphi(x) \in Z(E(R))$. But $E(R)$ has trivial center, since an $n \times n$ matrix can't commute with each $e_{ij}(1)$ unless it is a diagonal matrix and all its diagonal entries are equal, and $E(R)$ consists of infinite matrices whose diagonal entries are eventually 1. So $\varphi(x) = 1$, showing that $Z(\mathrm{St}(R)) \subseteq K_2(R)$.

For the reverse inclusion, suppose $x \in K_2(R)$, and write

$$x = x_{i_1j_1}(a_1)\cdots x_{i_nj_n}(a_n), \text{ where } e_{i_1j_1}(a_1)\cdots e_{i_nj_n}(a_n) = 1$$

in $E(R)$. Choose N larger than all the indices $i_1, \dots, i_n, j_1, \dots, j_n$. Then for any $l \le n$, any $k < N$, and any $a \in R$,

$$x_{i_lj_l}(a_1)x_{kN}(a)x_{i_lj_l}(a_1)^{-1} = \begin{cases} x_{kN}(a), & k \ne j_l, \\ x_{i_lN}(a_1a)x_{kN}(a), & k = j_l, \end{cases}$$

so x normalizes the subgroup A_N generated by the $x_{kN}(a)$, $k < N$ and $a \in R$. But by Lemma 4.2.3, the restriction of φ to A_N is injective. So for $y \in A_N$, $\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = \varphi(y)\varphi(y)^{-1} = 1$ and $xyx^{-1}y^{-1} = 1$. This shows x commutes with $x_{kN}(a)$ for any N larger than all the indices $i_1, \dots, i_n$ and $j_1, \dots, j_n$, for any $k < N$, and for any $a \in R$. Since these generate $\mathrm{St}(R)$ because of relation 4.2.1(c), x is central. Thus $K_2(R) \subseteq Z(\mathrm{St}(R))$. □

4.2.5. Example. Let R be any ring, and let

$$x = (x_{12}(1)x_{21}(-1)x_{12}(1))^4.$$

Then

$$\begin{aligned}\varphi(x) &= \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)^4 \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

so $x \in K_2(R)$ and hence x is central in $\mathrm{St}(R)$. It will turn out that when $R = \mathbb{Z}$, x has order 2 and generates $K_2(\mathbb{Z})$.

We are almost ready for the main theorem of this section, but first we prove a number of easy group-theoretic identities.

4.2.6. Lemma. *Let G be a group and let $u, v, w \in G$. Denote the commutator $uvu^{-1}v^{-1}$ by $[u, v]$. Then*

(a) $[u, v] = [v, u]^{-1}$.
(b) $[u, v][u, w] = [u, vw][v, [w, u]]$.
(c) *(Jacobi identity) If $G' = [G, G]$ is commutative, then*

$$[u, [v, w]][v, [w, u]][w, [u, v]] = 1.$$

Proof. (a) is trivial. For (b), note that

$$\begin{aligned}[u, vw][v, [w, u]] &= u(vw)u^{-1}(vw)^{-1}v(wuw^{-1}u^{-1})v^{-1}(uwu^{-1}w^{-1}) \\ &= (uv)(wu^{-1}w^{-1}v^{-1})(vwuw^{-1})u^{-1}v^{-1}(uwu^{-1}w^{-1}) \\ &= [u, v][u, w].\end{aligned}$$

For (c), first rewrite (b) as

$$[v, [w, u]] = [u, vw]^{-1}[u, v][u, w].$$

Cyclically permuting u, v, w and multiplying gives (provided commutators commute)

$$\begin{aligned}&[u, [v, w]][v, [w, u]][w, [u, v]] \\ &\quad = [w, uv]^{-1}[w, u][w, v][u, vw]^{-1}[u, v][u, w][v, wu]^{-1}[v, w][v, u] \\ &\quad = \left([w, uv]^{-1}[u, vw]^{-1}[v, wu]^{-1}\right)\left([w, u][u, w][w, v][v, w][u, v][v, u]\right) \\ &\quad = [uv, w][wu, v][vw, u] \\ &\quad = uvw(v^{-1}u^{-1}w^{-1}wuv)(u^{-1}w^{-1}v^{-1}vwu)w^{-1}v^{-1}u^{-1} \\ &\quad = 1. \quad \square\end{aligned}$$

4.2.7. Theorem. *Let R be a ring. Then $\mathrm{St}(R)$ is the universal central extension of $E(R)$.*

Proof. By Theorem 4.2.2, $\mathrm{St}(R)$ is a central extension of $E(R)$, and relation 4.2.1(c) shows $\mathrm{St}(R)$ is a perfect group. By Theorem 4.1.3, it suffices to show that every central extension of $\mathrm{St}(R)$ is trivial. Let (U, ψ) be a central extension of $\mathrm{St}(R)$. If $x, y \in \mathrm{St}(R)$ and we choose $X, Y \in U$ with $\psi(X) = x$ and $\psi(Y) = y$, then $[X, Y]$ is independent of the choices of X and Y, since changing X or Y by an element of $Z(U)$ will not affect the commutator. Thus it makes sense to refer to $[\psi^{-1}(x), \psi^{-1}(y)]$ as a well-defined element of U. We will define a splitting map $s : \mathrm{St}(R) \to U$ to ψ by sending

$$[x, y] \mapsto [\psi^{-1}(x), \psi^{-1}(y)]$$

for suitable x and y.

4.2.8. Lemma. *In this situation, if $j \neq k$, $i \neq l$, and $a, b \in R$, then*

$$[\psi^{-1}(x_{ij}(a)),\, \psi^{-1}(x_{kl}(b))] = 1 \text{ in } U.$$

Proof. Choose h distinct from i, j, k, l and choose

$$u \in \psi^{-1}(x_{ih}(1)), \quad v \in \psi^{-1}(x_{hj}(a)), \quad \text{and } w \in \psi^{-1}(x_{kl}(b)).$$

Then $[u, v] \in \psi^{-1}(x_{ij}(a))$. There must be elements $c, c' \in Z(U)$ such that $cuw = wu$, $c'vw = wv$. Then

$$\begin{aligned}
[[u, v], w] &= (uvu^{-1}v^{-1})w(vuv^{-1}u^{-1})w^{-1} \\
&= (uvu^{-1})c'w(uv^{-1})(u^{-1}w^{-1}) \\
&= c'uvu^{-1}w(uv^{-1})(w^{-1}u^{-1}c^{-1}) \\
&= c'c^{-1}uv(cw)(w^{-1}v^{-1}(c')^{-1})u^{-1} \\
&= (uvw)(w^{-1}v^{-1}u^{-1}) = 1. \quad \square
\end{aligned}$$

4.2.9. Lemma. *In this situation, if h, i, j, k are distinct and $a, b \in R$, then*

$$[\psi^{-1}(x_{hj}(a)),\, \psi^{-1}(x_{jk}(b))] = [\psi^{-1}(x_{hi}(1)),\, \psi^{-1}(x_{ik}(ab))] \text{ in } U.$$

Proof. By relations 4.2.1,

$$[x_{hj}(a),\, x_{jk}(b)] = [x_{hi}(1),\, x_{ik}(ab)] = x_{hk}(ab) \text{ in } \mathrm{St}(R).$$

Choose $u \in \psi^{-1}(x_{hi}(1))$, $v \in \psi^{-1}(x_{ij}(a))$, and $w \in \psi^{-1}(x_{jk}(b))$. Then $[u, v] \in \psi^{-1}(x_{hj}(a))$, $[v, w] \in \psi^{-1}(x_{ik}(ab))$, and $[u, w] = 1$ by Lemma 4.2.8. Furthermore, $[u, v]$ commutes with u, with v, and with $[v, w]$ by Lemma 4.2.8. So if G is the group generated by u, v, w, $[G, G]$ is commutative. Now apply (c) of Lemma 4.2.6. We obtain

$$[u, [v, w]][v, [w, u]][w, [u, v]] = 1,$$

or since $[w, u] = 1$, $[[u, v], w] = [u, [v, w]]$, which is what we want. $\square$

Proof of Theorem 4.2.7 (continued). Recall that we want to define a splitting map $s : \mathrm{St}(R) \to U$. Since $\mathrm{St}(R)$ is given by generators and relations, it will be enough to define elements $s_{ij}(a) \in U$, $i \neq j$ and $a \in R$, satisfying the same relations as the $x_{ij}(a) \in \mathrm{St}(R)$. Then there will be a unique homomorphism $s : U \to \mathrm{St}(R)$ sending $x_{ij}(a) \mapsto s_{ij}(a)$, and provided we choose $s_{ij}(a) \in \psi^{-1}(x_{ij}(a))$, s will split ψ and thus demonstrate that (U, ψ) is a trivial extension.

So let $a \in R$, $i \neq j$. Choose k distinct from i and j and define

$$s_{ij}(a) = [\psi^{-1}(x_{ik}(1)),\, \psi^{-1}(x_{kj}(a))].$$

By Lemma 4.2.9, this is an element of $\psi^{-1}(x_{ij}(a))$ **independent of the choice of** k. We will show that the elements $s_{ij}(a)$ satisfy the relations 4.2.1. Lemma 4.2.8 immediately gives relation 4.2.1(b). To check 4.2.1(a), let a, $b \in R$ and choose k distinct from i and j. Choose $u \in \psi^{-1}(x_{ik}(1))$, $v \in \psi^{-1}(x_{kj}(a))$, $w \in \psi^{-1}(x_{kj}(b))$. Then by Lemma 4.2.6(b),

$$s_{ij}(a)s_{ij}(b) = [u, v][u, w] = [u, vw][v, [w, u]].$$

But $[w, u]$ commutes with v by Lemma 4.2.8 and $vw \in \psi^{-1}(x_{kj}(a+b))$, so

$$s_{ij}(a)s_{ij}(b) = [u, vw] = [\psi^{-1}(x_{ik}(1)), \psi^{-1}(x_{kj}(a+b))],$$

which by definition is $s_{ij}(a+b)$. Finally, we need to check relation 4.2.1(c), but this follows immediately from Lemma 4.2.9. □

4.2.10. Corollary. *If R is any ring, there is a natural isomorphism $K_2(R) \to H_2(E(R), \mathbb{Z})$.*

Proof. This follows immediately from Theorems 4.1.19 and 4.2.7. □

4.2.11. Remark. Something that comes out of the construction used in the proof of Theorem 4.2.7 is that if x and y are commuting elements of $E(R)$, then $[\varphi^{-1}(x), \varphi^{-1}(y)]$ is a well-defined element of $\mathrm{St}(R)$ which maps to $[x, y] = 1$ under φ, in other words, an element of $K_2(R)$. In fact, this is the most useful way of constructing elements in $K_2(R)$, and under favorable circumstances, $K_2(R)$ is generated by such elements.

A case of particular interest is when R is a commutative ring. Then the units of R, $R^\times$, form an abelian group, and for $u \in R^\times$, $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in E(2, R)$ by Corollary 2.1.3.

4.2.12. Definition. Let R be a commutative ring, and let u, $v \in R^\times$. The **Steinberg symbol** $\{u, v\}$ is defined to be the element $[\varphi^{-1}(d_{12}(u)), \varphi^{-1}(d_{13}(v))]$ of $K_2(R)$ (as in Remark 4.2.11), where

$$d_{12}(u) = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d_{13}(v) = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}.$$

(Since $d_{12}(u)$ and $d_{13}(v)$ commute in $E(R)$, this indeed defines an element of $K_2(R)$.)

Note from the identities used in Corollary 2.1.3 that

$$d_{12}(u) = e_{12}(u)e_{21}(-u^{-1})e_{12}(u)e_{12}(-1)e_{21}(1)e_{12}(-1),$$
$$d_{13}(v) = e_{13}(v)e_{31}(-v^{-1})e_{13}(v)e_{13}(-1)e_{31}(1)e_{13}(-1),$$

so that if we define $w_{ij}(u) \in \mathrm{St}(R)$ and $h_{ij}(u) \in \mathrm{St}(R)$ by

$$w_{ij}(u) =_{\mathrm{def}} x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u), \quad h_{ij}(u) =_{\mathrm{def}} w_{ij}(u)w_{ij}(-1),$$

then

$$\varphi(w_{12}(u)) = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}, \quad \varphi(h_{12}(u)) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},$$

and

$$\{u, v\} =_{\text{def}} [h_{12}(u), h_{13}(v)].$$

The Steinberg symbols can also be described in terms of group homology. Note that if G is a free abelian group on two generators s and t, then $\mathbb{Z}G \cong \mathbb{Z}[t, t^{-1}, s, s^{-1}]$, and the trivial G-module $\mathbb{Z}$ has the free resolution

$$0 \to \mathbb{Z}G \xrightarrow{(s-1,\,1-t)} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{(t-1,\,s-1)} \mathbb{Z}G \xrightarrow{t\mapsto 1,\, s\mapsto 1} \mathbb{Z}.$$

Furthermore, if σ is the automorphism of G interchanging t and s, then the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}G & \xrightarrow{(s-1,\,1-t)} & \mathbb{Z}G \oplus \mathbb{Z}G & \xrightarrow{(t-1,\,s-1)} & \mathbb{Z}G & \xrightarrow{t\mapsto 1,\, s\mapsto 1} & \mathbb{Z} \\ \| & & {\scriptstyle -\sigma}\downarrow & & \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}\downarrow & & {\scriptstyle \sigma}\downarrow & & \| \\ 0 & \longrightarrow & \mathbb{Z}G & \xrightarrow{(s-1,\,1-t)} & \mathbb{Z}G \oplus \mathbb{Z}G & \xrightarrow{(t-1,\,s-1)} & \mathbb{Z}G & \xrightarrow{t\mapsto 1,\, s\mapsto 1} & \mathbb{Z}. \end{array}$$

Thus $H_\bullet(G, \mathbb{Z})$ is the homology of the complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z},$$

with σ acting by -1 on the first $\mathbb{Z}$ and interchanging the two summands in the $\mathbb{Z}^2$, and in particular $H_2(G, \mathbb{Z})$ is free abelian on a generator that is sent to its inverse if we interchange t and s. Thus $H_2(G, \mathbb{Z})$ is naturally isomorphic to $\bigwedge^2 G$, the alternating tensor product or second exterior power (this is a special case of the result of Exercise 4.1.31). The commuting elements $d_{12}(u)$ and $d_{13}(v)$ of $E(R)$ define a map $\alpha : G \to E(R)$ with $t \mapsto d_{12}(u)$, $s \mapsto d_{13}(v)$, and α_* sends the canonical generator $t \wedge s$ of $H_2(G, \mathbb{Z})$ to the Steinberg symbol $\{u, v\}$ in $H_2(E(R), \mathbb{Z}) \cong K_2(R)$. In fact, the diagonal matrices in $E(3, R)$ are an abelian subgroup isomorphic to $(R^\times)^2$ (the determinant must be 1, so the (3, 3)-entry is determined by the (1, 1) and (2, 2)-entries), generated by elements of the form $d_{12}(u)$ and $d_{13}(v)$. By Exercise 4.1.32, $H_2((R^\times)^2, \mathbb{Z}) \cong \bigwedge^2((R^\times)^2)$, which evidently contains $(R^\times \times \{1\}) \wedge (\{1\} \times R^\times)$ as a direct summand. The subgroup of $K_2(R)$ generated by the Steinberg symbols is the image of $(R^\times \times \{1\}) \wedge (\{1\} \times R^\times)$ in $H_2(E(R), \mathbb{Z}) \cong K_2(R)$.

4.2.13. Example. Of course, the whole definition would be a little silly if $\{u, v\}$ were always trivial. However, if $u = v = -1$ and $R = \mathbb{R}$, then $d_{12}(u)$ and $d_{13}(v)$ generate a Klein 4-group in $SO(3) \subset SL(3, \mathbb{R})$ (see Exercise 4.1.26), and the inverse image of this 4-group in the universal covering group $SU(2) \subset \widetilde{SL}(3, \mathbb{R})$ is a quaternion group Q. The same holds if we first embed $SL(3, \mathbb{R})$ in $SL(n, \mathbb{R})$ for any $n > 3$ and then take

the universal cover $\widetilde{SL(n, \mathbb{R})}$, since the embedding $SO(3) \hookrightarrow SL(n, \mathbb{R})$ induces an isomorphism on π_1 and thus the induced mapping $SU(2) \to \widetilde{SL(n, \mathbb{R})}$ is injective (compare Example 1.6.13 and Examples 4.1.5). Since $\varinjlim \widetilde{SL(n, \mathbb{R})}$ is a quotient group of $\mathrm{St}(\mathbb{R})$, this proves that $\{-1, -1\}$ maps to an element of order 2 in the corresponding quotient of $K_2(\mathbb{R})$. In fact this quotient group splits (see (2) of Exercise 4.1.26, or else note that by Lemma 4.2.14 below, $\{-1, -1\}$ can have order at most 2), so $\{-1, -1\} \in K_2(\mathbb{R})$ has order exactly 2.

4.2.14. Lemma. *Let R be a commutative ring. The Steinberg symbol map $R^\times \times R^\times \xrightarrow{\{\,,\,\}} K_2(R)$ is skew-symmetric and bilinear, that is, $\{u, v\} = \{v, u\}^{-1}$ and $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$.*

Proof. This is immediate from the homology approach, since as mentioned above and proved in Exercise 4.1.32, $H_2(G, \mathbb{Z}) \cong \bigwedge^2(G)$ for G an abelian group. Alternatively, we can check this directly from the definition above, since

$$\varphi(w_{23}(1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Thus $\varphi(w_{23}(1))$ conjugates $d_{12}(u)$ to $d_{13}(u)$ and vice versa. To prove skew-symmetry, note that

$$\begin{aligned}
\{v, u\} &=_{\text{def}} [\varphi^{-1}(d_{12}(v)), \varphi^{-1}(d_{13}(u))] \\
&= [w_{23}(1)\varphi^{-1}(d_{13}(v))w_{23}(1)^{-1}, w_{23}(1)\varphi^{-1}(d_{12}(u))w_{23}(1)^{-1}] \\
&= w_{23}(1)[\varphi^{-1}(d_{13}(v)), \varphi^{-1}(d_{12}(u))]w_{23}(1)^{-1} \\
&= w_{23}(1)\{u, v\}^{-1}w_{23}(1)^{-1} = \{u, v\}^{-1}.
\end{aligned}$$

Here we have used Lemma 4.2.6(a) and the fact that $K_2(R)$ is central in $\mathrm{St}(R)$.

To prove bilinearity, note that by Lemma 4.2.6(b),

$$\begin{aligned}
\{u, v_1v_2\} &=_{\text{def}} [\varphi^{-1}(d_{12}(u)), \varphi^{-1}(d_{13}(v_1v_2))] \\
&= [\varphi^{-1}(d_{12}(u)), \varphi^{-1}(d_{13}(v_1))\varphi^{-1}(d_{13}(v_2))] \\
&= [\varphi^{-1}(d_{12}(u)), \varphi^{-1}(d_{13}(v_1))][\varphi^{-1}(d_{12}(u)), \varphi^{-1}(d_{13}(v_2))] \\
&\qquad [\varphi^{-1}(d_{13}(v_1)), [\varphi^{-1}(d_{13}(v_2)), \varphi^{-1}(d_{12}(u))]]^{-1} \\
&= \{u, v_1\}\{u, v_2\}[\varphi^{-1}(d_{13}(v_1)), \{u, v_2\}^{-1}]^{-1} \\
&= \{u, v_1\}\{u, v_2\},
\end{aligned}$$

again since $K_2(R)$ is central. Bilinearity in the other variable follows from the skew-symmetry. ☐

Most of the rest of this section will be taken up with calculations using the relations (4.2.1), in order to give a slightly more convenient description of the Steinberg symbols and in order to prove that they satisfy two additional relations. These relations are important for the applications of $K_2(R)$ in the next section.

4.2.15. Lemma. *If R is any ring and $u \in R^\times$, the elements*

$$w_{ij}(u) =_{\text{def}} x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u), \quad h_{ij}(u) =_{\text{def}} w_{ij}(u)w_{ij}(-1)$$

of $\mathrm{St}(R)$ *defined in 4.2.12 satisfy*

$$(w_{ij}(u))^{-1} = (w_{ij}(-u)), \quad w_{ij}(u) = w_{ji}(-u^{-1}),$$

$h_{ij}(1) = 1$. *In addition, if* u, $v \in R^\times$ *and* $k \neq l$, $i \neq j$, *then*

$$w_{kl}(u)w_{ij}(v)\left(w_{kl}(u)\right)^{-1} = \begin{cases} w_{ij}(v), & i,\ j,\ k,\ l \text{ all distinct}, \\ w_{lj}(-u^{-1}v), & k = i, \quad i,\ j,\ l \text{ all distinct}, \\ w_{il}(-vu), & k = j, \quad i,\ j,\ k \text{ all distinct}, \\ w_{ji}(-u^{-1}vu^{-1}), & k = i,\ j = l. \end{cases}$$

Proof. To begin with, by (4.2.1)(a),

$$\begin{aligned} w_{ij}(u)w_{ij}(-u) &= x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)x_{ij}(-u)x_{ji}(u^{-1})x_{ij}(-u) \\ &= x_{ij}(u)x_{ji}(-u^{-1})x_{ji}(u^{-1})x_{ij}(-u) \\ &= x_{ij}(u)x_{ij}(-u) = 1, \end{aligned}$$

so $(w_{ij}(u))^{-1} = w_{ij}(-u)$. In particular, $h_{ij}(1) = w_{ij}(1)w_{ij}(-1) = 1$. The fact that $w_{kl}(u)$ and $w_{ij}(v)$ commute if i, j, k, l are all distinct is obvious from (4.2.1)(b). Next suppose i, j, l are all distinct and $k = i$. We have by (b), (c), and (d) of (4.2.1)

$$\begin{aligned} &w_{il}(u)x_{ij}(v)\left(w_{il}(u)\right)^{-1} \\ &\quad = x_{il}(u)x_{li}(-u^{-1})x_{il}(u)\left(x_{ij}(v)\right)x_{il}(-u)x_{li}(u^{-1})x_{il}(-u) \\ &\quad = x_{il}(u)x_{li}(-u^{-1})\left(x_{ij}(v)\right)x_{li}(u^{-1})x_{il}(-u) \\ &\quad = x_{il}(u)\left(x_{lj}(-u^{-1}v)x_{ij}(v)\right)x_{il}(-u) \\ &\quad = x_{ij}(-v)x_{lj}(-u^{-1}v)x_{ij}(v) \\ &\quad = x_{lj}(-u^{-1}v). \end{aligned}$$

Similarly

$$\begin{aligned} &w_{il}(u)x_{ji}(-v^{-1})\left(w_{il}(u)\right)^{-1} \\ &\quad = x_{il}(u)x_{li}(-u^{-1})x_{il}(u)\left(x_{ji}(-v^{-1})\right)x_{il}(-u)x_{li}(u^{-1})x_{il}(-u) \\ &\quad = x_{il}(u)x_{li}(-u^{-1})\left(x_{jl}(v^{-1}u)x_{ji}(-v^{-1})\right)x_{li}(u^{-1})x_{il}(-u) \\ &\quad = x_{il}(u)\left(x_{ji}(v^{-1})x_{jl}(v^{-1}u)x_{ji}(-v^{-1})\right)x_{il}(-u) \\ &\quad = x_{jl}(-v^{-1}u)x_{ji}(v^{-1})x_{jl}(v^{-1}u)x_{jl}(v^{-1}u)x_{ji}(-v^{-1}) \\ &\quad = x_{ji}(v^{-1})x_{jl}(v^{-1}u)x_{ji}(-v^{-1}) \\ &\quad = x_{jl}(v^{-1}u). \end{aligned}$$

So

$$\begin{aligned} w_{il}(u)w_{ij}(v)\,(w_{il}(u))^{-1} &= w_{il}(u)x_{ij}(v)x_{ji}(-v^{-1})x_{ij}(v)\,(w_{il}(u))^{-1} \\ &= x_{lj}(-u^{-1}v)x_{jl}(v^{-1}u)x_{lj}(-u^{-1}v) \\ &= w_{lj}(-u^{-1}v) \end{aligned}$$

and

$$\begin{aligned} w_{il}(u)w_{ji}(v)\,(w_{il}(u))^{-1} &= w_{il}(u)x_{ji}(v)x_{ij}(-v^{-1})x_{ji}(v)\,(w_{il}(u))^{-1} \\ &= x_{jl}(-vu)x_{lj}(u^{-1}v^{-1})x_{jl}(-vu) \\ &= w_{jl}(-vu), \end{aligned}$$

which gives the second and third relations.

Finally, to get the last relation, choose l distinct from i and j and note that by what we've already proved, $w_{ij}(v) = w_{il}(1)w_{lj}(v)w_{il}(-1)$. So

$$\begin{aligned} w_{ij}(u)w_{ij}(v)\,(w_{ij}(u))^{-1} &= w_{ij}(u)\,(w_{il}(1)w_{lj}(v)w_{il}(-1))\,(w_{ij}(u))^{-1} \\ &= w_{jl}(-u^{-1})w_{li}(vu^{-1})w_{jl}(u^{-1}) \\ &= w_{ji}(-u^{-1}vu^{-1}). \end{aligned}$$

Taking $u = v$ in this relation gives

$$w_{ij}(u) = w_{ji}(-u^{-1}uu^{-1}) = w_{ji}(-u^{-1}). \quad \square$$

4.2.16. Corollary. *If R is a commutative ring and $u,\ v \in R^\times$, then*

$$h_{12}(uv) = h_{12}(u)h_{12}(v)\{u,\ v\}^{-1}.$$

In other words, if we identify $R^\times$ with a subgroup of $E(R)$ via $u \mapsto d_{12}(u)$, then h_{12} gives a section $R^\times \to \mathrm{St}(R)$, and the Steinberg symbol is the inverse of the associated cocycle in $Z^2(R^\times,\ K_2(R))$ as defined in the proof of Theorem 4.1.16.

Proof. We have

$$\begin{aligned} \{u,\ v\} &= [h_{12}(u),\ h_{13}(v)] \\ &= h_{12}(u)h_{13}(v)(h_{12}(u))^{-1}(h_{13}(v))^{-1} \\ &= h_{12}(u)w_{13}(v)w_{13}(-1)\,(w_{12}(1)w_{12}(-u))\,w_{13}(1)w_{13}(-v) \\ &= h_{12}(u)w_{13}(v)\,(w_{32}(1)w_{32}(-u))\,w_{13}(-v) \\ &= h_{12}(u)w_{12}(v)w_{12}(-vu) \\ &= h_{12}(u)w_{12}(v)w_{12}(-1)w_{12}(1)w_{12}(-vu) \\ &= h_{12}(u)h_{12}(v)(h_{12}(vu))^{-1}. \quad \square \end{aligned}$$

4.2.17. Theorem. *If R is a commutative ring, the Steinberg symbol map $R^\times \times R^\times \xrightarrow{\{\,,\,\}} K_2(R)$ satisfies the additional relations*

(a) $\{u, -u\} = 1$ *for* $u \in R^\times$,
(b) $\{u, 1-u\} = 1$ *for* $u \in R^\times$, $1-u \in R^\times$.

Proof. (a) By Corollary 4.2.16, we need to show that $h_{12}(u)h_{12}(-u) = h_{12}(-u^2)$. But by the last identities of Lemma 4.2.15,

$$\begin{aligned} h_{12}(u)h_{12}(-u) &= w_{12}(u)w_{12}(-1)w_{12}(-u)w_{12}(-1) \\ &= w_{21}(u^{-2})w_{12}(-1) \\ &= w_{12}(-u^{-2})w_{12}(-1) = h_{12}(-u^2). \end{aligned}$$

(b) By Corollary 4.2.16, we need to show that

$$h_{12}(u)h_{12}(1-u) = h_{12}(u-u^2).$$

But

$$\begin{aligned} &h_{12}(u)h_{12}(1-u) \\ &\quad= w_{12}(u)w_{12}(-1)w_{12}(1-u)w_{12}(-1) \\ &\quad= w_{12}(u)w_{21}(1)w_{12}(1-u)w_{12}(-1) \\ &\quad= w_{12}(u)x_{21}(1)x_{12}(-1)x_{21}(1)w_{12}(1-u)w_{12}(-1) \\ &\quad= \left(w_{12}(u)x_{21}(1)w_{12}(-u)\right) w_{12}(u)x_{12}(-1)w_{12}(1-u) \\ &\qquad\qquad \left(w_{12}(u-1)x_{21}(1)w_{12}(1-u)\right) w_{12}(-1) \\ &\quad= x_{12}(-u^2)w_{12}(u)x_{12}(-1)w_{12}(1-u)x_{12}(-(1-u)^2)w_{12}(-1) \\ &\quad= x_{12}(-u^2)x_{12}(u)x_{21}(-u^{-1})x_{12}(u) \\ &\qquad\qquad x_{12}(-1)w_{12}(1-u)x_{12}(-(1-u)^2)w_{12}(-1) \\ &\quad= x_{12}(u-u^2)x_{21}(-u^{-1})x_{12}(u-1)w_{12}(1-u)x_{12}(-(1-u)^2)w_{12}(-1) \\ &\quad= x_{12}(u-u^2)x_{21}(-u^{-1})x_{12}(u-1)x_{12}(1-u) \\ &\qquad\qquad x_{21}(-(1-u)^{-1})x_{12}(1-u)x_{12}(-(1-u)^2)w_{12}(-1) \\ &\quad= x_{12}(u-u^2)x_{21}(-u^{-1})x_{21}(-(1-u)^{-1})x_{12}(u-u^2)w_{12}(-1) \\ &\quad= x_{12}(u-u^2)x_{21}(-u^{-1}(1-u)^{-1})x_{12}(u-u^2)w_{12}(-1) \\ &\quad= w_{12}(u(1-u))w_{12}(-1) = h_{12}(u-u^2). \quad \square \end{aligned}$$

4.2.18. Corollary. *If R is a finite field, all Steinberg symbols vanish in $K_2(R)$.*

Proof. Let $R = \mathbb{F}_q$, the finite field of q elements. Since $\mathbb{F}_q^\times$ is cyclic of order $q-1$, we may choose a generator u for $\mathbb{F}_q^\times$, and by bilinearity of the

symbol map, it suffices to prove that $\{u, u\} = 1$ (we are using multiplicative notation for K_2). By skew-symmetry of the Steinberg symbol, $\{u, u\} = \{u, u\}^{-1}$, *i.e.*, $\{u, u\}$ has order at most 2. If q is a power of 2, then $-1 = 1$ in $\mathbb{F}_q$, so by (a) of the Theorem, $\{u, u\} = \{u, -u\} = 1$. If q is odd, then by bilinearity and (a) of the Lemma,

$$\{u, u\} = \{u, -u\}\{u, -1\} = \{u, -1\} = \{u, u^{\frac{q-1}{2}}\} = \{u, u\}^{\frac{q-1}{2}}.$$

So if $\frac{q-1}{2}$ is even, we again conclude that $\{u, u\} = 1$. If $\frac{q-1}{2}$ is odd, then -1 is not a perfect square in $\mathbb{F}_q$. Suppose we can choose $w \in \mathbb{F}_q^\times$ such that neither w nor $1-w$ is a perfect square in $\mathbb{F}_q$. By (b) of the Theorem, $\{w, 1-w\} = 1$. But since neither w nor $1-w$ is a perfect square, they are both odd powers of u, so $\{w, 1-w\}$ is an odd power of $\{u, u\}$ and $\{u, u\} = 1$. So it's enough to show a suitable w exists. Since -1 is not a perfect square in $\mathbb{F}_q$, we need to show there is a w, not a perfect square, such that $w - 1$ **is** a perfect square. But such a w exists, since otherwise adding 1 to a perfect square would always give a perfect square in $\mathbb{F}_q$, and $1, 2, \ldots, -1$ would all be perfect squares, a contradiction. □

4.2.19. Example. If $R = \mathbb{Z}$, then $R^\times$ has only two elements, 1 and -1. We saw in Example 4.2.13 that $\{-1, -1\}$ has order 2, and this is the only non-trivial Steinberg symbol, since $\{u, v\} = 1$ if $u = 1$ or $v = 1$. In this particular case, relation (b) of Theorem 4.2.17 is vacuous and relation (a) is trivial. As mentioned in Example 4.2.5, the element

$$x = (x_{12}(1)x_{21}(-1)x_{12}(1))^4 = (w_{12}(1))^4$$

of $\mathrm{St}(\mathbb{Z})$ also lies in $K_2(\mathbb{Z})$. However

$$h_{12}((-1)\cdot(-1))(h_{12}(-1))^{-1}(h_{12}(-1))^{-1} = \{-1, -1\}^{-1} = \{-1, -1\}$$

by Corollary 4.2.16, and since $w_{12}(1) = (w_{12}(-1))^{-1}$ and $h_{12}(1) = 1$ by Lemma 4.2.15, the left-hand side simplifies to

$$(h_{12}(-1))^{-2} = (w_{12}(-1)w_{12}(-1))^{-2} = (w_{12}(-1))^{-4} = (w_{12}(1))^4 = x.$$

So the element of $K_2(\mathbb{Z})$ constructed in Example 4.2.5 is the same as $\{-1, -1\}$.

For more complicated rings, the relations in Theorem 4.2.17 are more interesting. For instance, if $R = \mathbb{Z}[\xi]$, where $\xi = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a primitive 6-th root of unity, then $R^\times$ is a cyclic group of order 6 generated by ξ, and the group of Steinberg symbols is generated by $\{\xi, \xi\}$, which can have order at most 2. But $1 - \xi = \bar{\xi} = \xi^{-1}$, so $1 = \{\xi, 1-\xi\} = \{\xi, \xi^{-1}\} = \{\xi, \xi\}^{-1}$, so $\{\xi, \xi\} = 1$. It follows that $\{-1, -1\} = \{\xi^3, \xi^3\} = 1$ in $K_2(R)$. Since $R \hookrightarrow \mathbb{C}$, this shows for instance that the map on K_2 induced by the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ kills $\{-1, -1\}$.

4.2.20. Exercise. Show that in Corollary 4.2.16, h_{12} can be replaced by h_{ij} for any $i \neq j$. In other words, if R is a commutative ring and $u, v \in R^\times$, then

$$h_{ij}(uv) = h_{ij}(u)h_{ij}(v)\{u, v\}^{-1}.$$

(Use the relations in Lemma 4.2.15.)

4.2.21. Exercise. Show that Corollary 4.2.16 implies that if R is a commutative ring, the subgroup of $K_2(R)$ generated by the Steinberg symbols contains the image of the corestriction map

$$H_2(R^\times, \mathbb{Z}) \to H_2(E(R), \mathbb{Z}),$$

where $R^\times \hookrightarrow E(2, R) \hookrightarrow E(R)$ via $d_{12} : u \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. By Exercise 4.1.32, $H_2(R^\times, \mathbb{Z}) \cong \bigwedge^2(R^\times)$, so in general one can't expect the map $H_2(R^\times, \mathbb{Z}) \to H_2(E(R), \mathbb{Z})$ to be injective; at the very least one has to factor $H_2(R^\times, \mathbb{Z}) \cong \bigwedge^2(R^\times)$ by the relations of Theorem 4.2.17. Show also by looking at the case of $R = \mathbb{Z}$ that the corestriction map $H_2(R^\times, \mathbb{Z}) \to H_2(E(R), \mathbb{Z})$ need not be surjective.

4.2.22. Exercise. Let R be the Dedekind domain $\mathbb{Z}[\sqrt{D}]$, the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{D})$ with $D \in \mathbb{N}$ square-free and not congruent to $1 \mod 4$. By Theorem 2.3.8, $R^\times$ is the product of the 2-element group $\{\pm 1\}$ and an infinite cyclic group. Compute the quotient of $(R^\times) \otimes (R^\times)$ by the relations of skew-symmetry and of Theorem 4.2.17. (First show relation (b) is vacuous; in other words, there is no $v \in R^\times$ with $1 - v \in R^\times$, since $a + b\sqrt{D}$ can only be a unit if $a^2 - Db^2 = \pm 1$.)

4.2.23. Exercise: Morita invariance of K_2. Show that for any ring, there is a natural identification of $E(M_n(R))$ with $E(R)$, and thus of the universal central extension of the former with the universal central extension of the latter. Obtain a "Morita invariance" isomorphism $K_2(M_n(R)) \cong K_2(R)$.

4.2.24. Exercise: a ring with vanishing K_2. Let k be a field and let V be an infinite-dimensional vector space over k. Let $R = \mathrm{End}_k(V)$. Show that $K_2(R) = 1$. Hint: V is isomorphic to an infinite direct sum of copies of itself. Thus if $A \in K_2(R) \subseteq \mathrm{St}(R)$, one can form "$\infty \cdot A$" by replacing each $x_{ij}(a)$ in the expression for A by $x_{ij}(\infty \cdot a)$ (cf. Exercise 2.1.7) and regard it also as an element of $K_2(R)$. Show that $A \oplus (\infty \cdot A)$ is conjugate to $(\infty \cdot A)$, hence that A represents the identity in $K_2(R)$. (Compare Example 1.2.6 and Exercise 2.1.7.)

3. Milnor's K_2

In the last section, we defined $K_2(R)$ both in terms of the Steinberg group and in terms of homology of $E(R)$. We also showed how to construct

elements of $K_2(R)$ (when R is commutative) using Steinberg symbols. In this section, we show how K_2 fits into the general framework of algebraic K-theory, via an exact sequence linking it with K_1 and K_0 and via a number of applications. The functor K_2 is unfortunately difficult to compute, but we deduce some information about it at least when R is a field.

4.3.1. Theorem. *Let R be a ring and $I \subseteq R$ an ideal. Then there is a natural exact sequence*

$$K_2(R) \xrightarrow{q_*} K_2(R/I) \xrightarrow{\partial} K_1(R, I) \to K_1(R) \xrightarrow{q_*} K_1(R/I) \\ \xrightarrow{\partial} K_0(R, I) \to K_0(R) \xrightarrow{q_*} K_0(R/I),$$

where q_* is induced by the quotient map $q : R \twoheadrightarrow R/I$, extending the exact sequence of Theorem 2.5.4 to the left.

Proof. We need to define the map $K_2(R/I) \xrightarrow{\partial} K_1(R, I)$ and to verify exactness at $K_1(R, I)$ and at $K_2(R/I)$. We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccccc}
 & & & & & & 1 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & 1 & \to & E(R, I) & \to & GL(R, I) & \to & K_1(R, I) & \to & 1 \\
 & & & & & & \downarrow & & \downarrow & & \| \\
1 & \to & K_2(R) & \to & \mathrm{St}(R) & \xrightarrow{\varphi_R} & GL(R) & \to & K_1(R) & \to & 1 \\
\| & & {\scriptstyle q_*}\downarrow & & {\scriptstyle q_*}\downarrow & & {\scriptstyle q_*}\downarrow & & {\scriptstyle q_*}\downarrow & & \| \\
1 & \to & K_2(R/I) & \to & \mathrm{St}(R/I) & \xrightarrow{\varphi_{R/I}} & GL(R/I) & \to & K_1(R/I) & \to & 1.
\end{array}$$

The map $q_* : \mathrm{St}(R) \to \mathrm{St}(R/I)$ is surjective, since if $\dot{a} \in R/I$ is $q(a)$ for $a \in R$, then $q_*(x_{ij}(a)) = x_{ij}(\dot{a})$, and thus every generator of $\mathrm{St}(R/I)$ is in the image of q_*. Therefore we can define ∂ by the usual "snake" process: if $x \in K_2(R/I)$, write $x = q_*(y)$ for some $y \in \mathrm{St}(R)$, chosen by lifting each $x_{ij}(\dot{a})$ appearing in an expression for x to $x_{ij}(a)$, where $a \in q^{-1}(\dot{a})$. Then $\varphi_R(y) \in E(R)$ and maps to 1 in $GL(R/I)$ (by commutativity of the diagram and exactness of the bottom line). So $\varphi_R(y) \in GL(R, I)$ and we define $\partial(x)$ to be its class in $GL(R, I)/E(R, I) \cong K_1(R, I)$. To show this is well defined, suppose $x = x_{i_1j_1}(\dot{a}_1) \cdots x_{i_rj_r}(\dot{a}_r)$ and let $y = x_{i_1j_1}(a_1) \cdots x_{i_rj_r}(a_r)$ and $y' = x_{i_1j_1}(a'_1) \cdots x_{i_rj_r}(a'_r)$, where $\dot{a}_k = q(a_k) = q(a'_k)$. We need to show that $\varphi_R(y)$ and $\varphi_R(y)$ differ by an element of $E(R, I)$, hence have the same class in $K_1(R, I)$. Let $b_k = a'_k - a_k \in I$. Then

$$\varphi_R(y) = e_{i_1j_1}(a_1) \cdots e_{i_rj_r}(a_r)$$

and

$$\begin{aligned}\varphi_R(y') &= e_{i_1j_1}(a'_1)\cdots e_{i_rj_r}(a'_r) = e_{i_1j_1}(b_1)e_{i_1j_1}(a_1)\cdots e_{i_rj_r}(b_r)e_{i_rj_r}(a_r)\\ &= e_{i_1j_1}(b_1)\left(e_{i_1j_1}(a_1)e_{i_1j_1}(b_2)\left(e_{i_1j_1}(a_1)\right)^{-1}\right)\\ &\quad\times\left(e_{i_1j_1}(a_1)e_{i_1j_1}(a_2)e_{i_1j_1}(b_2)\left(e_{i_1j_1}(a_2)\right)^{-1}\left(e_{i_1j_1}(a_1)\right)^{-1}\right)\\ &\qquad\cdots\varphi_R(y),\end{aligned}$$

so $\varphi_R(y)$ and $\varphi_R(y)$ differ by an element of the normal subgroup of $E(R)$ generated by the $e_{ij}(b)$, $b \in I$, *i.e.*, by an element of $E(R, I)$. Thus $\partial : K_2(R/I) \to K_1(R, I)$ is well defined.

Now we check exactness. The composite

$$K_2(R) \xrightarrow{q_*} K_2(R/I) \xrightarrow{\partial} K_1(R, I)$$

is trivial, since if $y \in K_2(R)$ and $x = q_*(y)$, then $\varphi_R(y) = 1$ and thus $\partial(x) = [\varphi_R(y)]$ is trivial. Conversely, if $\partial(x) = 1$, this means we can choose $y \in \mathrm{St}(R)$ such that $x = q_*(y) \in K_2(R/I)$ and $\varphi_R(y) \in E(R, I)$, *i.e.*, $\varphi_R(y)$ is a product of terms of the form

$$\left(e_{i_1j_1}(a_1)e_{i_1j_1}(a_2)\cdots e_{i_rj_r}(a_r)\right)e_{ij}(b)\left(e_{i_1j_1}(a_1)e_{i_1j_1}(a_2)\cdots e_{i_rj_r}(a_r)\right)^{-1}.$$

Changing y if necessary by an element of $K_2(R)$, this means we can assume y is a product of terms of the form

$$\left(x_{i_1j_1}(a_1)x_{i_1j_1}(a_2)\cdots x_{i_rj_r}(a_r)\right)x_{ij}(b)\left(x_{i_1j_1}(a_1)x_{i_1j_1}(a_2)\cdots x_{i_rj_r}(a_r)\right)^{-1},$$

$b \in I$. As we saw above, we are free to replace $x_{ij}(b)$ by $x_{ij}(0) = 1$, which then shows y can be made trivial after modification by an element of $K_2(R)$. So $\ker(\partial) \subseteq \mathrm{im}(q_*)$.

It remains to check exactness at $K_1(R, I)$. The composite

$$K_2(R/I) \xrightarrow{\partial} K_1(R, I) \to K_1(R)$$

is 1, since if $x \in K_2(R/I)$ and we choose $y \in K_2(R)$ with $x = q_*(y)$ as above, then $\varphi_R(y) \in E(R)$ and maps to 1 in $K_1(R) = GL(R)/E(R)$. Conversely, if $g \in GL(R, I)$ and the image $[g]$ of g in $K_1(R, I)$ maps to 1 in $K_1(R)$, this means $g \in E(R)$. So $g = \varphi_R(y)$ for some $y \in \mathrm{St}(R)$. If $x = q_*(y) \in \mathrm{St}(R/I)$, then $\varphi_{R/I}(x) = q_*(g)$, which is 1 since $g \in GL(R, I)$. So $x \in \ker(\varphi_{R/I}) = K_2(R/I)$, and $\partial(x) = [g]$ by construction. □

So far, we have not been able to compute K_2 in very many examples, though at least we've produced examples of rings where it is or is not trivial (Example 4.2.13 and Exercise 4.2.24). Our aim next is to study K_2 in the case of a (commutative) field. Unlike K_1 and K_0 which are not particularly interesting for fields, this is a decidedly non-trivial subject with a lot of applications. However, following ideas in [Keune], Theorem 4.3.1 now gives a way to relate the calculation of K_2 of a field to a problem about K_1, which can be studied using the theory of relative Mennicke symbols from Theorem 2.5.12.

4.3.2. Lemma. *It F is a field, there is a natural epimorphism $\partial : K_2(F) \to SK_1(F[t], (t^2-t))$.*

Proof. Let $R = F[t]$, which is a PID, and let $I = (t^2 - t) \subseteq R$. Then $R/I = F[t]/(t^2 - t) \cong F \times F$, with the quotient map $q : R \to F \times F$ corresponding to evaluation at 0 and at 1. By Corollary 2.3.3, $K_1(R) = F^\times$, and the map

$$q_* : K_1(R) \to K_1(R/I) \cong K_1(F \times F) \cong F^\times \times F^\times$$

is obviously the diagonal map, which is injective. Furthermore, since the map $R \to F$ corresponding to evaluation at 0 is split surjective via the inclusion of constant polynomials, we get a splitting $K_2(R) \cong K_2(F) \times NK_2(F)$ as in Theorem 3.2.22. The map

$$q_* : K_2(R) \to K_2(R/I) \cong K_2(F \times F) \cong K_2(F) \times K_2(F)$$

is obviously the diagonal injection on the $K_2(F)$ factor, so the cokernel is a quotient of $K_2(F)$. Then ∂ gives an isomorphism of this quotient of $K_2(F)$ onto $K_1(R, I) = SK_1(R, I)$ (since any unit in $F[t]$ is actually a unit in F, and thus can't be $\equiv 1 \mod I$ unless it is equal to 1). □

In fact, one can show that $K_2(R) \cong K_2(F)$, and ∂ is an isomorphism of $K_2(F)$ onto $SK_1(F[t], (t^2 - t))$. This makes the calculation of $K_2(F)$ essentially equivalent to the calculation of the relations among the relative Mennicke symbols from Theorem 2.5.12. (See Proposition 4.4.2 below.)

The key to getting more information is the following theorem, which can be proved either using calculations in the Steinberg group (for a proof along these lines, see [Milnor, §9]) or else using homology, as in [Hutchinson], on which the following proof is based.

4.3.3. Theorem. *If F is a field, then $K_2(F)$ is generated by Steinberg symbols.*

Proof. Recall that in the case of a field, $E(n, F) = SL(n, F)$ (Proposition 2.2.2). By Definition 4.2.12 and Exercise 4.2.21, the subgroup of $K_2(F)$ generated by the Steinberg symbols is precisely the image of the corestriction map $H_2(F^\times \times F^\times, \mathbb{Z}) \to H_2(SL(F), \mathbb{Z})$, where $F^\times \times F^\times \hookrightarrow SL(3, F) \hookrightarrow SL(F)$ via $(a, b) \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a^{-1}b^{-1} \end{pmatrix}$. We will show that except in the case of a few finite fields of small cardinality, the corestriction maps

$$H_2(F^\times \times F^\times, \mathbb{Z}) \to H_2(SL(3, F), \mathbb{Z}) \to H_2(SL(4, F), \mathbb{Z}) \to \cdots$$

are all surjective; in fact $H_2(SL(n, F), \mathbb{Z}) \to H_2(SL(n+1, F), \mathbb{Z})$ is an isomorphism for $n \geq 3$. Since $K_2(F) = \varinjlim H_2(SL(n, F), \mathbb{Z})$ (Exercise 4.1.29), this will prove the theorem and a bit more. We split the proof into

several steps; the Theorem is obtained by combining Propositions 4.3.6 and 4.3.11 and Theorem 4.3.12. The first Lemma involves some of the same ideas as Theorem 4.1.20. It (and similar results) is actually most easily proved using the theory of spectral sequences, but we give a direct proof, at least for the case we need. □

4.3.4. Lemma. *Let $G = N \ltimes H$ be the semidirect product of a normal subgroup N by a group H, and let M be a G-module. If $H_p(H, H_q(N, M)) = 0$ for all p and q with $p + q = j$, then $H_j(G, M) = 0$.*

Proof. This is clear if $j = 0$, since

$$H_0(H, H_0(N, M)) = (M/\{n_* - 1 : n \in N\})/\{h_* - 1 : h \in H\} = M/\{g_* - 1 : g \in G\} = H_0(G, M).$$

The general case is reduced to this case by induction on j, using resolutions. For example, we do the cases $j = 1$ and $j = 2$, which we will need below. Start by choosing an exact sequence of G-modules

$$0 \to M_1 \to F_0 \to M \to 0,$$

with F_0 free, and note that F_0 is free not just as a G-module but also as an N-module. The corresponding exact sequences in N-homology and in G-homology give

$$H_{j+1}(N, M) \cong H_j(N, M_1), \quad H_{j+1}(G, M) \cong H_j(G, M_1), \qquad j \geq 1,$$

as well as the exact sequences

$$0 \to H_1(N, M) \to H_0(N, M_1) \to H_0(N, F_0) \to H_0(N, M) \to 0,$$

$$0 \to H_1(G, M) \to H_0(G, M_1) \to H_0(G, F_0) \to H_0(G, M) \to 0.$$

Split the first of these into two exact sequences

$$0 \to H_1(N, M) \to H_0(N, M_1) \to K \to 0,$$

$$0 \to K \to H_0(N, F_0) \to H_0(N, M) \to 0.$$

Applying H-homology and assuming that $H_0(H, H_1(N, M)) = 0$ and $H_1(H, H_0(N, M)) = 0$, we see that

$$H_0(G, M_1) \cong H_0(H, H_0(N, M_1)) \cong H_0(H, K)$$

and that there is a short exact sequence

$$0 \to H_0(H, K) \to [H_0(G, F_0) \cong H_0(H, H_0(N, F_0))] \to [H_0(G, M) \cong H_0(H, H_0(N, M))] \to 0.$$

Comparing this with the exact sequence in G-homology, we see that $H_1(G, M) = 0$. Also, replacing M by M_1 lowers j by 1 and enables us to repeat the same trick, thus proving the Lemma by induction. □

4.3.5. Lemma. *If F is a field, $T \cong F^\times \times F^\times$ is the group of diagonal matrices in $GL(2, F)$, and $B \cong T \ltimes F$ is the group of upper-triangular matrices in $GL(2, F)$, then if F is infinite, the corestriction map*

$$H_\bullet(T, \mathbb{Z}) \to H_\bullet(B, \mathbb{Z})$$

coming from the inclusion of the diagonal matrices is an isomorphism. For finite F, this map is still an isomorphism in degrees 1 and 2 if F has more than 2 elements.

Proof. Note that $B \cong F^\times \times \mathrm{Aff}(F)$, where the first factor corresponds to the scalar matrices, and the second factor, the affine group or "$ax + b$ group" of F, is the group of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. The group B acts transitively on the set F by letting the scalar matrices act trivially and letting $\mathrm{Aff}(F)$ act by affine transformations. The stabilizer of the point 0 for this action is just the subgroup T. Consider the short exact sequence of B-modules

$$0 \to M \to \mathbb{Z}F \xrightarrow{\alpha} \mathbb{Z} \to 0,$$

where $\mathbb{Z}F$ denotes the free abelian group on the set F, with B-action coming from the B-action on the set F, α sends each point of F to $1 \in \mathbb{Z}$, and $M = \ker \alpha$. Since B acts transitively on F with T as one of the stability groups, $\mathbb{Z}F \cong \mathbb{Z}B \otimes_{\mathbb{Z}T} \mathbb{Z}$ as a B-module, and by Corollary 4.1.12, $H_\bullet(B, \mathbb{Z}F) \cong H_\bullet(T, \mathbb{Z})$. By the remarks in Definition 4.1.21, the map α_* can be identified with the corestriction map in the Lemma. So the Lemma will follow from the exact sequence of Proposition 4.1.9 if we can show that $H_\bullet(B, M) = 0$. For this we apply Lemma 4.3.4, so we need to show $H_\bullet(T, H_\bullet(F, M)) = 0$. Since $H_q(F, \mathbb{Z}F) = 0$ for $q > 0$ and $H_0(F, \mathbb{Z}F) \to H_0(F, \mathbb{Z})$ is an isomorphism, $H_q(F, M) \cong H_{q+1}(F, \mathbb{Z})$. For instance, $H_0(F, M) \cong F$ (and this isomorphism respects the T-module structure). If F has more than 2 elements, then there is some $a \neq 1$ in $F^\times$, and $a - 1$ is invertible in F. So $F/(a-1)F = 0$ and $H_0(T, H_0(F, M)) = 0$. Similarly, since T is abelian, $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ must act by the identity on all homology groups $H_s(T, H_0(F, M))$, whereas $a - 1$ is invertible, and thus the homology groups are all 0. When F is infinite, the fact that $H_s(T, H_q(F, M)) = 0$ for all q can be derived from this; for instance, if F has characteristic 0, F is torsion-free as an abelian group, and thus $H_q(F, M) \cong H_{q+1}(F, \mathbb{Z}) \cong \bigwedge^{q+1}(F)$ by Exercises 4.1.29 and 4.1.31, and a similar argument applies, since $a \in F^\times \subset \mathrm{Aff}(F)$ acts on $\bigwedge^{q+1}(F)$ by a^{q+1} but must act on homology by the identity. As pointed out by Suslin [SuslinLNM], a slightly different argument is required in the case F is of positive characteristic p. In this case F is a vector space over $\mathbb{F}_p$ and it's enough to show that $H_s(T, H_q(F, \mathbb{F}_p)) = 0$ for all $q > 0$. In this case, it turns out that $H_*(F, \mathbb{F}_p) \cong S(F) \otimes \bigwedge(F)$ (F viewed as a vector space over $\mathbb{F}_p$, the generators of the exterior algebra having degree 1 and those of the symmetric algebra having degree 2) if $p \neq 2$ and $H_*(F, \mathbb{F}_p) \cong$

$S(F)$ (with generators in degree 1) if $p = 2$. (See also (2) of Exercise 4.1.32.) If F is infinite, one can still prove vanishing of the cohomology by the same sort of argument as before. If F is finite with more than 2 elements, we still have vanishing of $H_0(B, M)$ and of $H_s(T, H_0(F, M))$ for all s. Since the quotient map $B \to T$ induces a left inverse to the corestriction map, we only need to show vanishing of $H_0(T, H_1(F, M))$ or of $H_0(F^\times, H_2(F, \mathbb{Z})) = H_0(F^\times, \bigwedge^2(F))$ to get an isomorphism through degree 2. Now $a \in F^\times$ acts on $\bigwedge^2(F)$ by multiplication by a^2, so vanishing of $H_0(F^\times, H_2(F, \mathbb{Z}))$ when F has at least 4 elements follows from the fact that there is an element $a \in F^\times$ with $a^2 - 1$ invertible. And when F has 3 elements, $H_2(F, \mathbb{Z}) = 0$ so the vanishing is automatic. □

4.3.6. Proposition. *If F is any field, the corestriction map*

$$H_2(F^\times \times F^\times, \mathbb{Z}) \to H_2(GL(2, F), \mathbb{Z})$$

coming from the inclusion of the diagonal matrices is surjective.

Proof. We consider the action of $G = GL(2, F)$ on $X = \mathbb{P}^1(F) = F \cup \{\infty\}$ by linear fractional transformations. (This may be defined by letting G act linearly on F^2 and taking the induced action on $\mathbb{P}^1(F) = \left(F^2 \smallsetminus \{(0, 0)\}\right)/F^\times$.) Note that G acts transitively on points of X, on ordered pairs of distinct points, and on ordered triples of distinct points. (When $F = \mathbb{F}_2$, X has exactly 3 points and G may be identified with the symmetric group of this set.) Let C_n be the free abelian group on ordered $(n+1)$-tuples $(x_0, \ldots, x_n)$ of distinct points of X, which is a G-module via the G-action on X. Define $\varepsilon : C_0 \to \mathbb{Z}$ by sending each $x \in X$ to 1 and define $d_n : C_{n+1} \to C_n$ by

$$d_n(x_0, \ldots, x_n) = \sum_{k=0}^{n+1} (x_0, \ldots, \hat{x}_k, \ldots, x_n).$$

Note that $d_n \circ d_{n+1} = 0$ and $\varepsilon \circ d_1 = 0$, so that $(C_\bullet, d) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is a chain complex. If F is infinite, this augmented complex $C_\bullet \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is algebraically the same as the augmented ordered simplicial chain complex of an infinite simplex, which is well known to be acyclic. (Or one can easily check this directly, see [Hutchinson, Lemma 1]: let $z \in \ker d_{n-1}$. Then z is a finite sum of terms $(x_0^k, \ldots, x_n^k)$ and we can choose x distinct from those x_j^k's which appear. If y is obtained from z by replacing each $(x_0^k, \ldots, x_n^k)$ by $(x, x_0^k, \ldots, x_n^k)$, then $d_{n+1}y = z$.) If F is finite, the complex is still exact at C_0 and C_1, and exact at C_2 if F has at least 3 elements. (However, the Proposition is true for $F = \mathbb{F}_2$ anyway since in this case $G \cong S_3$ and $H_2(G, \mathbb{Z}) = 0$ by Corollary 4.1.24 and Exercise 4.1.25.)

So assume F has at least 3 elements and look at the long exact homology sequences of the short exact sequences

$$0 \to M_0 \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \tag{4.3.7}$$

(4.3.8) $$0 \to M_1 \to C_1 \xrightarrow{d_0} M_0 \to 0,$$

(4.3.9) $$0 \to M_2 \to C_2 \xrightarrow{d_1} M_1 \to 0.$$

Let B be the upper-triangular subgroup of G, T the diagonal subgroup, and Z the center (scalar matrices). Since G is triply transitive on X, and B stabilizes ∞, T stabilizes $(\infty, 0)$, and Z stabilizes $(\infty, 0, 1)$, we may identify the G-modules C_0 with $\mathbb{Z}G \otimes_{\mathbb{Z}B} \mathbb{Z}$, C_1 with $\mathbb{Z}G \otimes_{\mathbb{Z}T} \mathbb{Z}$, and C_2 with $\mathbb{Z}G \otimes_{\mathbb{Z}Z} \mathbb{Z}$. Thus, by Corollary 4.1.12 and the comments in Definition 4.1.21, we may identify $H_\bullet(G, C_0)$ with $H_\bullet(B, \mathbb{Z})$, the map ε_* with corestriction $H_\bullet(B, \mathbb{Z}) \xrightarrow{\text{cores}} H_\bullet(G, \mathbb{Z})$, $H_\bullet(G, C_1)$ with $H_\bullet(T, \mathbb{Z}) \cong H_\bullet(F^\times \times F^\times, \mathbb{Z})$, and $H_\bullet(G, C_2)$ with $H_\bullet(Z, \mathbb{Z}) \cong H_\bullet(F^\times, \mathbb{Z})$. Since $d_0 : C_1 \to C_0$ sends

$$(\infty, 0) \mapsto (0) - (\infty) = w^{-1} \cdot (\infty) - (\infty), \qquad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we see that

$$H_\bullet(G, C_1) \cong H_\bullet(T, \mathbb{Z}) \xrightarrow{d_{0*}} H_\bullet(B, \mathbb{Z}) \cong H_\bullet(G, C_0)$$

is $z \mapsto \text{cores}(w \cdot z - z)$. Similarly, $d_1 : C_2 \to C_1$ sends

$$(\infty, 0, 1) \mapsto (0, 1) - (\infty, 1) + (\infty, 0) = g_1 \cdot (\infty, 0) - g_2 \cdot (\infty, 0) + (\infty, 0)$$

for suitable $g_1, g_2 \in G$, so

$$H_\bullet(G, C_2) \cong H_\bullet(Z, \mathbb{Z}) \xrightarrow{d_{1*}} H_\bullet(T, \mathbb{Z}) \cong H_\bullet(G, C_1)$$

is $z \mapsto \text{cores}(g_1^{-1} \cdot z - g_2^{-1} \cdot z + z)$, which since Z is central is just

$$\text{cores} - \text{cores} + \text{cores} = \text{cores}.$$

After making these substitutions, we obtain from (4.3.7) and (4.3.8) the exact sequences

(4.3.7′) $$\cdots \to H_{j+1}(G, \mathbb{Z}) \xrightarrow{\partial} H_j(G, M_0) \to H_j(B, \mathbb{Z}) \xrightarrow{\text{cores}} H_j(G, \mathbb{Z}) \xrightarrow{\partial} H_{j-1}(G, M_0) \to \cdots$$

and

(4.3.8′) $$\cdots \to H_{j+1}(G, M_0) \xrightarrow{\partial} H_j(G, M_1) \to H_j(T, \mathbb{Z}) \xrightarrow{d_{0*}} H_j(G, M_0) \xrightarrow{\partial} H_{j-1}(G, M_1) \to \cdots .$$

By Lemma 4.3.5, the corestriction map $H_2(T, \mathbb{Z}) \xrightarrow{\text{cores}} H_2(B, \mathbb{Z})$ is an isomorphism. So we only need to show $H_2(B, \mathbb{Z}) \xrightarrow{\text{cores}} H_2(G, \mathbb{Z})$ is surjective, which by the exact sequence (4.3.7′) means we need to show that

$\alpha : H_1(G, M_0) \to H_1(B, \mathbb{Z})$ in (4.3.7′) is injective. The image of this map is

$$\ker\left(H_1(B, \mathbb{Z}) \xrightarrow{\text{cores}} H_1(G, \mathbb{Z})\right) = \ker\left(T \xrightarrow{\det} G_{\text{ab}} = F^\times\right) \cong F^\times.$$

From (4.3.7′) and the facts that $H_1(B, \mathbb{Z}) \xrightarrow{\text{cores}} H_1(G, \mathbb{Z})$ is surjective and $H_0(B, \mathbb{Z}) \xrightarrow{\text{cores}} H_0(G, \mathbb{Z})$ is an isomorphism, $H_0(G, M_0) = 0$. From (4.3.8′), we have the exact sequence

$$\begin{aligned} H_1(G, M_1) &\xrightarrow{\beta} H_1(T, \mathbb{Z}) \cong T \xrightarrow{d_{0*}} H_1(G, M_0) \\ &\xrightarrow{\partial} H_0(G, M_1) \to H_0(T, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{d_{0*}} H_0(G, M_0) = 0. \end{aligned}$$

From (4.3.9), we have the exact sequence

$$\begin{aligned} \text{(4.3.9}'\text{)} \quad H_1(Z, \mathbb{Z}) \to H_1(G, M_1) &\xrightarrow{\partial} H_0(G, M_2) \\ &\to H_0(Z, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{d_{1*}} H_0(G, M_1) \to 0. \end{aligned}$$

Thus $H_0(G, M_1)$ is cyclic, and since by (4.3.8′),

$$H_0(G, M_1) \to H_0(T, \mathbb{Z}) \cong \mathbb{Z}$$

is surjective, this latter map must be an isomorphism. Thus $H_1(T, \mathbb{Z}) \cong T \xrightarrow{d_{0*}} H_1(G, M_0)$ is surjective. Since the composite

$$\begin{aligned} \beta \circ d_{1*} : H_1(Z, \mathbb{Z}) &\cong \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in F^\times \right\} \\ &\to H_1(T, \mathbb{Z}) \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in F^\times \right\} \end{aligned}$$

is the corestriction map, which is inclusion of the scalar matrices, $\operatorname{im} \beta$ contains the scalar matrices, and $T \xrightarrow{d_{0*}} H_1(G, M_0)$ kills the scalar matrices. Since we already concluded that d_{0*} is surjective and that $\operatorname{im} \alpha = T \cap SL(2, F)$, this means α must be injective. □

4.3.10. Lemma. *If F is any field, there is a natural splitting*

$$H_2(GL(n, F), \mathbb{Z}) \cong H_2(SL(n, F), \mathbb{Z}) \oplus H_2(F^\times, \mathbb{Z})$$

for $n \geq 3$. If F has at least 4 elements, there is an analogous fact for $n = 2$:

$$H_2(GL(2, F), \mathbb{Z}) \cong H_0(F^\times, H_2(SL(2, F), \mathbb{Z})) \oplus H_2(F^\times, \mathbb{Z}),$$

and the corestriction map of Proposition 4.3.6 maps onto the first factor in this decomposition.

Proof. For any n, $GL(n, F)$ is the semidirect product of the scalar matrices, isomorphic to $F^\times$, and of the normal subgroup $SL(n, F)$. So the

inclusion of the scalar matrices, together with the determinant map, gives a split copy of $H_2(F^\times, \mathbb{Z})$ inside $H_2(GL(n, F), \mathbb{Z})$. Also, we know $SL(n, F)$ is perfect for $n \geq 3$, and this also holds for $n = 2$ if F has at least 4 elements, by Exercise 4.1.28(1). Since the composite $SL(n, F) \hookrightarrow GL(n, F) \xrightarrow{\det} F^\times$ is trivial, the corestriction map

$$H_2(SL(n, F), \mathbb{Z}) \to H_2(GL(n, F), \mathbb{Z})$$

has its image contained in the complement of the split copy of $H_2(F^\times, \mathbb{Z})$. It in fact surjects onto this complement, and gives an isomorphism of the complement with $H_0(F^\times, H_2(SL(n, F), \mathbb{Z}))$, by an argument similar to that in Lemma 4.3.4, since $H_1(F^\times, H_1(SL(n, F), \mathbb{Z})) = 0$. (This is where we use the fact that $SL(n, F)$ is perfect.)

To conclude the proof, we need to show that $F^\times$ acts trivially on

$$H_2(SL(n, F), \mathbb{Z})$$

for $n \geq 3$. The case $n = 2$ follows from the next Proposition, since it will turn out that the corestriction map

$$H_2(GL(2, F), \mathbb{Z}) \to H_2(SL(3, F), \mathbb{Z})$$

induced by $A \hookrightarrow \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}$ is surjective. Since matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & a \end{pmatrix}$, $a \in F^\times$, give another complement to $SL(3, F)$ inside $GL(3, F)$ which commutes with matrices of the form $\begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}$, $A \in GL(2, F)$, the conjugation action of $F^\times$ on H_2 has to be trivial. The case of larger n will then follow from the Stability Theorem (Theorem 4.3.12). $\square$

4.3.11. Proposition. *If F is a field with more than 7 elements, then the corestriction map*

$$H_2(GL(2, F), \mathbb{Z}) \to H_2(SL(3, F), \mathbb{Z})$$

induced by $A \hookrightarrow \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}$ is surjective, and $H_2(SL(3, F), \mathbb{Z})$ is generated by Steinberg symbols. If F has more than 3 elements, the corestriction map is still surjective except perhaps for p-torsion, p the characteristic of the field.

Proof. The general idea is similar to that of Proposition 4.3.6. Consider the action of $G = SL(3, F)$ on $X = \mathbb{P}^2(F)$, the set of one-dimensional subspaces of F^3. Again let C_n be the free abelian group on ordered $(n+1)$-tuples $(x_0, \ldots, x_n)$ of distinct points of X, but with the extra condition

that if $n \geq 2$, no three x_j's are colinear. This is a G-module via the G-action on X. Define $\varepsilon : C_0 \to \mathbb{Z}$ and $d_n : C_{n+1} \to C_n$, getting an augmented chain complex as in the proof of Proposition 4.3.6. If F is infinite, this augmented complex $C_\bullet \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is acyclic by almost the same proof as before. If F is finite, the complex is still exact at C_0 and C_1. Note that G acts transitively on X, with the stability group at $[e_3]$ (e_1, e_2, e_3 the usual basis vectors for F^3)

$$P = \left\{ \begin{pmatrix} A & 0 \\ Y & (\det A)^{-1} \end{pmatrix} : A \in GL(2, F),\ Y \in F^2 \right\},$$

transitively on ordered pairs of distinct points in X, with the stability group at $([e_2], [e_3])$

$$P_1 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ y_1 & a_2 & 0 \\ y_2 & 0 & (a_1 a_2)^{-1} \end{pmatrix} : a_1,\ a_2 \in F^\times,\ y_1,\ y_2 \in F \right\},$$

and transitively on ordered generic triples of distinct points in X, with the stability group at $([e_1], [e_2], [e_3])$

$$P_2 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & (a_1 a_2)^{-1} \end{pmatrix} : a_1,\ a_2 \in F^\times \right\}.$$

We proceed as in the proof of Proposition 4.3.6, using short exact sequences of the form (4.3.7–4.3.9) and the corresponding long exact sequences in homology. The substitutes for (4.3.7′–4.3.9′) in our context are as follows:

$$(4.3.7'') \quad \cdots \to H_{j+1}(G, \mathbb{Z}) \xrightarrow{\partial} H_j(G, M_0) \to H_j(P, \mathbb{Z}) \xrightarrow{\text{cores}} H_j(G, \mathbb{Z}) \xrightarrow{\partial} H_{j-1}(G, M_0) \to \cdots,$$

$$(4.3.8'') \quad \cdots \to H_{j+1}(G, M_0) \xrightarrow{\partial} H_j(G, M_1) \to H_j(P_1, \mathbb{Z}) \xrightarrow{d_{0*}} H_j(G, M_0) \xrightarrow{\partial} H_{j-1}(G, M_1) \to \cdots,$$

and

$$(4.3.9'') \quad H_1(F^\times \times F^\times, \mathbb{Z}) \to H_1(G, M_1) \xrightarrow{\partial} H_0(G, M_2) \to H_0(F^\times \times F^\times, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{d_{1*}} H_0(G, M_1) \to 0.$$

We also compute d_{0*} and d_{1*} as in the proof of Proposition 4.3.6. Since $d_0 : C_1 \to C_0$ sends

$$([e_2], [e_3]) \mapsto [e_3] - [e_2] = [e_3] - w^{-1} \cdot [e_3], \qquad w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

we see that

$$H_\bullet(G, C_1) \cong H_\bullet(P_1, \mathbb{Z}) \xrightarrow{d_{0*}} H_\bullet(P, \mathbb{Z}) \cong H_\bullet(G, C_0)$$

is $z \mapsto \mathrm{cores}(w \cdot z - z)$. Similarly, $d_1 : C_2 \to C_1$ sends

$$\begin{aligned}([e_1], [e_2], [e_3]) \mapsto &([e_2], [e_3]) - ([e_1], [e_3]) + ([e_1], [e_2]) \\ &= ([e_2], [e_3]) + g_1 \cdot ([e_2], [e_3]) - g_2 \cdot ([e_2], [e_3])\end{aligned}$$

with

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; g_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G,$$

so

$$H_\bullet(G, C_2) \cong H_\bullet(F^\times \times F^\times, \mathbb{Z}) \xrightarrow{d_{1*}} H_\bullet(P_1, \mathbb{Z}) \cong H_\bullet(G, C_1)$$

is $z \mapsto \mathrm{cores}(g_1^{-1} \cdot z - g_2^{-1} \cdot z + z)$.

The proof is then exactly the same as that of Proposition 4.3.6, with the following exceptions:

(1) Since G is perfect, $H_1(G, \mathbb{Z}) = 0$. Assuming F has at least 4 elements, $SL(2, F)$ is also perfect, so $H_1(P, \mathbb{Z}) \cong F^\times$. One also has $H_1(P_1, \mathbb{Z}) \cong F^\times \times F^\times$.

(2) The idea of the proof will be to show that

$$H_2(P, \mathbb{Z}) \xrightarrow{\text{cores}} H_2(G, \mathbb{Z})$$

is surjective, by using (4.3.7″) and showing that

$$\alpha : H_1(G, M_0) \to H_1(P, \mathbb{Z}) \cong F^\times$$

is injective (α is automatically surjective since $H_1(G, \mathbb{Z}) = 0$). So one needs to know as well that

$$H_2(H, \mathbb{Z}) \xrightarrow{\text{cores}} H_2(P, \mathbb{Z})$$

is surjective, where

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} : A \in GL(2, F) \right\} \cong GL(2, F).$$

Since $P = H \ltimes F^2$, we need an analogue of Lemma 4.3.5. This is proved with the same technique, the only difference being that the action of $GL(2, F)$ on F^2 is by $A \cdot \xi = (\det A) A\xi$. So a scalar matrix $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ acts by multiplication by a^3 instead of by multiplication by a as in the proof of Lemma 4.3.5. This is no problem as long as

$|F^\times|$ does not divide 6, in particular, if F has at least 8 elements. For a smaller finite field of characteristic p, it's still true that

$$H_2(H, \mathbb{Z}[\frac{1}{p}]) \xrightarrow{\text{cores}} H_2(P, \mathbb{Z}[\frac{1}{p}])$$

is an isomorphism, since $H_\bullet(F^2, \mathbb{Z}[\frac{1}{p}])$ vanishes except in degree 0 (by Theorem 4.1.23).

(3) In the last step of the proof, one has to examine the composite

$$\beta \circ (d_1)_* : H_1(F^\times \times F^\times, \mathbb{Z}) \to H_1(P_1, \mathbb{Z}).$$

This time, this is **not** the corestriction map (which is an isomorphism) but rather the map $z \mapsto \text{cores}(g_1^{-1} \cdot z - g_2^{-1} \cdot z + z)$. If we identify the H_1 groups with groups of diagonal matrices, this becomes the map

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}$$
$$\mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} \begin{pmatrix} (ab)^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} b^{-2} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

On the other hand, the map $(d_0)_* : H_1(P_1, \mathbb{Z}) \to H_1(P, \mathbb{Z})$ becomes the map

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & (ab)^{-1} & 0 \\ 0 & 0 & b \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & ab^2 & 0 \\ 0 & 0 & a^{-1}b^{-2} \end{pmatrix},$$

whose kernel is precisely the image of the previous map. So α is again an isomorphism as in the proof of Proposition 4.3.6.

The fact that $H_2(SL(3, F), \mathbb{Z})$ is generated by Steinberg symbols now follows from combining this result with Proposition 4.3.6. □

4.3.12. Theorem (Stability for K_2). *If F is a field, the corestriction maps $H_2(SL(n, F), \mathbb{Z}) \to H_2(SL(n+1, F), \mathbb{Z})$ and $H_2(GL(n, F), \mathbb{Z}) \to H_2(GL(n+1, F), \mathbb{Z})$ are isomorphisms for $n \geq 3$ if F is infinite, and are at least surjective after inverting the characteristic of the field if F is finite.*

Hence (for an infinite field) $K_2(F) \cong H_2(SL(3, F), \mathbb{Z})$ and is generated by Steinberg symbols.

Proof. Because of Lemma 4.3.10 and the fact that the split copy of $H_2(F^\times, \mathbb{Z})$ in $H_2(GL(n, F), \mathbb{Z})$ clearly maps to the corresponding copy in $H_2(GL(n+1, F), \mathbb{Z})$, while $H_0(F^\times, H_2(SL(n, F), \mathbb{Z}))$ maps to

$$H_0(F^\times, H_2(SL(n+1, F), \mathbb{Z})),$$

it is enough to treat the case of $GL(n)$. The proof of surjectivity, or of surjectivity after inverting the characteristic of the field if F is finite, is virtually identical to the proof of Proposition 4.3.11, except that we use the action of $GL(n+1, F)$ on $X = \mathbb{P}^n(F)$, the set of one-dimensional subspaces of F^{n+1}. (Inverting the characteristic trivially yields the analogue of Lemma 4.3.5, that the corestriction map

$$H_\bullet(GL(n, F), \mathbb{Z}[\frac{1}{p}]) \to H_\bullet(GL(n, F) \ltimes F^n, \mathbb{Z}[\frac{1}{p}])$$

is an isomorphism, since F^n is a p-group and thus its homology with coefficients in $\mathbb{Z}[\frac{1}{p}]$ vanishes by Corollary 4.1.24. For an infinite field, the old proof still works.) Note incidentally that surjectivity of the corestriction map $H_2(SL(3, F), \mathbb{Z}) \to H_2(SL(n+1, F), \mathbb{Z})$ implies because of Proposition 4.3.11 that the latter is generated by Steinberg symbols.

The proof of injectivity is only slightly more delicate. For this part of the argument, assume F is infinite, let $G = GL(n+1, F)$, and let C_k be the free abelian group on ordered $(k+1)$-tuples $(x_0, \ldots, x_k)$ of distinct points of $X = \mathbb{P}^n(F)$, but with the extra conditions that if $k \geq 2$, no three x_j's are colinear, if $k \geq 3$, no four x_j's are coplanar, etc. This yields an augmented complex $C_\bullet \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ which is acyclic by almost the same proof as before. As in the proof of Proposition 4.3.11, let $P = P_0$ be the stabilizer of $[e_{n+1}]$, P_1 the stabilizer of $([e_n], [e_{n+1}])$, P_2 the stabilizer of $([e_{n-1}], [e_n], [e_{n+1}])$, etc. Then $C_0 \cong \mathbb{Z}G \otimes_{\mathbb{Z}P} \mathbb{Z}$, $C_1 \cong \mathbb{Z}G \otimes_{\mathbb{Z}P_1} \mathbb{Z}$, $C_2 \cong \mathbb{Z}G \otimes_{\mathbb{Z}P_2} \mathbb{Z}$, $C_3 \cong \mathbb{Z}G \otimes_{\mathbb{Z}P_3} \mathbb{Z}$, $C_4 \cong \mathbb{Z}G \otimes_{\mathbb{Z}P_4} \mathbb{Z}$. By Corollary 4.1.12 (Shapiro's Lemma), $H_\bullet(G, C_j) \cong H_\bullet(P_j, \mathbb{Z})$ for $j \leq 4$, and by the analogue of Lemma 4.3.5, the corestriction maps

$$H_\bullet(GL(n, F) \times F^\times, \mathbb{Z}) \to H_\bullet(P, \mathbb{Z}),$$

$$H_\bullet(GL(n-1, F) \times F^\times \times F^\times, \mathbb{Z}) \to H_\bullet(P_1, \mathbb{Z}),$$

$$H_\bullet(GL(n-2, F) \times F^\times \times F^\times \times F^\times, \mathbb{Z}) \to H_\bullet(P_2, \mathbb{Z}), \text{ etc.}$$

are isomorphisms. So $H_2(P, \mathbb{Z}) \cong H_2(GL(n, F), \mathbb{Z}) \oplus (F^\times \otimes F^\times) \oplus \bigwedge^2(F^\times)$. We again use the exact sequences (4.3.7″–4.3.9″), so we need to show the image of $H_2(G, M_0) \to H_2(P, \mathbb{Z})$ does not meet the copy of $H_2(GL(n, F), \mathbb{Z})$ in the latter.

Now the composite

$$H_\bullet(P_2, \mathbb{Z}) \to H_\bullet(G, M_1) \to H_\bullet(P_1, \mathbb{Z})$$

induced by d_1 is given by $z \mapsto \operatorname{cores}(g_1^{-1} \cdot z - g_2^{-1} \cdot z + z)$ and the composite

$$H_\bullet(P_1, \mathbb{Z}) \to H_\bullet(G, M_0) \to H_\bullet(P, \mathbb{Z})$$

induced by d_0 is given by $z \mapsto \operatorname{cores}(w \cdot z - z)$, as in the last proof. A long diagram chase then shows that the map $H_1(P_2, \mathbb{Z}) \to H_1(G, M_1)$ is surjective and that the map $H_0(G, M_3) \to H_0(P_3, \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism, hence that the map $H_1(P_3, \mathbb{Z}) \to H_1(G, M_2)$ is surjective. So the kernel of the map $H_1(P_2, \mathbb{Z}) \to H_1(G, M_1)$, which by the exact sequence (4.3.9″) is the image of the map $H_1(G, M_2) \to H_1(P_2, \mathbb{Z})$, is also the image of the map $H_1(P_3, \mathbb{Z}) \to H_1(P_2, \mathbb{Z})$. A calculation shows that this coincides with the kernel of the map $H_1(P_2, \mathbb{Z}) \to H_1(P_1, \mathbb{Z})$. So the map $H_2(G, M_0) \to H_1(G, M_1)$ must be zero and $H_2(P_1, \mathbb{Z}) \to H_2(G, M_0)$ is surjective. Finally, the image of the map

$$H_2(G, M_0) \to H_2(P, \mathbb{Z})$$

in the exact sequence (4.3.7″) is the same as the image of $(d_0)_* : H_2(P_1, \mathbb{Z}) \to H_2(P, \mathbb{Z})$. From the description of $(d_0)_*$ as $\operatorname{cores} \circ (w_* - 1)$, this has trivial intersection with the copy of $H_2(GL(n, F), \mathbb{Z})$ in the latter, which proves what we wanted. □

4.3.13. Corollary. *If F is a finite field (with the possible exception of $\mathbb{F}_2$), then $K_2(F) = 0$.*

Proof. By Corollary 4.2.18, all Steinberg symbols vanish, yet $K_2(F)[\frac{1}{p}]$ (where p is the characteristic of F) is generated by Steinberg symbols by Theorem 4.3.3. On the other hand, for any $n \geq 3$, $SL(n, F)$ is a finite group, so $H_2(SL(n, F), \mathbb{Z})$ is a finite abelian group by Theorem 4.1.23, whose p-primary part comes from the Sylow p-subgroup by Corollary 4.1.24. Now if the order of F is $q = p^r$, the order of $SL(n, F)$ is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-2})q^{n-1} = q^{1+2+\cdots+(n-1)}(q^n - 1) \cdots (q^2 - 1),$$

so the largest power of p dividing this is $q^{1+2+\cdots+(n-1)}$, which is the order of the subgroup $N(n, F)$ of upper-triangular matrices with 1's down the diagonal. Thus $N(n, F)$ is a Sylow p-subgroup of $SL(n, F)$. However, by Lemma 4.2.3, there is a homomorphism $N(n, F) \to \operatorname{St}(n, F)$ which splits the canonical map $\varphi : \operatorname{St}(n, F) \to SL(n, F)$ over $N(n, F)$. This shows that the central extension $\varphi : \operatorname{St}(F) \to SL(F)$ is trivial over $N(F) = \varinjlim N(n, F)$, and thus that the p-primary part of $K_2(F)$ vanishes. □

4.3.14. Remark. In fact there are no exceptional cases; $K_2(F)$ vanishes for any finite field. To prove this for $F = \mathbb{F}_2$, one can merely note that $H_2(SL(3, F), \mathbb{Z})$ is a finite abelian 2-group (see Exercise 4.1.28(5)), and then use Theorem 4.3.12 to deduce that $K_2(F)$ is a 2-group. The argument in the proof of Corollary 4.3.13 then shows that $K_2(F)$ has to vanish.

With somewhat more work, Proposition 4.3.6, Proposition 4.3.11, and Theorem 4.3.12 can be turned into a proof of the following famous (and difficult) theorem of Matsumoto.

4.3.15. Theorem (Matsumoto). *If F is any (commutative) field, $K_2(F)$ is the free (multiplicative) abelian group* Symb(F) *on generators $\{u, v\}$, $u, v \in F^\times$, subject to the relations of bilinearity in both variables and the relation $\{u, 1-u\} = 1$.*

Proof [Hutchinson]. First of all, the given relations imply the other relations we know about, namely skew-symmetry ($\{u, v\} = \{v, u\}^{-1}$) and the relation $\{u, -u\} = 1$, since

$$-u = (1-u)(1-u^{-1})^{-1},$$

hence

$$\{u, -u\} = \{u, 1-u\}\{u, 1-u^{-1}\}^{-1} = \{u^{-1}, 1-u^{-1}\} = 1$$

and

$$\begin{aligned}\{u, v\} &= \{u, v\}\{u, -u\} = \{u, -uv\}\\ &= \{uvv^{-1}, -uv\} = \{uv, -uv\}\{v^{-1}, -uv\}\\ &= \{v, -uv\}^{-1} = \{v, u\}^{-1}\{v, -v\}^{-1}\\ &= \{v, u\}^{-1}.\end{aligned}$$

Next, because of Corollary 4.2.18, Corollary 4.3.13, and Remark 4.3.14, the case where F is finite is already proved. So it's enough to show that when F is infinite, $H_0(F^\times, H_2(GL(2, F), \mathbb{Z}))$ has the indicated presentation, and that corestriction maps this isomorphically onto $H_2(GL(3, F), \mathbb{Z})$. By Lemma 4.3.10, it's enough for the second statement to show that the corestriction map $H_2(GL(2, F), \mathbb{Z}) \to H_2(GL(3, F), \mathbb{Z})$ is injective.

We begin with the first step, the identification of $H_2(G, \mathbb{Z})$, $G = GL(2, F)$, with the direct sum of $\bigwedge^2(F^\times)$ and the group Symb(F) on symbols $\{u, v\}$, $u, v \in F^\times$ satisfying the indicated relations. For this we have to go back to the exact sequence (4.3.7′) in the proof of Proposition 4.3.6 and identify the image of the map $H_2(G, M_0) \to H_2(B, \mathbb{Z}) \cong \bigwedge^2(F^\times \times F^\times)$. We also need to use the short exact sequences

$$0 \to M_3 \to C_3 \xrightarrow{d_2} M_2 \to 0, \tag{4.3.16}$$

$$0 \to M_4 \to C_4 \xrightarrow{d_3} M_3 \to 0, \tag{4.3.17}$$

and the corresponding exact sequences (4.3.16′) and (4.3.17′) in homology, in addition to (4.3.7–4.3.9). Since the orbits of G on 4-tuples and 5-tuples of distinct points in $\mathbb{P}^1(F)$ all have stablizer $Z \cong F^\times$, one finds that

$$H_\bullet(G, C_3) \cong \bigoplus_{x \neq 0, 1, \infty} H_\bullet(F^\times, \mathbb{Z}) \cdot \{x\},$$

$$H_\bullet(G,\, C_4) \cong \bigoplus_{x_1 \neq x_2,\, x_j \neq 0,\, 1,\, \infty} H_\bullet(F^\times,\, \mathbb{Z}) \cdot \{x_1,\, x_2\},$$

and computing $(d_2)_*$ and $(d_3)_*$ as in the proof of Proposition 4.3.6 yields that

$$(d_2)_* : H_\bullet(G,\, C_3) \to H_\bullet(G,\, C_2) = H_\bullet(Z,\, \mathbb{Z})$$

is the 0-map and that

$$\begin{aligned}(d_3)_* :& H_\bullet(G,\, C_4) \to H_\bullet(G,\, C_3) : z \cdot \{x_1,\, x_2\} \\ &\mapsto z \cdot \left(\left\{ \frac{x_1(1-x_2)}{x_2(1-x_1)} \right\} - \left\{ \frac{1-x_2}{1-x_1} \right\} + \left\{ \frac{x_2}{x_1} \right\} - \{x_2\} + \{x_1\} \right).\end{aligned}$$

Since $H_0(G,\, C_4) \to H_0(G,\, M_3)$ is surjective, the cokernel $H_0(G,\, M_2)$ of the map $H_0(G,\, M_3) \to H_0(G,\, C_3)$ is the same as that of the map $(d_3)_* : H_0(G,\, C_4) \to H_0(G,\, C_3)$, *i.e.*, $H_0(G,\, M_2)$ is the free abelian group $P(F)$ on generators $\{x\}$, $x \in F \setminus \{0,\, 1\}$, subject to the relations that for $x_1 \neq x_2$,

$$\left\{ \frac{x_1(1-x_2)}{x_2(1-x_1)} \right\} - \left\{ \frac{1-x_2}{1-x_1} \right\} + \left\{ \frac{x_2}{x_1} \right\} - \{x_2\} + \{x_1\} = 0.$$

Furthermore, since $(d_1)_*$ coincides with the corestriction map $H_\bullet(Z,\, \mathbb{Z}) \to H_\bullet(T,\, \mathbb{Z})$, which is a split injection, a simple diagram chase yields split short exact sequences

$$0 \to F^\times \to H_1(G,\, M_1) \to H_0(G,\, M_2) = P(F) \to 0,$$

$$0 \to \bigwedge^2(F^\times) \to H_2(G,\, M_1) \to H_1(G,\, M_2) \to 0.$$

We also know that the map $(d_0)_* : H_2(T,\, \mathbb{Z}) \to H_2(B,\, \mathbb{Z}) \underset{\cong}{\xleftarrow{\text{cores}}} H_2(T,\, \mathbb{Z})$ is given by $\text{cores} \circ (1 - w_*)$, so when we identify $H_2(T,\, \mathbb{Z})$ with $\bigwedge^2(T)$, the cokernel of $(d_0)_*$ can be computed to be $AS^2(F^\times) \oplus \bigwedge^2(F^\times)$. (Here $AS^2(F^\times)$ denotes the second antisymmetric tensor power, *i.e.*, $(F^\times \otimes F^\times)/\langle x \otimes y + y \otimes x \rangle$.)

Now consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
H_2(Z,\, \mathbb{Z}) & & & & H_3(G,\, \mathbb{Z}) & & H_1(Z,\, \mathbb{Z}) & & \\
\downarrow & & & & \partial \downarrow & & \downarrow & & \\
H_2(G,\, M_1) & \to & H_2(T,\, \mathbb{Z}) & \xrightarrow{(d_0)_*} & H_2(G,\, M_0) & \xrightarrow{\partial} & H_1(G,\, M_1) & \to & H_1(T,\, \mathbb{Z}). \\
\uparrow\downarrow & & & & \downarrow & & \uparrow\downarrow & & \\
H_1(G,\, M_2) & & & & H_2(B,\, \mathbb{Z}) & & H_0(G,\, M_2) & &
\end{array}$$

Chasing the diagram, we see that the cokernel of the map $H_2(G,\, M_0) \to H_2(B,\, \mathbb{Z})$, which is $H_2(G,\, \mathbb{Z})$, is the direct sum of $\bigwedge^2(F^\times)$, corresponding

to the split copy of $F^\times$ in G, and the cokernel of a certain map $P(F) \to AS^2(F^\times)$. Disentangling the various identifications made (see [Hutchinson, pp. 188–190]) shows that this map sends $\{x\} \in P(F)$ to $(1-x^{-1}) \wedge x^{-1}$. ($\wedge$ denotes the antisymmetric tensor product.) Thus

$$H_0(F^\times, H_2(SL(2, F), \mathbb{Z})) \cong AS^2(F^\times \times F^\times)/\langle (1-z) \wedge z : z \in F^\times \setminus \{1\} \rangle,$$

which is exactly the group with generators $\{x, y\}$ subject to bilinearity, antisymmetry, and the relation $\{1-z, z\} = 1$.

To finish the proof, it's enough to show that the corestriction map $H_2(GL(2, F), \mathbb{Z}) \to H_3(GL(3, F), \mathbb{Z})$ is an injection. The proof is quite similar to that of the injectivity part of Theorem 4.3.12. As in that proof we let $G = GL(3, F)$ and let C_j be the free abelian group on distinct $(j+1)$-tuples of points in $\mathbb{P}^2(F)$ such that no three are colinear if $j \geq 2$. However, in this case we have $P_2 = (F^\times)^3$ (the diagonal matrices) and we can take P_3 to be the stabilizer of $([e_1], [e_2], [e_3], [e_1 + e_2 + e_3])$, which is just the group $Z \cong F^\times$ of scalar matrices. The map $(d_2)_* : H_\bullet(P_3, \mathbb{Z}) \to H_\bullet(P_2, \mathbb{Z})$ turns out to be the 0-map since P_3 is central. So the proof proceeds as before, except that this time it turns out that the map $H_1(G, M_2) \to H_1(P_2, \mathbb{Z})$ is the 0-map, $H_1(P_2, \mathbb{Z}) \xrightarrow{\cong} H_1(G, M_1) \cong (F^\times)^3$, and the map $H_2(G, M_0) \to H_1(G, M_1)$ has image $\cong F^\times$. Write $H_2(P, \mathbb{Z}) \cong H_2(GL(2, F) \times F^\times, \mathbb{Z})$ as

$$\underbrace{\operatorname{Symb}(F) \oplus \bigwedge^2(F^\times)}_{H_2(GL(2,F),\mathbb{Z})} \oplus (F^\times \otimes F^\times) \oplus \underbrace{\bigwedge^2(F^\times)}_{H_2(F^\times,\mathbb{Z})}$$

and $H_2(P_1, \mathbb{Z}) \cong H_2((F^\times)^3, \mathbb{Z})$ as $\bigwedge^2((F^\times)^3)$. Then $\operatorname{cores} \circ (w-1)$ sends

$$\begin{aligned}(a, b, c) \wedge (a', b', c') \mapsto \Big(\{a, b'^{-1}c'\} - \{a', b^{-1}c\},\ & c \wedge c' - b \wedge b', \\ & a' \otimes b^{-1}c - a \otimes b'^{-1}c' + c \otimes b' + b' \otimes c \\ & - c' \otimes b - b \otimes c',\ b \wedge b' - c \wedge c'\Big).\end{aligned}$$

Thus the cokernel of $d_{0*} : H_2(P_1, \mathbb{Z}) \to H_2(P, \mathbb{Z})$ is isomorphic to

$$\operatorname{Symb}(F) \oplus \bigwedge^2(F^\times).$$

Going back to the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
H_2(P_1, \mathbb{Z}) & & & & & & \\
\downarrow & \searrow^{d_{0*}} & & & & & \\
H_2(G, M_0) & \longrightarrow & H_2(P, \mathbb{Z}) & \longrightarrow & H_2(G, \mathbb{Z}) & \xrightarrow{0} & H_1(G, M_0) \\
\downarrow & & & & & & \\
F^\times & & & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

we see that $H_2(G, \mathbb{Z})$ is the cokernel of a certain map $F^\times \to \mathrm{Symb}(F) \oplus \bigwedge^2(F^\times)$. A messy diagram chase shows that this map is actually the 0-map (in other words, the image of $H_2(G, M_0)$ in $H_2(P, \mathbb{Z})$ is contained in the image of d_{0*}), so $H_2(G, \mathbb{Z}) \cong \mathrm{Symb}(F) \oplus \bigwedge^2(F^\times)$, as asserted. □

4.3.18. Exercise. Show that $K_2(R_1 \times R_2) \cong K_2(R_1) \oplus K_2(R_2)$ for any two rings R_1 and R_2.

4.3.19. Exercise. This exercise concerns $K_2(\mathbb{Z}/(m))$ when m is a positive integer.

(1) Show from Theorem 4.3.1 and Exercise 2.5.17 that a proof that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ (see Exercise 4.3.20 below) would imply that $K_2(\mathbb{Z}/(m))$ has order at most 2 for any positive integer m, and would have to be generated by the Steinberg symbol $\{-1, -1\}$.
(2) If p is an odd prime and $r > 1$, $R = \mathbb{Z}/(p^r)$ is a local ring, and the quotient of this ring by its maximal ideal is the field $\mathbb{F}_p = \mathbb{Z}/(p)$. Observe that $R^\times$ is a group of order $p^{r-1}(p-1)$, and that the quotient map $R \twoheadrightarrow \mathbb{F}_p$ induces a map $R^\times \to \mathbb{F}_p^\times$ which must be an isomorphism after inverting elements of order a power of p. Thus this map splits. Show also that $R^\times$ contains an element of order p^{r-1}, hence that its Sylow p-subgroup is cyclic. Since $\mathbb{F}_p^\times$ is cyclic of order prime to p, deduce that $R^\times \cong \mathbb{F}_p^\times \oplus \mathbb{Z}/(p^{r-1})$ is cyclic.
(3) Show by an analogue of the argument in the proof of Corollary 4.2.18 that all Steinberg symbols must be trivial for $R = \mathbb{Z}/(p^r)$, p an odd prime.
(4) Deduce from (3), from the Chinese Remainder Theorem, and from Exercise 4.3.18 that all Steinberg symbols are trivial for $\mathbb{Z}/(m)$, m odd. Deduce from (1) that a proof that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ would imply that $K_2(\mathbb{Z}/(m))$ is trivial for m odd. (It is known that $\{-1, -1\}$ is non-trivial in $K_2(\mathbb{Z}/(2^r))$, $r > 1$.)

4.3.20. Exercise. This exercise concerns $K_2(\mathbb{Z})$. A proof that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ is given in [Milnor, §10]. We outline here another method of attack.

(1) Apply the same method of proof used in the proof of Theorem 4.3.12 to show that for any $n \geq 4$, the corestriction map

$$H_2(SL(n, \mathbb{Z}), \mathbb{Z}) \to H_2(SL(n+1, \mathbb{Z}), \mathbb{Z})$$

is surjective. Use the action of $SL(n+1, \mathbb{Z})$ on

$$X = \left\{a \in \mathbb{Z}^{n+1} : \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_{n+1} = \mathbb{Z}\right\} / \{\pm 1\}.$$

Identify points of X with vectors in $\mathbb{Z}^{n+1}$ (up to a sign), and let C_k be the free abelian group on ordered $(k+1)$-tuples of distinct points in X, with the extra condition that any subset consisting of $\leq n+1$ such vectors should be a set of rows in a matrix in $GL(n+1, \mathbb{Z})$. The rest of the proof should be extremely similar to that of Theorem 4.3.12.

(2) The same ideas apply to the cases $n = 2$ and $n = 3$; however, things are more complicated because of the fact that $SL(2, \mathbb{Z})$ is not perfect. In fact, it is a classical fact that $SL(2, \mathbb{Z})$ is generated by the elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

(this follows immediately from Theorem 2.3.2 and the relations $e_{12}(1) = ST^{-1}$, $e_{21}(1) = S^{-1}T$); furthermore, this gives a presentation of $SL(2, \mathbb{Z})$ as an amalgamated free product

$$\langle S, T \mid S^4 = T^6 = 1;\ S^2 = T^3 \rangle.$$

(The freeness is proved using the action of $SL(2, \mathbb{Z})$ on the upper-half plane—see [SerreTrees, p. 35].) Thus $H_1(SL(2, \mathbb{Z}), \mathbb{Z})$ is the free *abelian* group on S and T satisfying the same relations, and so is cyclic of order 12. Examining the action of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on $SL(2, \mathbb{Z})$, show that $H_1(GL(2, \mathbb{Z}), \mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2)^2$, with one of generators coming from $SL(2, \mathbb{Z})$.

(3) Plugging the results of (2) into the argument of (1), show that $H_2(SL(n, \mathbb{Z}), \mathbb{Z})$ is a finite 2-group for $n = 3$ or 4. (In fact, it is $(\mathbb{Z}/2)^2$ in both cases, but the corestriction map

$$H_2(SL(3, \mathbb{Z}), \mathbb{Z}) \to H_2(SL(4, \mathbb{Z}), \mathbb{Z})$$

is not an isomorphism [vandenKallen].)

(4) Deduce from (1) and (3) that $K_2(\mathbb{Z})$ is, up to at worst a finite 2-group, generated by the Steinberg symbol $\{-1, -1\}$. Careful analysis shows in fact that there is nothing else. Since we know that this Steinberg symbol is an element of order 2 (Example 4.2.13), $K_2(\mathbb{Z})$ is of order 2.

4. Applications of K_2

In this section we discuss applications of K_2 in several quite different fields. First are the rather direct applications to K_1 calculations that follow from the long exact sequence of Theorem 4.3.1. Then we briefly introduce the applications of K_2 to number theory, which have attracted considerable recent attention. Finally, we mention some applications of K_2 in analysis and topology.

Computing Certain Relative K_1 Groups. One of the first applications of K_2 follows from Corollary 4.3.13 and Remark 4.3.14. Namely, we obtain a new proof of the following.

4.4.1. Theorem. *Let R be the ring of integers in a number field, and let $\mathfrak{p}$ be a non-zero prime ideal in R. Then $SK_1(R, \mathfrak{p}) = 1$.*

Proof. We use the exact sequence of Theorem 4.3.1:

$$K_2(R/\mathfrak{p}) \to SK_1(R, \mathfrak{p}) \to SK_1(R) \to SK_1(R/\mathfrak{p}).$$

Since $R/\mathfrak{p}$ is a finite field (see the proof of Theorem 1.4.18), $SK_1(R/\mathfrak{p})$ vanishes by Proposition 2.2.2 and $K_2(R/\mathfrak{p})$ vanishes by Corollary 4.3.13 and Remark 4.3.14. So $SK_1(R, \mathfrak{p}) \cong SK_1(R)$. This vanishes by [Milnor, Corollary 16.3]. While this is a hard result, vanishing of $SK_1(R)$ is elementary when R is a Euclidean ring (Theorem 2.3.2), so for instance we obtain relatively elementary proofs of the vanishing of $SK_1(R, \mathfrak{p})$ when $R = \mathbb{Z}$ or $R = \mathbb{Z}[i]$ or $R = \mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]$. Proving this directly is not so easy even when $R = \mathbb{Z}$ (the proof sketched in Exercise 2.5.17 uses Dirichlet's Theorem on primes in arithmetic progressions). □

Similarly, we already know from Lemma 4.3.2 that when F is a field, there is a close relationship between $K_2(F)$ and $SK_1(F[t], (t^2-t))$. In fact, granted the non-trivial fact (which we haven't proved) that $K_2(R) \cong K_2(F)$ for $R = F[t]$, Matsumoto's Theorem for F is basically equivalent to a proof that there are no non-trivial relations (i.e., relations not consequences of the relations in Theorem 2.5.12), among the relative Mennicke symbols for $SK_1(F[t], (t^2 - t))$.

4.4.2. Proposition [Keune]. *The map $\partial : K_2(F) \to SK_1(R)$ of Lemma 4.3.2, where $R = F[t]$ and $I = (t^2 - t)$, maps*

$$\{a, b\} \mapsto \left[1 + (a-1)\frac{(b-1)^2}{b}\left(t^2 - t\right) \quad (1 + (b-1)t)\left(t^2 - t\right)\right]_I.$$

Proof. Using the notation of Lemma 4.2.15, let

$$\left.\begin{aligned} \alpha(t) &= w_{12}(a)x_{12}((a-1)t)w_{12}(-a), \\ \beta(t) &= w_{21}(-a^{-1})x_{21}((1-a^{-1})t)w_{21}(a^{-1}) \end{aligned}\right\} \quad \text{in } \mathrm{St}(R).$$

Then $\alpha(0) = w_{12}(a)w_{12}(-a) = 1$ and $\beta(0) = w_{21}(-a^{-1})w_{21}(a^{-1}) = 1$. Let

$$\gamma(t) = \alpha(t)x_{21}(a^{-1})\beta(t)x_{21}(-a^{-1})x_{12}((a-1)t),$$

so that $\gamma(0) = x_{21}(a^{-1})x_{21}(-a^{-1}) = 1$. Then

$$\begin{aligned} \gamma(1) &= w_{12}(a)x_{12}(a-1)w_{12}(-a)x_{21}(a^{-1}) \\ &\qquad w_{21}(-a^{-1})x_{21}(1-a^{-1})w_{21}(a^{-1})x_{21}(-a^{-1})x_{12}(a-1) \\ &= w_{12}(a)x_{12}(-1)x_{21}(a^{-1})x_{12}(-a)x_{12}(a) \\ &\qquad x_{21}(-a^{-1})x_{21}(1-a^{-1})x_{21}(a^{-1})x_{12}(-a)x_{12}(a-1) \\ &= w_{12}(a)x_{12}(-1)x_{21}(a^{-1})x_{21}(-a^{-1})x_{21}(1)x_{12}(-1) \\ &= w_{12}(a)x_{12}(-1)x_{21}(1)x_{12}(-1) \\ &= w_{12}(a)w_{12}(-1) = h_{12}(a). \end{aligned}$$

So if $\delta(t) = [\gamma(t), h_{13}(b)]$, $\delta(0) = [1, h_{13}(b)] = 1$ and $\delta(1) = [h_{12}(a), h_{13}(b)] = \{a, b\}$. Thus $\partial(\{a, b\})$ can be computed by tracing what happens when we apply the "snake" process in the proof of Theorem 4.3.1 to $\delta(t) \in \mathrm{St}(R)$. Now

$$\begin{aligned}
\varphi_R(\alpha(t)) &= \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & (a-1)t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -a^{-1}(1-a^{-1})t & 1 \end{pmatrix}, \\
\varphi_R(\beta(t)) &= \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (1-a^{-1})t & 1 \end{pmatrix} \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -a(a-1)t \\ 0 & 1 \end{pmatrix}, \\
\varphi_R(\gamma(t)) &= \begin{pmatrix} 1 & 0 \\ -a^{-1}(1-a^{-1})t & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -a(a-1)t \\ 0 & 1 \end{pmatrix} \\
&\qquad \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & (a-1)t \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1+(a-1)t & (a-1)^2(t^2-t) \\ -(a^{-1}-1)^2(t^2-t) & * \end{pmatrix} \in SL(2, R),
\end{aligned}$$

$$\begin{aligned}
&\varphi_R(h_{13}(b)\gamma(t)^{-1}h_{13}(b)^{-1}) \\
&= \begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1} \end{pmatrix} \begin{pmatrix} * & -(a-1)^2(t^2-t) & 0 \\ (a^{-1}-1)^2(t^2-t) & 1+(a-1)t & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad \begin{pmatrix} b^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix} \\
&= \begin{pmatrix} * & -b(a-1)^2(t^2-t) & 0 \\ b^{-1}(a^{-1}-1)^2(t^2-t) & 1+(a-1)t & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&\varphi_R(\delta(t)) \\
&= \begin{pmatrix} 1+(a-1)t & (a-1)^2(t^2-t) \\ -(a^{-1}-1)^2(t^2-t) & * \end{pmatrix} \\
&\qquad \begin{pmatrix} * & -b(a-1)^2(t^2-t) \\ b^{-1}(a^{-1}-1)^2(t^2-t) & 1+(a-1)t \end{pmatrix} \\
&= \begin{pmatrix} 1-(1-b^{-1})\frac{(a-1)^4}{a^2}(t^2-t)^2 & (1-b)(a-1)^2\,(1+(a-1)t)\,(t^2-t) \\ * & * \end{pmatrix}.
\end{aligned}$$

The result then follows after simplifying. □

Keune [Keune] used this to obtain a new proof of Matsumoto's Theorem along the following lines:

(1) First prove that there are no non-trivial relations (i.e., relations not consequences of the relations in Theorem 2.5.12), among the relative Mennicke symbols for $SK_1(R, I)$. This is done in [Bass, Ch. VI, §2].
(2) Then show that $K_2(F)$ is generated by Steinberg symbols, *i.e.*, the natural map $\psi : \text{Symb}(F) \to K_2(F)$ is surjective. This uses only the easier part (surjectivity) of Theorem 4.3.12.
(3) Then construct a map $\rho : SK_1(R, I) \to \text{Symb}(F)$ using the presentations of the two groups.
(4) Check by direct calculation that $\partial \circ \psi \circ \rho$ is the identity on generators of $SK_1(R, I)$, using Proposition 4.4.2.
(5) It follows that ∂ and ψ have to be injective, proving in particular Theorem 4.3.15.

We omit the details since we have not proved the hard fact that the relations of Theorem 2.5.12 give a presentation for $SK_1(R, I)$. We see also that this fact must be of difficulty comparable to that of Matsumoto's Theorem.

K_2 of Fields and Number Theory. The study of K_2 of fields is intimately connected with certain questions in number theory. The reader who wants to learn more about this relationship is referred to [Milnor, §11 and §14–§16] and to [Srinivas, §8 (The Mercurjev-Suslin Theorem)] for a much deeper discussion, but we will try here to sketch at least a few basic ideas. To motivate everything, recall that our proof (Example 4.2.13) of the non-triviality of $\{-1, -1\} \in K_2(\mathbb{R})$ depended on the use of the quaternions $\mathbb{H}$. In addition, as related circumstantial evidence, recall that $\{-1, -1\}$ is trivial in $K_2(\mathbb{C})$ (Example 4.2.19), and that there is no non-trivial finite-dimensional division algebra over $\mathbb{C}$ ($\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$). And note as well that we have shown that $K_2(\mathbb{F}_q) = 1$ for any finite field $\mathbb{F}_q$, while it is a classical fact due to Wedderburn that there are no non-commutative finite division algebras. All these facts suggest a close relationship between $K_2(F)$ for a field F and the existence of non-commutative finite-dimensional division algebras over F, which is measured by the *Brauer group* $\text{Br}(F)$, an important invariant of the arithmetic of the field. We will see that group homology makes an appearance in this subject as well.

Before getting to the quaternion and division algebras, we start with something quite classical, and in fact closely related (see [SerreCourseArith, Ch. III]). As a by-product of our work, we will obtain a proof of the Law of Quadratic Reciprocity.

4.4.3. Definition. Let F be a field of characteristic $\neq 2$. The **Hilbert symbol** of F is the map $(\ ,\ )_F : F^\times \times F^\times \to \{\pm 1\}$ defined as follows: if $a, b \in F^\times$, $(a, b)_F = 1$ if there exist $x, y, z \in F$, not all zero, such that $z^2 = ax^2 + by^2$, and $(a, b)_F = -1$ otherwise. It is clear that (a, b) only depends on the images of a and b in $F^\times/(F^\times)^2$. (Here $(F^\times)^2$ is the subgroup of $F^\times$ consisting of perfect squares.) Thus the Hilbert symbol is identically 1 if every element of F is a perfect square, for instance, if F is

algebraically closed. It is also clear that if $F = \mathbb{R}$, $(a, b)_F = 1$ if and only if a and b are not both negative.

4.4.4. Lemma. *Let F be a field of characteristic $\neq 2$, and let $a, b \in F^\times$. The Hilbert symbol $(a, b)_F$ is 1 if and only if a lies in the image of the norm map $N : F(\sqrt{b})^\times \to F^\times$.*

Proof. If $b = c^2$ is a perfect square in F, then $F(\sqrt{b}) = F$ and N is the identity, so the condition is always satisfied. But in this case $c^2 = a \cdot 0^2 + b \cdot 1^2$ so $(a, b)_F = 1$. So suppose b is not a perfect square in F. Then $F(\sqrt{b}) = \{x + y\sqrt{b} : x, y \in F\}$ and

$$N\left(F\left(\sqrt{b}\right)^\times\right) = \{x^2 - by^2 : x, y \in F, \quad \text{not both } 0\}.$$

If $a = x^2 - by^2$, then $x^2 = a \cdot 1^2 + by^2$, so $(a, b)_F = 1$. Conversely, if there exist $x, y, z \in F$, not all zero, such that $z^2 = ax^2 + by^2$, then $ax^2 = z^2 - by^2$. We can't have $x = 0$, since then $N(z + y\sqrt{b}) = 0$ and $z + y\sqrt{b} = 0$ (the norm is the product of the conjugates, so it vanishes only on the 0-element), so

$$a = \frac{N(z + y\sqrt{b})}{x^2} = N\left(\frac{z + y\sqrt{b}}{x}\right).$$

Thus a lies in the image of the norm map $N : F(\sqrt{b})^\times \to F^\times$. □

4.4.5. Proposition. *Let F be a field of characteristic $\neq 2$, and suppose that for any quadratic extension $F(\sqrt{b})$ of F, $N(F(\sqrt{b})^\times)$ has index at most 2 in $F^\times$. Then the Hilbert symbol $(a, b)_F$, for $a, b \in F^\times$, only depends on the Steinberg symbol $\{a, b\} \in K_2(F)$, and defines a homomorphism $K_2(F) \to \{\pm 1\}$.*

Proof. Because of Matsumoto's Theorem (4.3.15), it's enough to show the Hilbert symbol satisfies the relations in Symb(F). Obviously the Hilbert symbol is symmetric (or anti-symmetric, since it takes values in $\{\pm 1\}$). If $a \neq 0, 1$, then $(a, 1-a)_F = 1$ since $a \cdot 1^2 + (1-a) \cdot 1^2 = 1^2$. So we have only to prove bilinearity in the first variable. If $(a_1, b)_F = 1$ and $(a_2, b)_F = 1$, then by Lemma 4.4.4, a_1 and a_2 lie in the image of the norm map $N : F(\sqrt{b})^\times \to F^\times$, hence so does their product. Similarly, if $(a_1, b)_F = 1$ and $(a_2, b)_F = -1$ or *vice versa*, then one of a_1 and a_2 lies in the image of the norm map but the other does not, so their product cannot lie in the image of the norm map and $(a_1 a_2, b)_F = -1$. Finally, if $(a_1, b)_F = (a_2, b)_F = -1$, then b cannot be a perfect square in F, and a_1 and a_2 both represent non-trivial elements of the quotient group $F^\times / N(F(\sqrt{b})^\times)$. However, by the hypothesis on F, this quotient group has only two elements, so $a_1 a_2$ is trivial in $F^\times / N(F(\sqrt{b})^\times)$ and $(a_1 a_2, b)_F = 1$. □

The hypothesis of Proposition 4.4.5 appears very special, but is satisfied in a non-trivial case of great interest, that of a local field.

4.4.6. Theorem. *Let F be a local field of characteristic $\neq 2$, that is, $\mathbb{R}$, $\mathbb{C}$, a finite extension of the p-adic numbers $\mathbb{Q}_p$, or the field $\mathbb{F}_q((t))$ of formal Laurent power series over a finite field $\mathbb{F}_q$ (with q not a power of 2). Then for any non-trivial quadratic extension $F(\sqrt{b})$ of F, $N(F(\sqrt{b})^\times)$ has index exactly 2 in $F^\times$.*

Proof. When $F = \mathbb{C}$, there are no non-trivial quadratic extensions. When $F = \mathbb{R}$, there is only one, namely $\mathbb{C}$, and $N(z) = |z|^2$ for $z \in \mathbb{C}$, so $N(\mathbb{C}^\times) = \mathbb{R}_+^\times$, which has index 2 in $\mathbb{R}^\times$. Thus we may assume F is non-archimedean. Let R be the ring of integers in F and let $\mathfrak{p}$ be its maximal ideal. The finite field $R/\mathfrak{p}$ is called the residue-class field. Any quadratic extension $\bar{F} = F(\sqrt{b})$ of F is also a non-archimedean local field with its own ring of integers $\bar{R}$ and maximal ideal $\mathfrak{P}$. Without loss of generality we may assume $b \in R^\times$ and $b \notin \mathfrak{p}^2$. Choose generators $\pi \in R$ of $\mathfrak{p}$ and $\bar{\pi} \in \bar{R}$ of $\mathfrak{P}$. The quadratic extensions are of two types: **unramified**, that is, those for which $[\bar{R}/\mathfrak{P} : R/\mathfrak{p}] = 2$, and **ramified**, those for which $[\bar{R}/\mathfrak{P} : R/\mathfrak{p}] = 1$ (these are the only two possibilities since it is easy to see that $[\bar{R}/\mathfrak{P} : R/\mathfrak{p}] \leq [\bar{F} : F] = 2$). Note that $F^\times \cong \{\pi^n u : n \in \mathbb{Z},\ u \in R^\times\}$ and $F(\sqrt{b})^\times \cong \{\bar{\pi}^n v : n \in \mathbb{Z},\ v \in \bar{R}^\times\}$. Then it turns out that in the unramified case, $N(\bar{R}^\times) = R^\times$ and $N(\bar{\pi}) = \pi^2 u$ for some $u \in R^\times$, whereas in the ramified case, $N(\bar{\pi}) = \pi u$ for some $u \in R^\times$ and $N(\bar{R}^\times)$ is of index 2 in $R^\times$. In either case, $N(F(\sqrt{b})^\times)$ has index 2 in $F^\times$.

To prove this, we have to do a calculation. Since we're assuming the characteristic of F is not 2, the extension $F(\sqrt{b})$ is separable with Galois group $G = \mathrm{Gal}(F(\sqrt{b})/F)$ cyclic of order 2, with generator $\sigma : \sqrt{b} \mapsto -\sqrt{b}$. First suppose $F = \mathbb{F}_q((t))$ with q odd. Then equating coefficients of power series shows that any element $b = \sum_{i=0}^{\infty} b_i t^i$ of R with leading coefficient $b_0 = 1$ is a perfect square, so there are only two kinds of non-trivial quadratic extensions of F: $\bar{F} = \mathbb{F}_{q^2}((t))$, corresponding to taking b to be a constant power series $b = b_0 \notin (\mathbb{F}_q^\times)^2$ (this is the unramified case), and $\bar{F} = \mathbb{F}_q((\sqrt{b_1 t}))$, corresponding to taking $b = b_1 t$ (the ramified case). Since $N : \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$ is surjective, it is easy to compute that $N(\bar{R}^\times) = R^\times$ and $N(t) = t^2$ in the first case, whereas in the second case, $N(\sqrt{b_1 t}) = -b_1 t$ but $R^\times / N(\bar{R}^\times) \cong \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$. In either case, $N(\bar{F}^\times)$ has index 2 in $F^\times$.

It remains to deal with the case where F is non-archimedean of characteristic 0, *i.e.*, a finite extension of the p-adic numbers $\mathbb{Q}_p$ for some p. In this case we can use the fact that the power series for the exponential and logarithm functions converge in a small enough disk and give an isomorphism of groups from some small compact open subgroup U of R to a compact open subgroup e^U of $R^\times$. Similarly, the exponential map gives an isomorphism from $\theta_1 U + \theta_2 U$, with, say, $\theta_1 = 1+\sqrt{b}$, $\theta_2 = 1-\sqrt{b}$, to an open σ-invariant subgroup V of $\bar{R}^\times$. View $\bar{F}^\times$, $\bar{R}^\times$, and V as G-modules via the action of σ. If we consider the maps $N : x \mapsto x\sigma(x)$, $\alpha : x \mapsto x\sigma(x)^{-1}$, then by Exercise 4.1.25, the chain complex whose maps are alternately N and σ gives a calculation of the G-homology, where $G = \{1, \sigma\}$, with H_{2n}, $n > 0$, being $\ker N / \operatorname{im} \alpha$, and with H_{2n+1}, $n > 0$, being $\ker \alpha / \operatorname{im} N$. Note

that $\ker\alpha$ consists of the fixed points for σ, which just gives the intersection with F. Also, in the case of the G-module $\bar{F}^\times$, $\operatorname{im}\alpha = \ker N$, *i.e.*, $H_{2n}(G, \bar{F}^\times) = 1$, $n > 0$. This is the simplest case of Hilbert's "Theorem 90"—in this case, the proof is immediate, since obviously $\operatorname{im}\alpha \subseteq \ker N$, while if

$$N(x_0 + x_1\sqrt{b}) = x_0^2 - bx_1^2 = 1,$$

then either $x_0 = 1$ and $x_1 = 0$, so $x_0 + x_1\sqrt{b} = 1$, or else

$$\begin{aligned} x_0 + x_1\sqrt{b} &= \left(bx_1 + (x_0 - 1)\sqrt{b}\right)\left(bx_1 - (x_0 - 1)\sqrt{b}\right)^{-1} \\ &= \alpha\left(bx_1 + (x_0 - 1)\sqrt{b}\right). \end{aligned}$$

Consider the long exact homology sequences (Proposition 4.1.9) applied to the short exact sequences of G-modules

(4.4.7) $$1 \to \bar{R}^\times \to \bar{F}^\times \to \{\bar{\pi}^n : n \in \mathbb{Z}\} \cong \mathbb{Z} \to 1,$$

(4.4.8) $$1 \to V \to \bar{R}^\times \to A \to 1,$$

where A is a finite abelian group since V is open in the compact group $\bar{R}^\times$. From (4.4.7) we obtain for n large the exact sequence

(4.4.7′)
$$\begin{aligned} H_{2n}(G, \mathbb{Z}) = 1 \to H_{2n-1}(G, \bar{R}^\times) \to H_{2n-1}(G, \bar{F}^\times) \to H_{2n-1}(G, \mathbb{Z}) \cong G \\ \to H_{2n-2}(G, \bar{R}^\times) \to H_{2n-2}(G, \bar{F}^\times) = 1. \end{aligned}$$

Since by construction G permutes θ_1 and θ_2, Shapiro's Lemma (Corollary 4.1.12) shows that $H_\bullet(G, V) \cong H_\bullet(1, U)$, so the higher homology vanishes. Thus from (4.4.8) we obtain for n large the exact sequences

(4.4.8′)
$$\begin{cases} H_{2n}(G, V) = 1 \to H_{2n}(G, \bar{R}^\times) \to H_{2n}(G, A) \\ \qquad\qquad \to H_{2n-1}(G, V) = 1, \\ H_{2n-1}(G, V) = 1 \to H_{2n-1}(G, \bar{R}^\times) \to H_{2n-1}(G, A) \\ \qquad\qquad \to H_{2n-2}(G, V) = 1. \end{cases}$$

Since A is finite, $H_{2n}(G, A)$ and $H_{2n-1}(G, A)$ are finite and (non-canonically) isomorphic (this is a consequence of the fact that for an endomorphism of a finite abelian group, the kernel and cokernel are non-canonically isomorphic). So by (4.4.8′), $H_{2n}(G, \bar{R}^\times)$ and $H_{2n-1}(G, \bar{R}^\times)$ are finite and non-canonically isomorphic. Substituting in (4.4.7′), we see that

$$H_{2n-1}(G, \bar{F}^\times) = \ker\alpha / \operatorname{im} N = F^\times / N(\bar{F}^\times)$$

has the same order as G, namely 2. One can also see from (4.4.7′) that there are two cases, the unramified case where $H_{2n-1}(G, \bar{F}^\times) \to H_{2n-1}(G, \mathbb{Z}) \cong G$ is an isomorphism and

$$H_{2n-1}(G, \bar{R}^\times) = \ker\alpha|_{\bar{R}^\times} / \operatorname{im} N|_{\bar{R}^\times} = R^\times / N(\bar{R}^\times) = 1,$$

and the ramified case where $H_{2n-1}(G, \bar{F}^\times) \to H_{2n-1}(G, \mathbb{Z}) \cong G$ is the 0-map and

$$H_{2n-1}(G, \bar{R}^\times) = \ker \alpha|_{\bar{R}^\times} / \operatorname{im} N|_{\bar{R}^\times} = R^\times / N(\bar{R}^\times)$$

has order 2. □

Proposition 4.4.5 and Theorem 4.4.6 can often be used to construct nontrivial homomorphisms from K_2 of a field to $\{\pm 1\}$. For instance, in the case of $\mathbb{Q}$, we obtain the following.

4.4.9. Theorem. *$K_2(\mathbb{Q})$ is a direct limit of finite abelian groups, and $K_2(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is an infinite direct sum of cyclic groups of order two, one for each prime number p. The Hilbert symbol $(\ ,\ )_{\mathbb{Q}_p}$ of the p-adic numbers, when restricted to $\mathbb{Q}$, kills the summands of $K_2(\mathbb{Q})$ corresponding to primes other than p, and maps the summand corresponding to p* **onto** *$\{\pm 1\}$. The Hilbert symbol $(\ ,\ )_{\mathbb{R}}$ of the real numbers, when restricted to $\mathbb{Q}$, is given by the product formula*

$$(\ ,\ )_{\mathbb{R}} = \prod_{p \text{ prime}} (\ ,\ )_{\mathbb{Q}_p}.$$

The product converges in the sense that for $a, b \in \mathbb{Q}^\times$, $(a, b)_{\mathbb{Q}_p} = 1$ for all but finitely many values of p.

Proof (partially attributed by Milnor to Tate [Milnor, §11]*).* By Theorem 4.3.12, $K_2(\mathbb{Q})$ is generated by Steinberg symbols; furthermore, by the Fundamental Theorem of Arithmetic, $\mathbb{Q}^\times$ is generated by -1 (of order 2) and by the prime numbers p (linearly independent and each of infinite order). For each positive integer m, let A_m be the subgroup of $K_2(\mathbb{Q})$ generated by Steinberg symbols $\{u, v\}$ with $u, v \in \mathbb{Z}$, $|u|, |v| \leq m$. Then A_m is an increasing sequence of groups and $K_2(\mathbb{Q}) = \varinjlim A_m$. Note that A_1 is the subgroup generated by $\{-1, -1\}$, which we know to have order exactly 2. (It can't have order greater than 2, but it maps to an element of order 2 in $K_2(\mathbb{R})$ by Example 4.2.13.) Since any integer can be factored into primes, $A_m = A_{m-1}$ if m is not prime. Also, $A_2 = A_1$ since $\{2, -2\} = 1$ by 4.2.17(a) and $\{2, -1\} = \{2, 1-2\} = 1$ by 4.2.17(b). For p an odd prime, again $\{p, -p\} = 1$ and $\{p, 1-p\} = 1$ by Theorem 4.2.17, so that $\{p, p\}$ and $\{p, p-1\}$ coincide with $\{p, -1\}$, which has order at most 2. We claim there is a surjective homomorphism $\mathbb{F}_p^\times \to A_p/A_{p-1}$, given by $x \mapsto \{x, p\}$ mod A_{p-1} for $x = 1, \ldots, p-1$. This will show A_p/A_{p-1} is finite cyclic with order at most $p-1$. Indeed, if x and y are positive integers $\leq p-1$ and $xy = kp + r$, where the remainder r is a positive integer $\leq p-1$, then

$$1 = \left\{\frac{kp}{xy}, 1 - \frac{kp}{xy}\right\} = \left\{\frac{kp}{xy}, \frac{r}{xy}\right\},$$

or by bilinearity,

$$\{kp, r\}\{kp, xy\}^{-1}\{xy, r\}^{-1}\{xy, xy\} = 1.$$

Since x, y, k, $r \le p-1$, this shows

$$\{p, r\}\{p, xy\}^{-1} = 1 \mod A_{p-1},$$

or $\{xy, p\} = \{r, p\} \mod A_{p-1}$, and so the homomorphism is well defined. It's surjective since $\{p, p\}$ and $\{p, -1\}$ coincide with $\{p, p-1\} = \{p-1, p\}$.

By Proposition 4.4.5 and Theorem 4.4.6, $(\, , \,)_{\mathbb{Q}_p}$ defines a homomorphism from $K_2(\mathbb{Q})$ to $\{\pm 1\}$. Next we show that $(\, , \,)_{\mathbb{Q}_p}$ is non-trivial on A_p and, for p an odd prime, also trivial on A_{p-1}. For the case $p = 2$, it's enough (by Lemma 4.4.4) to note that -1 is not a square in $\mathbb{Q}_2$, and also does not lie in $N(\mathbb{Q}_2(\sqrt{-1})^\times)$. Indeed,

$$\mathbb{Q}_2^\times = \{2^n : n \in \mathbb{Z}\} \times \{\pm 1\} \times U,$$

where $U = \{u \in \mathbb{Z}_2^\times : u \equiv 1 \mod 4\}$ [SerreCourseArith, §I.3.2], so -1 is not a square or a sum of two squares (*i.e.*, a norm from $\mathbb{Q}_2(\sqrt{-1})$) in $\mathbb{Q}_2^\times$, and $(-1, -1)_{\mathbb{Q}_2} = -1$.

Now suppose p is an odd prime. We claim $(-1, -1)_{\mathbb{Q}_p} = +1$, which will show $(-1, -1)_{\mathbb{Q}_p}$ is trivial on $A_1 = A_2$. To see this, note that

$$\mathbb{Q}_p^\times = \{p^n : n \in \mathbb{Z}\} \times \mathbb{F}_p^\times \times U,$$

where $U = \{u \in \mathbb{Z}_p^\times : u \equiv 1 \mod p\}$ (again see [SerreCourseArith, §I.3.2]). One can solve the equation $x^2 + y^2 = -1$ in $\mathbb{F}_p$, since either -1 is a square mod p (when $p \equiv 1 \mod 4$), hence is a square in $\mathbb{Q}_p$, or else -1 is not a square mod p, $\mathbb{F}_p(\sqrt{-1}) = \mathbb{F}_{p^2}$, and $N : \mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times$ is surjective. In either event, it follows from Lemma 4.4.4 that $(-1, -1)_{\mathbb{Q}_p} = 1$. Furthermore, p is not a square in $\mathbb{Q}_p$, and $\mathbb{Q}_p(\sqrt{p})$ is a ramified quadratic extension of $\mathbb{Q}_p$, so that $\mathbb{Z}_p^\times / N(\mathbb{Z}_p(\sqrt{p})^\times)$ is of order 2 by the proof of Theorem 4.4.6. Since everything in $U \hookrightarrow \mathbb{Z}_p^\times$ is a square, there is some positive integer k with $1 \le k \le p-1$ such that the image of k in $\mathbb{F}_p^\times \hookrightarrow \mathbb{Z}_p^\times$ is not in $N(\mathbb{Z}_p(\sqrt{p})^\times)$, and $(k, p)_{\mathbb{Q}_p} = -1$. Thus $(\, , \,)_{\mathbb{Q}_p}$ is non-trivial on A_p. On the other hand, if k and m are positive integers relatively prime to p, we claim that $(k, m)_{\mathbb{Q}_p} = +1$. Indeed, if m or k is a square mod p, then it is also a square in $\mathbb{Q}_p$ and this is obvious, whereas otherwise $\mathbb{Q}_p(\sqrt{m})$ is an unramified quadratic extension of $\mathbb{Q}_p$, so that $\mathbb{Z}_p^\times / N(\mathbb{Z}_p(\sqrt{m})^\times) = 1$ and $k \in N(\mathbb{Z}_p(\sqrt{m})^\times)$. In particular, this shows $(\, , \,)_{\mathbb{Q}_p}$ is trivial on A_{p-1}. Since A_p/A_{p-1} is cyclic and we see now that the various $(\, , \,)_{\mathbb{Q}_p}$'s are linearly independent homomorphisms to $\{\pm\}$, it follows by induction on p that A_p is a direct sum of cyclic groups, each of even order, one for each prime $p' \le p$, and that we may arrange for $(\, , \,)_{\mathbb{Q}_{p'}}$ to be trivial except on the summand corresponding to p'. Passing to the limit, we get the desired structure theorem for $K_2(\mathbb{Q})$.

It remains to prove the product formula for $(\, , \,)_{\mathbb{R}}$. Since $(\, , \,)_{\mathbb{R}}$ gives a homomorphism from $K_2(\mathbb{Q})$ to $\{\pm 1\}$, it follows from the structure theorem just proved that it must be a product of $(\, , \,)_{\mathbb{Q}_l}$'s for various primes l. So we just need to check that each $(\, , \,)_{\mathbb{Q}_l}$ occurs in the expansion. By bilinearity

and skew-symmetry of Steinberg symbols, it's enough to check the formula on $\{-1, -1\}$, on $\{-1, p\}$ for p prime, and on $\{q, p\}$ for p and q prime. We already know $\{-1, p\}$ and $\{p, p\}$ coincide in $K_2(\mathbb{Q})$ and that $\{2, 2\} = 1$, so we can dispense with the generators $\{2, 2\}$ and $\{-1, p\}$ for p prime. We know $(-1, -1)_{\mathbb{R}} = (-1, -1)_{\mathbb{Q}_2} = -1$ and $(-1, -1)_{\mathbb{Q}_p} = 1$ when p is an odd prime, so $(\ ,\ )_{\mathbb{Q}_2}$ must occur in the expansion of $(\ ,\ )_{\mathbb{R}}$. Also, for any primes p and q, we have $(q, p)_{\mathbb{R}} = 1$. On the other hand, given any prime l, then either p is a square in $\mathbb{Q}_l$, in which case $(q, p)_{\mathbb{Q}_l} = 1$ for any q, or else $\mathbb{Q}_l(\sqrt{p})$ is a quadratic extension of $\mathbb{Q}_l$. If this extension is unramified, which is the case if the image of p is not a square in $\mathbb{F}_l$, in particular if l is odd and $p \neq l$, then $N(\mathbb{Z}_l(\sqrt{p})^\times) = \mathbb{Z}_l^\times$ but $l \notin N(\mathbb{Z}_l(\sqrt{p})^\times)$. So we see that $(q, p)_{\mathbb{Q}_l} = 1$ for l odd, $q \neq l$ and $p \neq l$, and $(q, p)_{\mathbb{Q}_q} = (p, q)_{\mathbb{Q}_q} = -1$ for $p \neq q$, q odd and p not a square mod q. If $l = 2$, then p is a square in ${\mathbb{Q}_2}^\times$ exactly when $p \equiv 1 \mod 8$. If $p \equiv 3, 5, 7 \mod 8$, then p is a square mod 2 but not a square in ${\mathbb{Q}_2}^\times$, so $\mathbb{Q}_2(\sqrt{p})$ is a ramified extension of $\mathbb{Q}_2$. In this case, for q an odd prime, $q \in N(\mathbb{Q}_2(\sqrt{p})^\times)$ exactly when $q \equiv 1$ or $-p \mod 4$. The extension $\mathbb{Q}_2(\sqrt{2})$ of $\mathbb{Q}_2$ is also ramified, and for q an odd prime, $q \in N(\mathbb{Q}_2(\sqrt{2})^\times)$ exactly when $q \equiv \pm 1 \mod 8$. We still have to compute $(-1, p)_{\mathbb{Q}_p} = (p, p)_{\mathbb{Q}_p}$ for p odd. This is 1 exactly when -1 is a square mod p, which happens if and only if $p \equiv 1 \mod 4$.

Now we can check that each $(\ ,\ )_{\mathbb{Q}_l}$, l odd, occurs in the expansion of $(\ ,\ )_{\mathbb{R}}$. For p an odd prime, $(p, p)_{\mathbb{Q}_l} = 1$ except perhaps for $l = 2$ and $l = p$. We have $(p, p)_{\mathbb{Q}_p} = -1$ exactly when $p \equiv 3 \mod 4$, and $(p, p)_{\mathbb{Q}_2} = -1$ exactly when $p \equiv 3, 5, 7 \mod 8$ and $p \not\equiv 1$ or $-p \mod 4$, *i.e.*, when $p \equiv 3 \mod 4$. So since we already know $(\ ,\ )_{\mathbb{Q}_2}$ occurs in the expansion of $(\ ,\ )_{\mathbb{R}}$, $(\ ,\ )_{\mathbb{Q}_l}$ must also occur for $l \equiv 3 \mod 4$ to give the correct value on $\{l, l\}$. Similarly, for p an odd prime, $(p, 2)_{\mathbb{Q}_l} = 1$ except perhaps for $l = 2$ and $l = p$. We have $(p, 2)_{\mathbb{Q}_2} = -1$ exactly when $p \equiv \pm 3 \mod 8$, so since we already know $(\ ,\ )_{\mathbb{Q}_2}$ occurs in the expansion of $(\ ,\ )_{\mathbb{R}}$, $(\ ,\ )_{\mathbb{Q}_l}$ must also occur for $l \equiv 5 \mod 8$ to give the correct value on $\{l, 2\}$. Finally, suppose p is a prime with $p \equiv 1 \mod 8$. We can show by induction on p that $(\ ,\ )_{\mathbb{Q}_p}$ must occur in the expansion of $(\ ,\ )_{\mathbb{R}}$. Suppose inductively that $(\ ,\ )_{\mathbb{Q}_l}$ occurs in the expansion of $(\ ,\ )_{\mathbb{R}}$ for all $l < p$. (To start the induction, this is true for $p = 17$ since no smaller prime is $\equiv 1 \mod 8$.) Since $p \equiv 1 \mod 8$, p is a square in ${\mathbb{Q}_2}^\times$, so $(q, p)_{\mathbb{Q}_2} = 1$ for any prime q. For $q \neq p$ odd, $(q, p)_{\mathbb{Q}_l} = 1$ except perhaps for $l = q$ and/or $l = p$. Also, $(q, p)_{\mathbb{Q}_q} = -1$ exactly when p is not a square mod q, and $(q, p)_{\mathbb{Q}_p} = -1$ exactly when q is not a square mod p. If there is a prime $q < p$ for which p is not a square mod q, then since we already know $(\ ,\ )_{\mathbb{Q}_q}$ occurs in the expansion of $(\ ,\ )_{\mathbb{R}}$, $(\ ,\ )_{\mathbb{Q}_p}$ must also occur to give the correct value on $\{q, p\}$.

So we need to show there is a prime $q < p$ for which p is not a quadratic residue mod q. The following proof of this by contradiction is due to Gauss [Gauss, *Disquisitiones Arithmeticæ*, §129]. Namely, let $m = [\sqrt{p}]$. Since $p \geq 17$, $2m+1 < p$. If p is a quadratic residue for all odd primes $q < p$, then it is also a quadratic residue modulo all odd prime powers $< p$ (because of

Exercise 4.3.19(2)), and since $p \equiv 1 \mod 8$, p is also a quadratic residue modulo any power of 2. From this one can see that any prime power $q^s < p$ must divide

$$p(p-1^2)(p-2^2)\cdots(p-m^2)$$

at least as many times as it divides $(2m+1)!$. Indeed, q^s divides $\left[\frac{2m+1}{q^s}\right]$ of the numbers $1, 2, \ldots, 2m+1$, and q^s must divide $p-j^2$ for some j since p is a quadratic residue mod q^s. Gauss now argues that q^s must divide at least $\left[\frac{2m+1}{q^s}\right]$ of the numbers

$$\frac{1}{2}(p-1),\ 2(p-4),\ \frac{1}{2}(p-9),\ 2(p-16),\ \ldots,\ 2(p-m^2) \text{ or } \frac{1}{2}(p-m^2).$$

This is clear if $q^s = 2$ or 4, since $\frac{1}{2}(p-1)$, $\frac{1}{2}(p-9), \ldots$ are all divisible by 4. But also $[\frac{m}{4}]$ of these terms are divisible by 8, etc., so 2 divides the product of these terms as often as it divides $(2m+1)!$. If q is an odd prime, then in any consecutive q^s integers j, there must be two for which q^s divides $p-j^2$, and thus q^s must divide at least $\left[\frac{2m+1}{q^s}\right]$ of the numbers $p-j^2$, $1 \leq j \leq m$. Thus, putting all this together, we see that $\prod_{j=0}^{m}(p-j^2)$ is divisible by $(2m+1)!$. But

$$\begin{aligned}(2m+1)! &= (m+1)(m+2)m(m+3)(m-1)\cdots(2m+1)(1)\\ &= (m+1)[(m+1)^2-1][(m+1)^2-4]\cdots[(m+1)^2-m^2],\end{aligned}$$

so since $\prod_{j=1}^{m}(p-j^2)$ is divisible by $(2m+1)!$, we see that

$$\frac{1}{m+1}\,\frac{p-1}{(m+1)^2-1}\,\frac{p-4}{(m+1)^2-4}\cdots\frac{p-m^2}{(m+1)^2-m^2}$$

is an integer. Since $m+1 > \sqrt{p}$, however, all factors in this product are less than 1, and this is a contradiction of the assumption that p was a quadratic residue modulo all smaller primes. Thus $(\ ,\ )_{\mathbb{Q}_p}$ must also occur in the expansion. □

4.4.10. Corollary (Gauss' Law of Quadratic Reciprocity). *If p and q are odd primes not both $\equiv 3 \mod 4$, then p is a square mod q if and only if q is a square mod p. If p and q are odd primes both $\equiv 3 \mod 4$, then p is a square mod q if and only if q is* **not** *a square mod p.*

Proof. This follows immediately from the product formula

$$1 = (q,p)_{\mathbb{R}} = (q,p)_{\mathbb{Q}_2}(q,p)_{\mathbb{Q}_q}(q,p)_{\mathbb{Q}_p}\prod_{\substack{l \text{ prime}\\ l\neq 2,q,p}}(q,p)_{\mathbb{Q}_l}.$$

The terms with $l \neq 2, q, p$ are all $= 1$, and the term $(q,p)_{\mathbb{Q}_2}$ is 1 unless p and q are both $\equiv 3 \mod 4$, in which case it's -1. Finally, we've seen $(q,p)_{\mathbb{Q}_q}$ is 1 exactly when p is a square mod q, and $(q,p)_{\mathbb{Q}_p}$ is 1 exactly when q is a square mod p. □

Now we can explain the connection between K_2 and the Brauer group. We start with the case of the Hilbert symbol, which is related to algebras of quaternions.

4.4.11. Definition. Let F be a field of characteristic $\neq 2$ and let $a, b \in F^\times$. The **quaternion algebra** $A_F(a, b)$ is the (non-commutative) associative algebra over F obtained by dividing the free associative algebra on two generators x, y over F by the relations $x^2 = a$, $y^2 = b$, $xy = -yx$. For example, when $F = \mathbb{R}$ and $a = b = -1$, $x \mapsto i$ and $y \mapsto j$ give an isomorphism from $A_F(a, b)$ to the Hamilton quaternions $\mathbb{H}$. In general, it is clear that $A_F(a, b)$ has dimension 4 as a vector space over F, with basis $1, x, y, xy$, and that F is precisely the center of $A_F(a, b)$ (here we are using the condition that the characteristic be $\neq 2$).

4.4.12. Lemma. *Let F be a field of characteristic $\neq 2$ and let $a, b \in F^\times$. If $(a, b)_F = 1$, then $A_F(a, b) \cong M_2(F)$, whereas if $(a, b)_F = -1$, then $A_F(a, b)$ is a non-commutative division algebra.*

Proof. If $(a, b)_F = 1$, then either a and b are both perfect squares in $F^\times$, or else we may assume that b is not a square but $a \in N(F(\sqrt{b})^\times)$. In the first case, suppose $a_0^2 = a$ and $b_0^2 = b$. Then the matrices

$$X = \begin{pmatrix} a_0 & 0 \\ 0 & -a_0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & b_0 \\ b_0 & 0 \end{pmatrix}$$

satisfy the same relations as x and y, and so define an isomorphism $x \mapsto X$ and $y \mapsto Y$ from $A_F(a, b)$ to $M_2(F)$. In the second case, suppose $a = u^2 - bv^2$. Then the matrices

$$X = \begin{pmatrix} -u & -bv \\ v & u \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

satisfy the same relations as x and y, and so define an isomorphism $x \mapsto X$ and $y \mapsto Y$ from $A_F(a, b)$ to $M_2(F)$.

Now suppose $(a, b)_F = -1$. Then in particular, b is not a perfect square in $F^\times$. Define an F-linear automorphism $\overline{}$ of $A_F(a, b)$ by requiring that $\bar{1} = 1$, $\bar{x} = -x$, $\bar{y} = -y$, $\overline{xy} = -xy = yx$. Then $\overline{}$ is an algebra anti-automorphism (*i.e.*, it reverses the order of multiplication) and if $u_0, u_1, u_2, u_3 \in F$,

$$\begin{aligned}(u_0 + u_1x + u_2y + u_3xy)&\overline{(u_0 + u_1x + u_2y + u_3xy)}\\ &= (u_0 + u_1x + u_2y + u_3xy)(u_0 - u_1x - u_2y - u_3xy)\\ &= u_0^2 - u_1^2a - u_2^2b + u_3^2ab.\end{aligned}$$

Thus if the quadratic form $u_0^2 - u_1^2a - u_2^2b + u_3^2ab$ is definite (*i.e.*, is 0 only if all the u's are 0), this shows $A_F(a, b)$ is a division algebra, and if not, $A_F(a, b)$ contains a zero-divisor. However, if $u_0^2 - u_1^2a - u_2^2b + u_3^2ab = 0$, then $a(u_1^2 - u_3^2b) = u_0^2 - u_2^2b$, or

$$aN(u_1 + u_3\sqrt{b}) = N(u_0 + u_2\sqrt{b}).$$

If one side of this equation vanishes, so does the other, and so all the u's are 0. Otherwise, this shows $a \in N(F(\sqrt{b})^\times)$, contradicting the assumption

that $(a, b)_F = -1$. So the quadratic form $u_0^2 - u_1^2 a - u_2^2 b + u_3^2 ab$ is definite and $A(a, b)$ is a division algebra. □

Thus Proposition 4.4.5 can be reinterpreted as saying that existence of non-trivial quaternion algebras can be used to prove non-triviality of elements of K_2. This seems consistent with the philosophy behind Example 4.2.13, where Hamilton's quaternions $\mathbb{H}$ were used to prove non-triviality of $K_2(\mathbb{R})$.

In fact, even for fields not satisfying the somewhat stringent hypothesis of Proposition 4.4.5, the Hilbert symbol can be viewed as giving a non-trivial homomorphism

$$(a, b)_F \mapsto [A_F(a, b)]$$

from $K_2(F)$ to another group, but the difference in this case is that the target group is no longer $\{\pm 1\}$ but rather the abelian group which one can provisionally call $\text{Quat}(F)$, with generators being the isomorphism classes of quaternion algebras $A_F(a, b)$, and relations

$$[A_F(a, b)][A_F(a', b')] = [A_F(a'', b'')]$$

if

$$A_F(a, b) \otimes_F A_F(a', b') \cong M_2(A_F(a'', b'')).$$

Instead of working out the details of this theory, which we leave to the reader as an exercise (Exercise 4.4.28 below), we go on to the natural generalization, which is the theory of the Brauer group of a field F.

4.4.13. Definition. Let F be a field. A finite-dimensional F-algebra A (associative and with unit, but in general not commutative) is called **central simple** if its center is exactly $A \cdot 1$ and it has no two-sided ideals other than 0 and all of A. The classical Wedderburn structure theory implies that any such algebra A is F-isomorphic to $M_n(D)$, for some $n \geq 1$ and some finite-dimensional F-division algebra D with center F. The D is uniquely determined up to isomorphism, since $D^{\text{op}} \cong \text{End}_A(M)$ for any simple A-module M. It is easy to see that the tensor product $A \otimes_F B$ of two central simple F-algebras A and B, with multiplicative structure determined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2),$$

is again central simple.

Call two central simple F-algebras A and B **stably isomorphic** if for some r and s, $M_r(A) \cong M_s(B)$ (as F-algebras). Since we may assume $A = M_{n_1}(D_1)$ and $B = M_{n_2}(D_2)$ for some division algebras D_1 and D_2, this is equivalent to assuming $M_{rn_1}(D_1) \cong M_{sn_2}(D_2)$, which is possible if and only if $D_1 \cong D_2$. Thus each stable isomorphism class of central simple F-algebras contains a unique isomorphism class of central division algebras. The **Brauer group** of F, denoted $\text{Br}(F)$, is the set of stable isomorphism class of central simple F-algebras, with product coming from the tensor product of algebras. This is obviously an abelian monoid with

identity element $[F]$; it is a group since if A^{op} denotes A with multiplication reversed, then

$$A \otimes_F A^{\mathrm{op}} \cong \mathrm{End}_F(A)$$

via the identification $(a \otimes b^{\mathrm{op}})(c) = acb$, so that $[A][A^{\mathrm{op}}] = [\mathrm{End}_F(A)] = [F]$ or $[A^{\mathrm{op}}] = [A]^{-1}$. For example, the anti-automorphism constructed in the proof of Lemma 4.4.12 shows that for a quaternion algebra,

$$A_F(a, b)^{\mathrm{op}} \cong A_F(a, b),$$

so that $[A_F(a, b)]$ has order at most 2 in $\mathrm{Br}(F)$. By Lemma 4.4.12, it has order exactly 2 if and only if $(a, b)_F = 1$.

If $\bar{F}$ is an extension field of F and A is a central simple F-algebra, then A is said to be **split** by $\bar{F}$ if $\bar{F} \otimes_F A$ is $\bar{F}$-isomorphic to $M_n(\bar{F})$ for some n. It is a classical fact that every central simple F-algebra is split by some finite Galois extension of F. One can define a relative Brauer group $\mathrm{Br}(F;\, \bar{F})$ out of the stable isomorphism classes of $\bar{F}$-split central simple F-algebras, and $\mathrm{Br}(F) = \mathrm{Br}(F;\, \bar{F})$ if $\bar{F}$ is a separable closure of F.

Brauer groups are actually cohomology groups of Galois groups in disguise, because of the following result.

4.4.14. Theorem. *If $\bar{F}$ is a finite Galois extension of a field F, and if $G = \mathrm{Gal}(\bar{F}/F)$, then there is an isomorphism $H^2(G, \bar{F}^{\times}) \to \mathrm{Br}(F;\, \bar{F})$ which sends the class of a (normalized) 2-cocycle ω to $[A_F(\omega)]$, where $A_F(\omega)$ is the "crossed product" which as an $\bar{F}$-vector space is $\bar{F}G$, but with "twisted" multiplication: $u_\sigma x = \sigma(x)u_\sigma$, $u_\sigma u_\rho = \omega(\sigma, \rho)u_{\sigma\rho}$. (Here u_σ is the basis element of $A_F(\omega)$ corresponding to $\sigma \in G$.)*

Proof. First we check that the indicated multiplication makes $A_F(\omega)$ into a central simple F-algebra, whose isomorphism class only depends on the cohomology class of ω. Then we show that $A_F(\omega)$ is a matrix algebra over F if and only if ω is a coboundary, and that every element of $\mathrm{Br}(F;\, \bar{F})$ can be realized as some $A_F(\omega)$. Finally we check that $[\omega] \mapsto [A_F(\omega)])$ defines a homomorphism $H^2(G, \bar{F}^{\times}) \to \mathrm{Br}(F;\, \bar{F})$. First of all, $A_F(\omega)$ is an associative algebra because of the 2-cocycle identity (recall the formulas in Definition 4.1.7)

$$\omega(\sigma, \rho)\omega(\sigma\rho, \xi) = \sigma(\omega(\rho, \xi))\omega(\sigma, \rho\xi),$$

which gives

$$\begin{aligned}(u_\sigma u_\rho)u_\xi &= (\omega(\sigma, \rho)u_{\sigma\rho})u_\xi = \omega(\sigma, \rho)\omega(\sigma\rho, \xi)u_{\sigma\rho\xi} \\ &= \sigma(\omega(\rho, \xi))\omega(\sigma, \rho\xi)u_{\sigma\rho\xi} \\ &= \sigma(\omega(\rho, \xi))u_\sigma u_{\rho\xi} \\ &= u_\sigma\omega(\rho, \xi)u_{\rho\xi} \\ &= u_\sigma(u_\rho u_\xi).\end{aligned}$$

Since we've normalized our cocycle to have $\omega(1, \sigma) = \omega(\sigma, 1) = 1$, $u_1 = 1$ is a unit element, and $u_\sigma^{-1} = u_{\sigma^{-1}}\omega(\sigma, \sigma^{-1})^{-1}$. Notice also that $A_F(\omega)$ has dimension $|G| = [\bar{F} : F]$ over $\bar{F}$ and thus dimension $[\bar{F} : F]^2$ over F.

Next we check that $A_F(\omega)$ is a central simple F-algebra. Clearly F lies in the center. On the other hand, if $z = \sum_{\sigma \in G} x_\sigma u_\sigma$ is central, then for any $y \in \bar{F}$ we obtain

$$\sum_{\sigma \in G} (y x_\sigma u_\sigma) = y \left(\sum_{\sigma \in G} x_\sigma u_\sigma \right) = \left(\sum_{\sigma \in G} x_\sigma u_\sigma \right) y = \sum_{\sigma \in G} (x_\sigma \sigma(y) u_\sigma),$$

so $y x_\sigma = x_\sigma \sigma(y)$ for all σ. Since for any $\sigma \neq 1$ we can choose y with $y \neq \sigma(y)$, this implies $x_\sigma = 0$ for all $\sigma \neq 1$, so $z \in \bar{F}$. Then since z commutes with u_σ for all σ, we see $z \in F$. This proves F is precisely the center. To check simplicity, suppose I is a proper two-sided ideal in $A_F(\omega)$, and let $R = A_F(\omega)/I$. Then the quotient map $A_F(\omega) \to R$ must be injective when restricted to the copy of $\bar{F}$ inside $A_F(\omega)$, so R is an algebra over $\bar{F}$ with invertible generators $\dot{u}_\sigma$ again satisfying the relations $\dot{u}_\sigma x = \sigma(x)\dot{u}_\sigma$, $\dot{u}_\sigma \dot{u}_\rho = \omega(\sigma, \rho)\dot{u}_{\sigma\rho}$, $x \in \bar{F}$ and $\sigma, \rho \in G$. Now the inner automorphism $\operatorname{Ad} \dot{u}_\sigma$ of R given by conjugation by $\dot{u}_\sigma$ restricts to σ on $\bar{F}$, and since the σ's are linearly independent over $\bar{F}$, the $\dot{u}_\sigma$'s are linearly independent over $\bar{F}$ (acting on the left). Thus $\dim_F R = [\bar{F} : F]^2 = \dim_F A_F(\omega)$, so $I = 0$ and $A_F(\omega)$ is simple.

Next we observe that the F-isomorphism class of $A_F(\omega)$ only depends on the cohomology class of ω in $H^2(G, \bar{F}^\times)$. Indeed, suppose we replace ω by $\omega' = d^1(\varphi)\omega$, where $\varphi : G \to \bar{F}^\times$ is an arbitrary map (which we can assume sends 1_G to 1_F). Then we can define a linear isomorphism from $A_F(\omega')$ to $A_F(\omega)$ by $u'_\sigma \mapsto \varphi(\sigma)u_\sigma$ (and by the identity on $\bar{F}$). Since

$$(\varphi(\sigma)u_\sigma)\, x = \varphi(\sigma)\sigma(x)u_\sigma = \sigma(x)\, (\varphi(\sigma)u_\sigma)$$

and

$$\begin{aligned}
(\varphi(\sigma)u_\sigma)\,(\varphi(\rho)u_\rho) &= \varphi(\sigma)\sigma(\varphi(\rho))u_\sigma u_\rho \\
&= \varphi(\sigma)\sigma(\varphi(\rho))\omega(\sigma, \rho)u_{\sigma\rho} \\
&= \left(\varphi(\sigma)\sigma(\varphi(\rho))\varphi(\sigma\rho)^{-1}\omega(\sigma, \rho)\right)(\varphi(\sigma\rho)u_{\sigma\rho}) \\
&= d^1(\varphi)(\sigma, \rho)\omega(\sigma, \rho)\,(\varphi(\sigma\rho)u_{\sigma\rho}) = \omega'(\sigma, \rho)\,(\varphi(\sigma\rho)u_{\sigma\rho}),
\end{aligned}$$

multiplication is preserved and we get an algebra isomorphism from $A_F(\omega')$ to $A_F(\omega)$.

Note also that if $\omega = 1$, then $A_F(\omega) \cong \operatorname{End}_F(\bar{F}) \cong M_{[\bar{F}:F]}(F)$, since in this case we have an isomorphism $\Phi : A_F(\omega) \to \operatorname{End}_F(\bar{F})$ defined by $\Phi(x)y = xy$, $\Phi(u_\sigma)y = \sigma(y)$ for $x, y \in \bar{F}$. (This is compatible with the multiplication in $A_F(1)$ since

$$\begin{aligned}
\Phi(u_\sigma)\Phi(x)y = \Phi(u_\sigma)(xy) &= \sigma(xy) = \sigma(x)\sigma(y) \\
&= \Phi(\sigma(x))(\sigma(y)) = \Phi(\sigma(x))\Phi(u_\sigma)(y)
\end{aligned}$$

and since $u_\sigma u_\rho = u_{\sigma\rho}$ in this case, and thus

$$\Phi(u_\sigma)\Phi(u_\rho)y = \Phi(u_\sigma)\rho(y) = \sigma \circ \rho(y) = \Phi(u_\sigma u_\rho)y.$$

The map Φ must be an isomorphism by dimension-counting.)

Conversely, if there is an isomorphism Ψ from $A_F(\omega)$ to $A_F(1)$, then the elements $v_\sigma = \Psi^{-1}(u_\sigma)$, $\sigma \in G$, and elements $x' = \Psi^{-1}(x)$, $x \in \bar{F}$, satisfy the relations $v_\sigma v_\rho = v_{\sigma\rho}$, $v_\sigma x' = (\sigma(x))' v_\sigma$. Since any two maximal commutative subfields of a matrix algebra are conjugate to one another, there must be an automorphism α of $A_F(\omega)$ sending x' to x. Let $w_\sigma = \alpha(v_\sigma)$. Then we have the relations $w_\sigma w_\rho = w_{\sigma\rho}$, $w_\sigma x = \sigma(x) w_\sigma$, while on the other hand $u_\sigma u_\rho = \omega(\sigma, \rho) u_{\sigma\rho}$, $u_\sigma x = \sigma(x) u_\sigma$. So each $u_\sigma w_\sigma^{-1}$ commutes with every element of $\bar{F}$. Since $\bar{F}$ is maximal abelian in $\mathrm{End}_F(\bar{F}) \cong A_F(1)$ and thus in $A_F(\omega)$, there are elements $\varphi(\sigma) \in \bar{F}^\times$ such that $u_\sigma w_\sigma^{-1} = \varphi(\sigma)$. From this it easily follows that $\omega = d^1(\varphi)$. Thus $A_F(\omega)$ is a matrix algebra over F (and represents the identity in the Brauer group) if and only if $[\omega] = 1$ in $H^2(G, \bar{F}^\times)$.

Next suppose we have some central simple algebra $A' = M_r(D)$ representing a class in $\mathrm{Br}(F; \bar{F})$. Since A' is split by $\bar{F}$, so is D, *i.e.*, $\bar{F} \otimes_F D \cong \mathrm{End}_{\bar{F}}(V)$ for some finite-dimensional $\bar{F}$-vector space V. Then the centralizer of $\bar{F}$ in $A = \mathrm{End}_D(V)$ must consist only of $\bar{F}$ and $\dim_F(A) = [\bar{F} : F]^2$. So A represents the same class in $\mathrm{Br}(F; \bar{F})$ as A' and has the same dimension as any $A_F(\omega)$, and $\bar{F}$ is maximal abelian in A. Furthermore, any F-automorphism $\sigma \in G$ of $\bar{F}$ can be extended to an inner automorphism of A. (To see this, look at the orbits of $A^\times$ on $\mathrm{Hom}_F(\bar{F}, A) \cong \bar{F} \otimes_F A \cong M_{[\bar{F}:F]}(\bar{F})$.) So there must be elements u_σ of A, $\sigma \in G$, with $u_\sigma x = \sigma(x) u_\sigma$ for all $x \in \bar{F}$. Since $u_\sigma u_\rho$ must differ from $u_{\sigma\rho}$ by an element of the centralizer of $\bar{F}$ in A, which is $\bar{F}$ itself, we must have $u_\sigma u_\rho = \omega(\sigma, \rho) u_{\sigma\rho}$ for some $\omega : G \times G \to \bar{F}^\times$. The associative law in A forces ω to obey the cocycle identity, and the $\bar{F}$-linear span of the u_σ's is a quotient of $A_F(\omega)$. Since we have already noted that this algebra is simple, $A = A_F(\omega)$ by dimension counting.

To conclude the proof, we need to see that the map $[\omega] \mapsto [A_F(\omega)]$ is multiplicative, in other words, that $A_F(\omega) \otimes_F A_F(\omega')$ is stably isomorphic to $A_F(\omega\omega')$. This will show that $[\omega] \mapsto [A_F(\omega)]$ is an injective homomorphism $H^2(G, \bar{F}^\times) \to \mathrm{Br}(F; \bar{F})$. Now $A_F(\omega) \otimes_F A_F(\omega')$ is generated by a copy of $\bar{F} \otimes_F \bar{F}$ (which looks like a direct sum of $n = [\bar{F} : F]$ copies of $\bar{F}$), together with elements $u_\sigma \otimes u'_\rho$ with $\sigma, \rho \in G$. Choose a partition $1 = \sum_{i=1}^n e_i$ of 1 into orthogonal minimal idempotents in $\bar{F} \otimes_F \bar{F}$. We can index these idempotents by elements of G so that conjugation by $u_\sigma \otimes 1$ sends e_ρ to $e_{\sigma\rho}$ and conjugation by $1 \otimes u'_\sigma$ sends e_ρ to $e_{\rho\sigma^{-1}}$. Then one can see that $A = A_F(\omega) \otimes_F A_F(\omega')$ is isomorphic to $M_n(A')$, where $A' = e_1 A e_1$. In particular, A' contains the $e_1(u_\sigma \otimes u'_\sigma)e_1$, which satisfy the same relations as the generators of $A_F(\omega\omega')$, and by simplicity of the latter and dimension counting, $A' = A_F(\omega\omega')$ so that $A \cong M_n(A_F(\omega\omega'))$. This proves that $[\omega] \mapsto [A_F(\omega)]$ gives a homomorphism $H^2(G, \bar{F}^\times) \to \mathrm{Br}(F; \bar{F})$. Since this homomorphism is surjective with trivial kernel, it's an isomorphism. $\square$

Remark. Although Theorem 4.4.14 was only formulated for finite Galois extensions, it is easy to see that it can be applied to infinite Galois extensions by passing to the limit. In particular, $\mathrm{Br}(F) \cong H^2(G, \bar{F}^\times)$, where $\bar{F}$ is a separable closure of F (the inductive limit of a maximal chain of finite Galois extensions, which exists by an application of Zorn's Lemma).

Using Theorem 4.4.14, we can now generalize Proposition 4.4.5, and construct more homomorphisms from $K_2(F)$ to various torsion groups. We begin with some classical facts about Galois cohomology which are of independent interest.

4.4.15. Theorem. *Let F be any field and let $\bar{F}$ be any finite Galois extension of F with Galois group $G = \mathrm{Gal}(\bar{F} : F)$. Note that this acts on $\bar{F}$ and on $\bar{F}^\times$ by automorphisms. Then $H^1(G, \bar{F}^\times) = 1$.*

Proof. Let $u : G \to \bar{F}^\times$ be a 1-cocycle. Since the elements of G are linearly independent over $\bar{F}$ as maps from $\bar{F}$ to itself, there is some $x \in \bar{F}^\times$ for which

$$y = \sum_{\sigma \in G} u(\sigma)\sigma(x) \neq 0.$$

Then for $\rho \in G$,

$$\begin{aligned} \rho(y) &= \sum_{\sigma \in G} \rho(u(\sigma))\rho(\sigma(x)) \\ &= \sum_{\sigma \in G} u(\rho)^{-1}u(\rho\sigma)(\rho\sigma(x)) \\ &= u(\rho)^{-1} \sum_{\sigma' \in G} u(\sigma')\sigma(x') = u(\rho)^{-1}y, \end{aligned}$$

so $u(\rho) = \rho(y^{-1})y = d^0(y^{-1})(\rho)$ and u is a coboundary. □

4.4.16. Corollary ("Hilbert's Theorem 90"). *Let F be any field and let $\bar{F}$ be any finite cyclic Galois extension of F. Let σ be a generator of $G = \mathrm{Gal}(\bar{F} : F)$. Then any element in the kernel of the norm map $N : \bar{F}^\times \to F^\times$ is of the form $x\sigma(x)^{-1}$ for some $x \in \bar{F}^\times$.*

Proof. If $y \in \ker N$, then there is a unique 1-cocycle $u : G \to \bar{F}^\times$ sending 1 to 1 and σ to y. (The cocycle identity forces $u(\sigma^2) = y\sigma(y)$, $u(\sigma^3) = u(\sigma^2)\sigma^2(y) = y\sigma(y)\sigma^2(y)$, etc., and then if $n = [\bar{F} : F] = |G|$,

$$u(\sigma^n) = y\sigma(y)\sigma^2(y) \cdots \sigma^{n-1}(y) = N(y).$$

Since $N(y) = 1$, this agrees with $u(1) = 1$ and u is well defined.) By the Theorem, u is a coboundary, which just says $y = x\sigma(x)^{-1}$ for some $x \in \bar{F}^\times$. □

4.4.17. Theorem (Kummer). *Let n be a positive integer, and let F be a field of characteristic 0 or of characteristic p not dividing n, containing a primitive n-th root of unity ξ. If $\bar{F}$ is a sufficiently large Galois extension of F with Galois group $G = \mathrm{Gal}(\bar{F} : F)$, in particular if $\bar{F}$ is a separable*

closure of F, then there is an isomorphism $\varphi : F^\times/(F^\times)^n \to \mathrm{Hom}(G, \mu_n)$, where μ_n is the multiplicative group of n-th roots of unity, defined by $\varphi(y)(\sigma) = \sigma(y)y^{-1}$, where $y^n = x$ in $\bar{F}$. (Note that y is well defined up to multiplication by a power of ξ, and since $\xi \in F$, $\sigma(\xi^j)\xi^{-j} = 1$ so that this definition is independent of the choice of y. Furthermore, $\sigma(y)$ must differ from y by a root of unity, so that $\sigma(y)y^{-1} \in \mu_n$, and we get a homomorphism $G \to \mu_n$ since

$$(\sigma \circ \rho(y)y^{-1}) = (\sigma(\rho(y)))(\rho(y)^{-1})(\rho(y)y^{-1}) = (\sigma(y)y^{-1})(\rho(y)y^{-1}),$$

because of the fact that σ fixes all roots of unity.)

Proof. Consider the short exact sequence of G-modules

$$1 \to \mu_n \to \bar{F}^\times \xrightarrow{x \mapsto x^n} \bar{F}^\times \to 1.$$

This gives an exact sequence

$$H^0(G, \mu_n) = \mu_n \to H^0(G, \bar{F}^\times) = F^\times \xrightarrow{x \mapsto x^n} F^\times$$
$$\xrightarrow{\delta} H^1(G, \mu_n) = \mathrm{Hom}(G, \mu_n) \to H^1(G, \bar{F}^\times),$$

and the last group in this sequence vanishes by Theorem 4.4.15. The Theorem follows upon decoding the definition of the connecting map δ. □

4.4.18. Theorem. *Let n be a positive integer, and let F be a field of characteristic 0 or of characteristic p not dividing n, containing a primitive n-th root of unity ξ. Then there is a homomorphism, sometimes called the "norm residue symbol" or "Galois symbol," $K_2(F) \to \{n\text{-torsion in } \mathrm{Br}(F)\}$ defined as follows. Use Theorem 4.4.14 to identify $\mathrm{Br}(F)$ with $H^2(G, \bar{F}^\times)$, where $\bar{F}$ is a separable closure of F and $G = \mathrm{Gal}(\bar{F}/F)$. View the Kummer isomorphism of Theorem 4.4.17, $\varphi : F^\times/(F^\times)^n \to \mathrm{Hom}(G, \mu_n)$, as taking its values in $\mathrm{Hom}(G, \mathbb{Z}/n) = H^1(G, \mathbb{Z}/n)$, by identifying ξ with $1 \mod n$, and let $\beta : H^1(G, \mathbb{Z}/n) \to H^2(G, \mathbb{Z})$ be the "Bockstein homomorphism," i.e., the connecting map in the long exact cohomology sequence of the short exact sequence of G-modules*

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0.$$

Then send $\{u, v\} \in K_2(F)$ to

$$(u, v) =_{\mathrm{def}} v_*(\beta \circ \varphi(u)),$$

where we think of v as giving a map of G-modules $\mathbb{Z} \to \bar{F}^\times$ with $1 \mapsto v$. (This map is equivariant since v is fixed by the Galois group.) Note that (u, v) is an n-torsion class since $\varphi(u)$ is an n-torsion class. (Note: the map $(\ ,\ )$ as we've defined it depends on the choice of a primitive n-th root of unity ξ. This choice is canonical if $n = 2$ but not otherwise; to remove this

dependence on the choice of ξ, one can give a fancier definition with values in $H^2(G, \mu_n^{\otimes 2})$.)

Furthermore, $(u, v) = 1$ if and only if v lies in the image of the norm map $N : F(u^{\frac{1}{n}})^\times \to F^\times$. (This explains the name "norm residue symbol.")

Proof. The indicated formula for (u, v) is clearly bilinear in each variable. So by Theorem 4.3.15, it's enough to show that $(u, 1-u) = 1$ if $u \neq 0, 1$; anti-symmetry (or symmetry in the case $n = 2$, which was the case of the Hilbert symbol, since $(\pm 1)^{-1} = \pm 1$) will follow automatically. Incidentally, the continuation of the cohomology exact sequence used in the proof of Theorem 4.4.17 has the form

$$1 = H^1(G, \bar{F}^\times) \to H^2(G, \mu_n) \to H^2(G, \bar{F}^\times) \xrightarrow{(x \mapsto x^n)_*} H^2(G, \bar{F}^\times),$$

which shows that $H^2(G, \mu_n)$ can be identified with the n-torsion in

$$H^2(G, \bar{F}^\times) \cong \mathrm{Br}(F).$$

Next we show (in analogy with Lemma 4.4.4) that $(u, v) = 1$ if and only if v lies in the image of the norm map $N : F(u^{\frac{1}{n}})^\times \to F^\times$. Of course, if $u \in (F^\times)^n$, then any $v \in F^\times$ is a norm, while $\varphi(u) = 1$ so $(u, v) = 1$ for any v. So we can assume $F(u^{\frac{1}{n}})$ is a proper Galois extension of F, say of degree d, where $d|n$, and the conjugates of $u^{\frac{1}{n}}$ in $F(u^{\frac{1}{n}})$ are $u\xi^j$ with $j = 0, \frac{n}{d}, \ldots, \frac{(d-1)n}{d}$. Let $H = \mathrm{Gal}(F(u^{\frac{1}{n}})/F)$, which is a cyclic quotient of G of order d, say with generator σ mapping $u^{\frac{1}{n}}$ to $u\xi^{\frac{n}{d}}$. Then $\varphi(u)$ factors through H and $\varphi(u)(\sigma) = \xi^{\frac{n}{d}}$, or if we identify μ_n with $\mathbb{Z}/n$ and use additive notation, $\varphi(u)(\sigma) = \frac{n}{d}$. So the cohomology class (u, v) factors through $H^2(H, F(u^{\frac{1}{n}})^\times)$, which by the proof of Theorem 4.4.6 is just $F^\times/N\left(F\left(u^{\frac{1}{n}}\right)^\times\right)$. Under this isomorphism, (u, v) just goes to the class of v in $F^\times/N\left(F\left(u^{\frac{1}{n}}\right)^\times\right)$, proving our claim.

Now we prove that $(u, 1-u) = 1$. We may assume as before that the conjugates of $u^{\frac{1}{n}}$ in $F(u^{\frac{1}{n}})$ are $u\xi^j$ with $j = 0, \frac{n}{d}, \ldots, \frac{(d-1)n}{d}$. So for $v \in F$,

$$N(v - u^{\frac{1}{n}}\xi^i) = \prod_{j=0}^{d-1}(v - u^{\frac{1}{n}}\xi^{i+\frac{jn}{d}})$$

and

$$v^n - u = \prod_{i=0}^{n-1}(v - u^{\frac{1}{n}}\xi^i) = \prod_{i=0}^{\frac{n}{d}-1} N(v - u^{\frac{1}{n}}\xi^i).$$

If we take $v = 1$, this shows $1 - u$ is a product of norms, hence a norm, and so $(u, 1-u) = 1$. □

Remark. Mercurjev and Suslin have shown that this map is actually an **isomorphism** from $K_2(F) \otimes_{\mathbb{Z}} \mathbb{Z}/n$ to the n-torsion in $\mathrm{Br}(F)$. However this

is quite a difficult result and we refer the reader to [Srinivas, §8] for the proof. We merely remark here that the map of Theorem 4.4.18 generalizes the map of Proposition 4.4.5 or the map sending $\{u, v\}$ to the class in the Brauer group of the quaternion algebra $A_F(u, v)$. See some of the exercises at the end of this section.

Almost Commuting Operators. As pointed out by Larry Brown [LBrown2], there is an interesting appearance of K_2 in operator theory, having to do with determinants of multiplicative commutators of almost commuting operators. This subject was first studied by Helton and Howe [HeltonHowe]. To discuss this application, we begin with a quick discussion of K_2 of rings of continuous functions.

4.4.19. Theorem [Milnor, §7]. *Let X be a compact Hausdorff space, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let R be a dense topological subalgebra of $C^{\mathbb{F}}(X)$ with the property that R is complete in its own Fréchet topology (stronger than the norm topology) and that if $f \in R$ and f has an inverse in $C^{\mathbb{F}}(X)$, then f has an inverse in R. (The main cases of interest are either $R = C^{\mathbb{F}}(X)$ or $R = C^\infty(X, \mathbb{F})$ with X a compact manifold.) Then if $\mathbb{F} = \mathbb{R}$, $K_0(R) \cong KO^0(X)$, $K_1(R)$ is an extension of $KO^{-1}(X)$ by the connected component of the identity in $R^\times$, and $K_2(R)$ surjects onto $KO^{-2}(X)$. Similarly, if $\mathbb{F} = \mathbb{C}$, $K_0(R) \cong KO^0(X)$, $K_1(R)$ is an extension of $KU^{-1}(X)$ by the connected component of the identity in $R^\times$, and $K_2(R)$ surjects onto $KU^{-2}(X)$.*

Proof. The part about K_0 and K_1 is proved in Exercise 3.1.23 in the case $R = C^{\mathbb{F}}(X)$. But, as noticed by Karoubi, basically the same proof works under the weaker hypothesis given here, which is enough to guarantee that the analogues of Theorem 1.6.3 and of Lemmas 1.6.6 and 1.6.7 will hold. So we just need to prove the part about K_2. Since $K_2(R)$ is the kernel of the universal central extension of $E(R)$, it will be enough by Remark 4.1.4 to construct a perfect central extension of $E(R)$ with kernel $K_{\mathbb{F}}^{-2}(X)$. First we note that $E(R)$ is the connected component of the identity in $SL(R)$. The inclusion one way is obvious, since every elementary matrix $e_{ij}(a)$ can be homotoped to the identity via $e_{ij}(ta)$, $t \in [0, 1]$. The converse is proved by Milnor [Milnor, Lemma 7.4] by using the continuity of the usual procedure (see the proof of Proposition 2.2.2) for writing a matrix in $SL(n, \mathbb{F})$ as a product of elementary matrices. This shows that if one has a function $X \to SL(n, \mathbb{F})$ which lies in $M_n(R)$ and is sufficiently close to the identity, then it is a product of elementary matrices $e_{ij}(a(t))$ with each function a close to the identity in the topology of R.

So we need to construct a perfect central extension of $SL(R)^0$ (the path-component of the identity) by $K_{\mathbb{F}}^{-2}(X)$. Because of Examples 4.1.5, it's enough to construct a surjection from $\pi_1(SL(R)^0)$ to $K_{\mathbb{F}}^{-2}(X)$. Now $SL(C^{\mathbb{F}}(X)) = \varinjlim C(X, SL(n, \mathbb{F}))$, and a loop in this space can be viewed as an element of $\varinjlim C(S^1 \times X, SL(n, \mathbb{F}))$. From this one can see that $\pi_1(SL(C^{\mathbb{F}}(X))^0)$ is the kernel of the map

$$[S^1 \times X, \varinjlim SL(n, \mathbb{F})] \xrightarrow{\text{res}} [X, \varinjlim SL(n, \mathbb{F})],$$

where $[A, B]$ denotes the set of homotopy classes of continuous maps from A to B. Now $[X, \varinjlim SL(n, \mathbb{F})]$ is naturally isomorphic to $K_{\mathbb{F}}^{-1}(X)$ (see Exercise 3.1.23 again), and similarly $[S^1 \times X, \varinjlim SL(n, \mathbb{F})]$ is naturally isomorphic to $K_{\mathbb{F}}^{-1}(S^1 \times X) = K_{\mathbb{F}}^{-1}(X) \oplus K_{\mathbb{F}}^{-2}(X)$. So $\pi_1(SL(C^{\mathbb{F}}(X))^0) \cong K_{\mathbb{F}}^{-2}(X)$. If R is only a dense subalgebra of $C^{\mathbb{F}}(X)$, the same holds since $SL(n, R)$ is a dense subspace of the path space $C(X, SL(n, \mathbb{F}))$ having the same fundamental group. □

4.4.20. Example. We are now ready for Brown's application of $K_2(R)$, where R is a ring such as in Theorem 4.4.19. Though everything could be done in greater generality, we concentrate here on the case $R = C^\infty(S^1, \mathbb{C})$, which illustrates all the main phenomena. Suppose we have a complex Hilbert space $\mathcal{H}$ and two invertible operators A and B in $\mathcal{H}$ such that the commutators $AA^* - A^*A$, $BB^* - B^*B$, $AB - BA$, and $AB^* - B^*A$ are all in the trace-class operators $\mathcal{L}^1(\mathcal{H})$ (see Exercise 2.2.10). Then the images modulo the compact operators $\pi(A)$ and $\pi(B)$ of A and B modulo the compact operators and their adjoints $\pi(A^*)$ and $\pi(B^*)$ generate a commutative subalgebra of $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ which is also closed under the adjoint operation $*$. By the basic structure theory of such algebras, the norm closure $\mathcal{A}$ of this algebra $\mathbb{C}[\pi(A), \pi(A^*), \pi(B), \pi(B^*)]$ is $*$-isomorphic to an algebra $C(X)$, for some compact subset $X \subset \mathbb{C}^2$ called the **joint essential spectrum** of A and B. (This is the set of points $(z_1, z_2) \in \mathbb{C}^2$ for which $\pi(A - z_1 \cdot 1)$ and $\pi(B - z_2 \cdot 1)$ generate a proper ideal of $\mathcal{A}$.) Suppose for instance that X is a smoothly embedded closed curve in $\mathbb{C}^2$, so that we may also think of X as S^1. This is the case, for instance, if $\mathcal{H} = L^2(S^1)$, where we think of S^1 as the unit circle in the complex plane, A is multiplication by z (the identity function $S^1 \to \mathbb{C}$), and B is the operator sending z^n to $c_n z^n$, where the c_n's are non-zero and bounded and $\sum_{n=-\infty}^{\infty}(c_n - c_{n-1})$ converges absolutely. (In this example $A^{-1} = A^*$ and $AB - BA = AT$, where $T = B - A^{-1}BA$ is a diagonal operator with eigenvalues $c_n - c_{n+1}$.)

Now by suitably closing the algebra of (non-commuting) polynomials in A, A^*, B, B^*, and the elements of $\mathcal{L}^1(\mathcal{H})$, we get a Fréchet algebra $\mathfrak{A}$ (sitting inside $\mathcal{B}(\mathcal{H})$) that fits into a short exact sequence

$$0 \to \mathcal{L}^1(\mathcal{H}) \to \mathfrak{A} \xrightarrow{s} R = C^\infty(X) \to 0, \tag{4.4.21}$$

where the "symbol map" s sends A and B to the functions induced on X by the coordinate functions z_1 and z_2 on $\mathbb{C}^2$. The algebra $\mathfrak{A}$ is what's called in [HeltonHowe] an **almost-commuting algebra**.

The problem now arises of computing the operator determinant

$$\det(ABA^{-1}B^{-1}),$$

where det is defined in Exercise 2.2.10. This is well defined since

$$s(ABA^{-1}B^{-1}) = s(A)s(B)s(A)^{-1}s(B)^{-1} = 1,$$

and thus $ABA^{-1}B^{-1} \equiv 1 \mod \mathcal{L}^1$.

4.4.22. Proposition [LBrown2]. *Suppose one is given an almost-commuting algebra $\mathfrak{A}$ as in (4.4.21) and A and B are invertible operators in $\mathfrak{A}$. (One doesn't need to assume anything about the space X.) Then* $\det(ABA^{-1}B^{-1})$ *only depends on the Steinberg symbol $\{s(A), s(B)\} \in K_2(R)$, and in fact is the image of this element under the composite*

$$K_2(R) \xrightarrow{\partial} K_1(\mathfrak{A}, \mathcal{L}^1(\mathcal{H})) \xrightarrow{\iota_*} K_1(\mathcal{B}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \xrightarrow{\det} \mathbb{C}^\times,$$

where ∂ is the connecting map in the K-theory exact sequence of (4.4.21), and det *is the operator determinant.*

Proof. We need to compute $\partial(\{s(A), s(B)\})$ according to the recipe in the proof of Theorem 4.3.1. By Corollary 4.2.16,

$$\{s(A), s(B)\} = h_{12}(s(AB))^{-1}h_{12}(s(A))h_{12}(s(B)).$$

This obviously lifts to

$$h_{12}(AB)^{-1}h_{12}(A)h_{12}(B)$$

in $\mathrm{St}(\mathfrak{A})$, whose image in $GL(\mathfrak{A})$ is the matrix

$$\begin{pmatrix} (AB)^{-1} & 0 \\ 0 & (AB) \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ABA^{-1}B^{-1} \end{pmatrix}.$$

So

$$\begin{aligned} \det \circ \iota_* \circ \partial(\{s(A), s(B)\}) &= \det \begin{pmatrix} 1 & 0 \\ 0 & ABA^{-1}B^{-1} \end{pmatrix} \\ &= \det(ABA^{-1}B^{-1}). \quad \square \end{aligned}$$

4.4.23. Corollary ([HeltonHowe], [LBrown1]). *If one is given an almost-commuting algebra $\mathfrak{A}$ as in (4.4.21) (but without having to assume anything about the essential joint spectrum X), and A and B are invertible operators in $\mathfrak{A}$, the determinant invariant* $\det(ABA^{-1}B^{-1})$ *only depends on $f = s(A)$ and on $g = s(B)$ in $C^\infty(X)$, and so can be written $d(f, g)$. It is defined for any invertible functions f and g in $C^\infty(X)$ which lift to invertible operators in $\mathfrak{A}$, and satisfies the following formal properties:*

(1) $d(1, g) = d(g, 1) = 1$,
(2) $d(f_1 f_2, g) = d(f_1, g)d(f_2, g)$,
(3) $d(f, g) = d(g, f)^{-1}, \quad d(f, \pm f) = 1$,
(4) $d(f, 1 - f) = 1$ *(assuming f and $1 - f$ are both invertible).*

Proof. Everything except the relation $d(f, f) = 1$ follows immediately from Proposition 4.4.22 and Theorem 4.2.17. Since $\det(AAA^{-1}A^{-1}) = \det 1 = 1$, we also have $d(f, f) = 1$. $\square$

4.4.24. Remarks. (1) Given an invertible function $f \in C^\infty(X)$, the question of when there is an invertible operator $A \in \mathfrak{A}$ with $s(A) = f$ depends on another K-theoretic invariant of (4.4.21), the **Brown-Douglas-Fillmore index invariant** $\gamma : K^{-1}(X) \to \mathbb{Z}$. In the simplest case where $X = S^1$, this just depends on the winding number of f as a map $S^1 \to \mathbb{C}^\times$ (see Exercise 4.4.30 below). Indeed, the "stable" obstruction to lifting f to an invertible element of $\mathfrak{A}$ is given by the boundary map $\partial : K_1(C^\infty(X)) \to K_0(\mathcal{L}^1(\mathcal{H})) \cong \mathbb{Z}$ of Theorem 2.5.4, and it is easy to see that this map must be trivial on the connected component of the identity in $GL(C^\infty(X))$, since if $s(A) = f$, $s(e^A) = e^f$ and e^A is invertible. So using Theorem 4.4.19 applied to the structure of $K_1(C^\infty(X))$, ∂ factors through $K^{-1}(X)$ and gives a map γ. We also know that if $\gamma(f) \neq 0$, then there is no invertible A with $s(A) = f$. The converse is true as well, since in this case we can remove the stabilization in the proof of Theorem 2.5.4, using the fact that a matrix of operators can be viewed as a single operator.

Thus Brown [LBrown1] observed that

$$d : (f, g) \mapsto \det \circ \iota_* \circ \partial(\{f, g\})$$

is a map satisfying all of the above relations except for $d(f, f) = 1$, which is now replaced by $d(f, f) = (-1)^{\gamma(f)}$ (which seems more natural from the point of view of K_2). The determinant invariant is the restriction of d to pairs (f, g) with $\gamma(f) = \gamma(g) = 0$.

(2) The original proof of Corollary 4.4.23 was operator-theoretic and considerably more complicated, and also obscured the relationship between the determinant invariant and K_2. This shows the power of K-theoretic methods.

(3) Helton-Howe and Brown actually gave a formula for d in the case where $X = S^1$, which shows as a consequence that $K_2(C^\infty(S^1))$ has to be extremely complicated. (In other words, the kernel of the map $K_2(C^\infty(S^1)) \to KU^{-2}(S^1) = \mathbb{Z}$ of Theorem 4.4.19 has to be quite large.) The formula is

$$d(f, g) = \exp\left(\frac{1}{2\pi i}\iint \frac{\frac{\partial \tilde{f}}{\partial x}\frac{\partial \tilde{g}}{\partial y} - \frac{\partial \tilde{f}}{\partial y}\frac{\partial \tilde{g}}{\partial x}}{\tilde{f}\tilde{g}}\, dm\right),$$

where $\tilde{f}$ and $\tilde{g}$ are smooth extensions of f and g from the circle to the disk, and m is Lebesgue measure on the disk multiplied by the integer $-\gamma(z)$, where z is the standard generator of $K^{-1}(S^1)$. This formula involves derivatives, which suggests that $K_2(C(S^1))$ may be rather different from $K_2(C^\infty(S^1))$.

Pseudo-isotopy. Finally, we briefly mention an application of K_2 in the topology of manifolds, which has been worked out by Hatcher, Wagoner, and Igusa. This will involve an obstruction group $\mathrm{Wh}_2(G)$, which is a certain quotient of $K_2(\mathbb{Z}G)$, where G is the fundamental group of the manifold in question.

4.4.25. Definition. Let G be a group and let $R = \mathbb{Z}G$ be its integral group ring. Let W_G be the subgroup of $\mathrm{St}(R)$ generated by all $w_{ij}(g)$, $g \in G$, where $w_{ij}(g)$ is defined as in 4.2.12. The **second Whitehead group** of G is by definition

$$\mathrm{Wh}_2(G) = K_2(R)/\left(K_2(R) \cap W_G\right).$$

Note that if G is trivial, $R = \mathbb{Z}$ and W_G is the group generated by the $w_{ij}(1)$'s. Since by Exercise 4.3.20, $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$, with a generator that by Example 4.2.19 can be written as $w_{12}(1)^4$, $K_2(\mathbb{Z}) \subset W_{\{1\}}$ and $\mathrm{Wh}_2(\{1\}) = 1$.

When G is of the form $\mathbb{Z} \times H$, then $\mathbb{Z}G \cong \mathbb{Z}H[t, t^{-1}]$, and in analogy with the Bass-Heller-Swan Theorem, Theorem 3.2.22, one can show that $K_2(\mathbb{Z}G) \cong K_2(\mathbb{Z}H) \oplus K_1(\mathbb{Z}H) \oplus (NK\text{-terms})$. This gives a decomposition of $\mathrm{Wh}_2(G)$ as $\mathrm{Wh}_2(H) \oplus \mathrm{Wh}(H) \oplus (NK\text{-terms})$. If H is trivial, the NK-terms vanish and one obtains $\mathrm{Wh}_2(\mathbb{Z}) = 1$. However, this calculation shows that $\mathrm{Wh}_2(G)$ does not vanish in general, since $\mathrm{Wh}_2(\mathbb{Z} \times \mathbb{Z}/p) \supseteq \mathrm{Wh}(\mathbb{Z}/p) \neq 1$ for p a prime ≥ 5.

$\mathrm{Wh}_2(G)$ shows up in topology basically in the following way. Suppose M^n is a compact connected smooth manifold with n sufficiently large (some of the theorems in this area work for $n \geq 5$, others only for $n \geq 7$), and for simplicity suppose M is without boundary. It is often interesting to be able to compute $\pi_0(\mathrm{Diff}(M))$, the group of components of the diffeomorphism group of M, or in other words, the group of **isotopy classes** of diffeomorphisms. (By definition, two diffeomorphisms are said to be isotopic if they lie in the same path-component of the diffeomorphism group. This group is locally contractible, so in particular, the path-components are the same as the components.) For instance, as we saw in the discussion following Corollary 2.4.5, $\pi_0(\mathrm{Diff}(S^{n-1}))$ parameterizes the set of smooth structures on S^n, since every homotopy n-sphere is obtained by gluing two standard n-balls by means of a diffeomorphism of S^{n-1}.

However, it is difficult to tell in practice when two diffeomorphisms h_0 and h_1 are isotopic to each other. It is often easier to tell when they are **pseudo-isotopic**, meaning that there is a diffeomorphism of the cylinder $M \times [0, 1]$ restricting to h_0 on $M \times \{0\}$ and to h_1 on $M \times \{1\}$. Of course, if there is an isotopy h_t from h_0 to h_1, then $h(m, t) = (h_t(m), t)$ defines a pseudo-isotopy, but a pseudo-isotopy comes from an isotopy only if it is "level-preserving," *i.e.*, sends $M \times \{t\}$ to itself for all $t \in [0, 1]$. Understanding the difference between isotopy and pseudo-isotopy depends on being able to compute $\pi_0(P(M))$, where $P(M)$ is the **pseudo-isotopy space** of M, that is, the group of diffeomorphisms h of $M \times [0, 1]$ restricting to the identity on $M \times \{0\}$. A famous theorem of Cerf [Cerf] showed that if M is simply connected, then $P(M)$ is path-connected. Since $P(M)$ acts continuously on $\mathrm{Diff}(M)$ by $h \cdot g = h_1 g$ and the orbit of the identity consists of all diffeomorphisms pseudo-isotopic to the identity, this shows that this set is path-connected, *i.e.*, that any two pseudo-isotopic diffeomorphisms are isotopic to each other in the simply connected case.

The problem arises as to what happens when M has a non-trivial fundamental group, say π. The answer is related to algebraic K-theory, as shown by the following result of Hatcher and Wagoner:

4.4.26. Theorem [HatWag]. *If M^n, $n \geq 5$, is a smooth compact connected manifold without boundary and with fundamental group π, then there is a surjection of $\pi_0(P(M))$ onto* $\mathrm{Wh}_2(\pi)$.

Remark. In some cases, [HatWag] also computed the kernel of the map $\pi_0(P(M)) \to \mathrm{Wh}_2(\pi)$ and identified it with a Wh_1 obstruction. However, one has to read the literature with caution since there was a mistake in the original results that was later corrected in [Igusa].

Since the proof of Theorem 4.4.26 is extremely complicated, we content ourselves here with only a brief hint of some of the ideas involved, in order to see the connection between $P(M)$ and K_2. The starting point of the proof is an observation of Cerf that $P(M)$ is homotopy-equivalent to the space $E(M)$ of functions $f : M \times [0, 1] \to [0, 1]$ which are smooth, have no critical points, and satisfy $f(x, 0) = 0$ and $f(x, 1) = 1$ for all $x \in M$. The homotopy equivalence is simply the map that sends $h \in P(M)$ to $f : (x, t) \mapsto p_2 \circ h(x, t)$, where $p_2 : M \times [0, 1] \to [0, 1]$ is projection onto the second coordinate. A homotopy inverse $E(M) \to P(M)$ to this map is constructed by fixing a Riemannian metric on M and sending $f \in E(M)$ to the pseudo-isotopy constructed from its gradient flow. So given $h \in P(M)$, its obstruction in $\mathrm{Wh}_2(\pi)$ will be constructed using a path f_t of smooth functions $M \times [0, 1] \to [0, 1]$ with $f_0 = p_2$ and $f_1 = f_0 \circ h$. If this path can be deformed to one with no critical points, then h must lie in the identity component of $P(M)$. One starts by using the usual ideas of differential topology to deform f to a "generic" function with non-degenerate isolated critical points, and then analyzes what happens as one goes from one critical point to the next (so far this is like the start of the proof of the h-cobordism theorem). In the simplest case where are the critical points are either of index i or index $i+1$, one gets for each t a realization of $M \times [0, 1]$ as being obtained from $M \times [0, 1]$ by attaching i-handles and $(i+1)$-handles. Since $M \times [0, 1]$ is topologically a product, these handles have to cancel as far as their effect on (π-equivariant) homology of the universal cover is concerned, so one gets an intersection matrix $A(t)$ in $GL(\mathbb{Z}\pi)$ measuring how the i-handles (coming from critical points of index i) are cancelled by the $(i+1)$-handles. For t close to 0, $A(t)$ is the identity matrix; near $t = 1$ it is a product of a permutation matrix and a diagonal matrix with entries of the form $\pm g$, $g \in \pi$; and in between it changes finitely many times by certain elementary matrices $e_{jk}(\pm g)$. So if one takes the Steinberg generators $x_{jk}(\pm g)$ corresponding to the $e_{jk}(\pm g)$, one finds that their product gives rise to an element of $\mathrm{St}(\mathbb{Z}\pi)$ which lifts $A(1)$. Using Lemma 4.2.15, one can find another lift of $A(1)$ as a product of the $w_{jk}(\pm g)$'s, and so as an element of W_π. Dividing, one gets an element of $K_2(\mathbb{Z}\pi)$ which is well defined modulo W_π, *i.e.*, an element of $\mathrm{Wh}_2(\pi)$. One can show that this element doesn't change under smooth deformation, so it gives an obstruction to being able to deform f to a function without critical points.

4.4.27. Exercise (Practice with the homological definition of the Brauer group). Use Theorem 4.4.14 to show that $\mathrm{Br}(\mathbb{R})$ is cyclic of order 2, with generator $[\mathbb{H}]$, and that the Brauer group of a finite field is trivial. (Hint: any finite extension of a finite field $\mathbb{F}_q$ is Galois and cyclic, with Galois group generated by the Frobenius automorphism $x \mapsto x^q$. Similarly, the only finite extension of $\mathbb{R}$ is $\mathbb{C}$, with cyclic Galois group generated by complex conjugation. For a cyclic group, the Galois cohomology is easy to compute as in the proof of Theorem 4.4.6.)

4.4.28. Exercise (The "quaternion algebra symbol"). Let F be a field and let $\mathrm{Quat}(F)$ be the subgroup of $\mathrm{Br}(F)$ generated by the quaternion algebras $A_F(a, b)$. Since, as remarked in Definition 4.4.13, $[A_F(a, b)]$ has order 2 whenever the Hilbert symbol $(a, b)_F$ is non-trivial, $\mathrm{Quat}(F)$ is an abelian group of exponent 2, and thus a direct sum of cyclic groups of order 2. Show directly that $\{a, b\} \mapsto [A_F(a, b)]$ gives a homomorphism $K_2(F) \to \mathrm{Quat}(F)$. Then show that this map coincides with the norm residue symbol of Theorem 4.4.18 in the case $n = 2$, $\xi = -1$.

4.4.29. Exercise. Generalize the result of Exercise 4.4.28 by showing that the norm residue symbol of Theorem 4.4.18 coincides with the map sending $\{a, b\}$ to the class in the Brauer group of the central simple F-algebra of dimension n^2 with generators x and y satisfying the relations $x^n = a$, $y^n = b$, and $xy = \xi yx$. Hint: let $\bar{F} = F(b^{\frac{1}{n}})$ and rewrite this central simple F-algebra as the algebra determined by a class in $H^2(\mathrm{Gal}(\bar{F}/F), \bar{F}^{\times})$.

4.4.30. Exercise (The "Toeplitz algebra"). This exercise will construct an explicit almost-commuting algebra of the form (4.4.21) with $X = S^1$ for which one can compute the determinant invariant of Corollary 4.4.23.

(1) Let $L^2(S^1)$ denote the (complex) L^2-space of the unit circle S^1 in $\mathbb{C}$, with respect to normalized Lebesgue measure $\frac{1}{2\pi}d\theta$. If $z : S^1 \to S^1$ is the identity map, then $L^2(S^1)$ has $\{z^n : n \in \mathbb{Z}\}$ as an orthonormal basis. Let $\mathcal{H} = H^2$, the "Hardy space," be the Hilbert subspace of $L^2(S^1)$ with orthonormal basis $\{z^n : n \geq 0\}$, and let p be orthogonal projection $L^2(S^1) \to \mathcal{H}$. Let $f \in C^\infty(S^1)$, and define the **Toeplitz operator** T_f on $\mathcal{H}$ by $T_f(g) = p(fg)$ for $g \in \mathcal{H}$, where fg is the ordinary pointwise product. Note that if f is "analytic," *i.e.*, has no non-zero negative Fourier coefficients, then T_f is simply multiplication by f. Show that $T_f^* = T_{\bar{f}}$, where $\bar{f}$ is the complex conjugate of f, and that the commutator of T_f and T_g, for $f, g \in C^\infty(S^1)$, is a trace-class operator. (You will need the fact that the Fourier coefficients c_n of a C^∞ function are "rapidly decreasing," *i.e.*, that for any positive integer k, $c_n|n|^k \to 0$ as $n \to \pm\infty$.) Deduce that the algebra $\mathfrak{A}$ generated by the T_f, $f \in C^\infty(S^1)$, together with all trace-class operators on $\mathcal{H}$, is an almost-commuting algebra with symbol map $s : T_f \mapsto f$.

(2) Show that for this example, the index invariant γ of Remark 4.4.24 (1) is (with a suitable choice of orientations) given by

$$\gamma(f) = -(\text{winding number of } f) = \frac{-1}{2\pi i}\int_{S^1} \frac{df(z)}{f(z)},$$

for $f \in C^\infty(S^1, \mathbb{C}^\times)$. Thus T_f is invertible if and only if ($f(z) \neq 0$ for all $z \in S^1$ and $\gamma(f) = 0$).

(3) Let

$$d : (f, g) \mapsto \det \circ \iota_* \circ \partial(\{f, g\})$$

as in Remark 4.4.24(1). Show by explicit computation that $d(z, z) = -1$. Then using the fact that $C^\infty(S^1, \mathbb{C}^\times)/\ker\gamma$ is infinite cyclic with z as generator, deduce the relation $d(f, f) = (-1)^{\gamma(f)}$.

(4) When $f, g : S^1 \to \mathbb{C}$ and $\gamma(f) = \gamma(g) = 0$, then f and g have continuous logarithms $\log(f)$ and $\log(g)$. Show [HeltonHowe] using the Campbell-Baker-Hausdorff formula (cf. Exercise 2.2.10) that, in this case,

$$\det(T_f T_g T_f^{-1} T_g^{-1}) = \exp\left(\operatorname{Tr}\left(T_{\log(f)}T_{\log(g)} - T_{\log(g)}T_{\log(f)}\right)\right).$$

Show that if the Fourier coefficients of $\log(f)$ and of $\log(g)$ are a_j and b_j, respectively, that this gives $\exp\left(\sum_{j=-\infty}^{\infty} j a_{-j} b_j\right)$. In particular, the determinant invariant is highly non-trivial on the kernel of the map $K_2(C^\infty(S^1)) \to KU^{-2}(S^1) \cong \mathbb{Z}$.

4.4.31. Exercise (K_2 of Laurent polynomial rings and Wh_2). Show that if R is a ring, then $K_1(R)$ is a direct summand in $K_2(R[t, t^{-1}])$. Here is a suggestion of how to proceed. First suppose one has a class in $K_1(R)$ represented by some matrix $a \in GL(n, R)$. Using Morita invariance of K_2 (Exercise 4.2.23), one can replace R by $M_n(R)$ and suppose $a \in R^\times$. Let S be the subring of $R[t, t^{-1}]$ generated by 1, a, a^{-1}, t, and t^{-1}. Then S is commutative and $a, t \in S^\times$, so the Steinberg symbol $\{a, t\}$ is well defined in $K_2(S)$. Map the class of a in $K_1(R)$ to the image in $K_2(R)$ of $\{a, t\}$, and show that this gives a well-defined homomorphism from $K_1(R)$ to $K_2(R[t, t^{-1}])$.

To get a homomorphism in the other direction, note that $R[t, t^{-1}] \cong R[s, t]/(st - 1)$, and use the boundary map in Theorem 4.3.1:

$$K_2(R[t, t^{-1}]) \to K_1(R[s, t], (st - 1))$$

composed on one side with the map $K_1(R[s, t], (st-1)) \to K_1(R)$ induced by mapping $R[s, t]$ to R and on the other side with the map $K_2(R) \to K_2(R[t, t^{-1}])$ induced by a suitable inclusion of R into $R[t, t^{-1}]$.

Use the identification of $K_1(R)$ as a direct summand in $K_2(R[t, t^{-1}])$ to get an identification of $\mathrm{Wh}(\pi)$ as a direct summand of $\mathrm{Wh}_2(\mathbb{Z}\times\pi)$. Deduce from Theorem 4.4.26 that for high-dimensional connected closed manifolds M with $\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}/p$, $P(M)$ is not path-connected.

5

The +-Construction and Quillen K-Theory

1. An introduction to classifying spaces

When the subject of algebraic K-theory first grew up, considerable effort went into the search for definitions of "higher K-functors" K_i, $i \geq 2$, that would fit nicely into exact sequences such as that of Theorem 4.3.1. This effort led to Milnor's definition of K_2, given in the last chapter, and to the study of its properties. However, for a while there seemed to be no good way to define the expected functors K_i, $i \geq 3$. (With hindsight, we now know that one could have given a straightforward definition of $K_3(R)$ as $H_3(\mathrm{St}(R), \mathbb{Z})$, for reasons which will be apparent in Theorem 5.2.7 and Corollary 5.2.8, but it's hard to see how one could have arrived at this without giving at the same time a reasonable definition of K_i for all $i \geq 3$.) This situation changed dramatically with the work of Daniel Quillen in the early 1970s, for which he was awarded the Fields Medal (the highest international honor in mathematics) in 1978. Quillen had the idea that one should try to construct the higher K-functors not one at a time but all at once, as the homotopy groups of a topological space (or, from a more sophisticated point of view, as the homotopy groups of a "generalized space" or "spectrum"). Thus one should have spaces $\mathbf{K}(R)$ for any ring R and $\mathbf{K}(R, I)$ for any ring R together with a two-sided ideal I, well defined up to homotopy equivalence, so that one could define $K_i(R) = \pi_i(\mathbf{K}(R))$ and $K_i(R, I) = \pi_i(\mathbf{K}(R, I))$. Of course, these would be required to coincide with the classical definitions when $i = 0$ or 1, and $R \rightsquigarrow \mathbf{K}(R)$, $(R, I) \rightsquigarrow \mathbf{K}(R, I)$ should be functors into the homotopy category of spaces (in which the morphisms are homotopy classes of continuous maps). The desired long exact sequence

$$\cdots \to K_{i+1}(R, I) \xrightarrow{\partial} K_i(R, I) \to K_i(R) \to K_i(R/I) \xrightarrow{\partial} K_{i-1}(R, I) \to \cdots$$

should then arise as the long exact homotopy sequence of a fibration (this

will be explained below in Theorem 5.1.24)

$$\mathbf{K}(R, I) \to \mathbf{K}(R) \to \mathbf{K}(R/I).$$

Quillen managed to carry out this program, in fact giving two very different constructions of the functor $R \rightsquigarrow \mathbf{K}(R)$, which he called the **+-construction** and the **Q-construction.** Since Quillen did his original work, the +-construction has remained basically unchanged, though there are now a whole slew of alternate versions of the Q-construction, sometimes called "infinite loop machines" (see [Adams]). Each of these constructions has its own advantages and disadvantages, and much of the power of K-theory comes from the rather hard theorem that they all give naturally equivalent functors. The one feature of all the constructions, however, is that they depend on the notion of a **classifying space.** Our objective in the first section of this chapter is to explain the notion of a classifying space and some of the algebraic topology needed to understand the general framework of Quillen's program. The rest of the chapter will focus on the details of the simplest of Quillen's constructions, the +-construction. The student who really wants to thoroughly learn about higher K-theory also needs to learn about the Q-construction, the proof that the +-construction and the Q-construction are equivalent, and the "resolution theorem," "devissage theorem," and "localization theorem" (these are the analogues for higher K-theory of the main results of Chapter 3), but these are difficult results and are well treated in §§2–7 of [Srinivas], so we will not do more than to give a very quick sketch of them in §3 below.

From the facts that $K_1(R) = GL(R)_{\text{ab}} \cong H_1(GL(R), \mathbb{Z})$ and that $K_2(R) \cong H_2([GL(R), GL(R)], \mathbb{Z})$, it should already be apparent that for any reasonable definition of a space $\mathbf{K}(R)$ that would give rise to the K-groups $K_i(R)$, the homology of $\mathbf{K}(R)$ (as a topological space) should somehow be related to the homology of $GL(R)$ (as a group). In fact, we will see that for any group G, one can construct a space BG, called the classifying space of the group G, whose homology as a space is identical to the homology of G as a group. The idea of the +-construction is then to construct $\mathbf{K}(R)$ as a suitable modification of the classifying space $BGL(R)$. A consequence will be that for all $i \geq 1$, there is a natural transformation, the **Hurewicz map**, from $K_i(R)$ to $H_i(GL(R), \mathbb{Z})$, which is an isomorphism for $i = 1$ and is the first main tool for calculating $K_i(R)$ for $i \geq 2$.

Since we don't assume that the reader has had a course in homotopy theory, and since such courses don't always go into the theory of classifying spaces, we begin with some preliminaries on the theory of fibrations and CW-complexes. A reader with a strong background in topology can skip ahead at this point to the statement of Theorem 5.1.15 and Definition 5.1.16, and can then go on to the next section of the chapter. The next few results were known in various forms by work of Hurewicz and Steenrod, but in the generality stated here were proved by Dold in the very elegant paper [Dold].

5.1.1. Definition. If E and B are topological spaces and $p : E \to B$ is a continuous map (in applications, always also open and surjective), then

a continuous map $s : B \to E$ with $p \circ s = id_B$ is called a **section** of p. If $A \subseteq B$, a **halo** around A (Dold's terminology) is a set V with $A \subseteq V \subseteq B$ such that there is a continuous function $f : B \to [0, 1]$ with $f = 1$ on A, $f = 0$ on the complement of V. For instance, B itself is a halo around $\emptyset$ (take $f = 0$), and if A_1 and A_2 are disjoint and closed in B, and B is normal, in particular if B is compact Hausdorff, A_1 has a halo which doesn't meet A_2 (Urysohn's Lemma). We also write $p_A : E_A \to A$ for $p|_{p^{-1}(A)} : p^{-1}(A) \to A$. We say $p : E \to B$ has the **section extension property** if whenever $A \subseteq B$ and $s : A \to E_A$ is a section of p_A which extends to a section s' of p_V for some halo V around A, then s (though not necessarily s') extends to a section $\tilde{s}$ of p. Sometimes in this context s is called a local section of p and $\tilde{s}$ is called a global section.

If $E = B \times F$ for some space F and if p is projection onto the first factor, then sections of p are of the form $y \mapsto (y, f(y))$ for continuous maps $f : B \to F$. Thus the section extension property in this case is equivalent to a Tietze-like extension property for maps into F.

5.1.2. Lemma. *If $E = B \times F$ for some contractible space F and if $p : E \to B$ is projection onto the first factor, then p has the section extension property.*

Proof. Let $h : F \times [0, 1] \to F$ be a contraction, that is, a map such that there is a point $x_0 \in F$ with $h(x, 1) = x$, $h(x, 0) = x_0$ for all x. Let $A \subseteq B$, let V be a halo around A, and let $f : B \to [0, 1]$ with $f = 1$ on A, $f = 0$ on the complement of V. If s is a section of p_A which extends to a section s' of p_V, let g and g' be the corresponding maps into F and define a global extension $\tilde{g}$ of g by

$$\tilde{g}(y) = \begin{cases} x_0, & y \notin V, \\ h(g'(y), f(y)), & y \in V. \end{cases} \qquad \square$$

5.1.3. Lemma. *If $p : E \to B$ is a continuous map with the section extension property and $f : B \to [0, 1]$ is continuous, then p_W has the section extension property, where W is the open set $f^{-1}((0, 1])$.*

Proof [Dold]. Suppose $g : W \to [0, 1]$ is continuous and s is a section of p over the open subset $g^{-1}((0, 1])$ of W (which is of course also open in B). We need to find a section $\tilde{s}$ of p_W which agrees with s on $A = g^{-1}(1)$. Inductively we construct sections s_n, $n = 2, 3, \ldots$, of p so that s_n converges on W to $\tilde{s}$. This will be done by guaranteeing at each step that $s_{n+1} = s_n$ on $f^{-1}((\frac{1}{n}, 1])$ (this ensures that the sequence $s_n(x)$ is eventually constant for $x \in W$) and that $s_n = s$ on $f^{-1}((\frac{1}{n+1}, 1]) \cap g^{-1}((\frac{n-1}{n}, 1])$ (this ensures that the sequence $s_n(x)$ eventually agrees with $s(x)$ for $x \in W \cap g^{-1}(1)$). To start the induction, we need a section s_2 agreeing with s where $f > \frac{1}{3}$ and $g > \frac{1}{2}$. The pointwise product fg is continuous on all of B (if we define it to be 0 on the complement of W), and s extends from the set where $f > \frac{1}{3}$ and $g > \frac{1}{2}$ to the halo in B where $fg > 0$, so s has a global extension s_2 by the section extension property for p. For the inductive

step, assume we've constructed $s_2, \ldots, s_n$ with the correct properties. We need to construct s_{n+1} agreeing with s_n on $f^{-1}((\frac{1}{n}, 1])$ and with s on $f^{-1}((\frac{1}{n+2}, 1])\cap g^{-1}((\frac{n}{n+1}, 1])$. Choose a continuous and decreasing function (it can be piecewise linear) $k_n : [0, 1] \to [\frac{1}{n+2}, \frac{1}{n}]$ with $k_n(t) = \frac{1}{n}$ if $t \leq \frac{n-1}{n}$, $k_n(t) = \frac{1}{n+2}$ if $t \geq \frac{n}{n+1}$. Let $A_n = \{x \in W : f(x) > k_n(g(x))\}$ and let $V_n = \{x \in W : f(x) > \frac{n}{n+1}k_n(g(x))\}$. Now s_n and s agree on the set where both $f > \frac{1}{n+1}$ and $g > \frac{n-1}{n}$, so we can define a section of p over A_n by using s_n where $f > \frac{1}{n+1}$ and by using s where $g > \frac{n-1}{n}$. (One or the other of these inequalities holds everywhere on A_n.) It will be enough to extend to a halo and then use the section extension property for p. Now if

$$h_n(x) = \frac{(n+1)f(x) - nk_n(g(x))}{k_n(g(x))}$$

for $x \in V_n \smallsetminus A_n$, $h_n(x) = 0$ for $x \notin V_n$, and $h_n(x) = 1$ for $x \in A_n$, then h_n is continuous on all of B. Thus V_n is a halo for A_n and we can do the inductive step using the section extension property. □

5.1.4. Theorem [Dold]. *Let B be a paracompact Hausdorff space, let E and F be Hausdorff spaces, and suppose $p : E \to B$ is an open continuous surjective map with the property that each $x \in B$ has a neighborhood U_x such that there is a homeomorphism $\varphi_x : p^{-1}(U_x) \xrightarrow{\simeq} U_x \times F$ with $p|_{p^{-1}(U_x)} = p_1 \circ \varphi_x$. (Here p_1 denotes projection onto the first factor.) Assume F is contractible. Then the map p is a homotopy equivalence. In fact, one can choose a section s of p such that there is a homotopy from id_E to $s \circ p$ which is "vertical," that is, commutes with p.*

Proof. The key to the proof is to construct a global section of p. For this purpose it is enough to show that p has the section extension property (then take $A = \emptyset$). Since B is paracompact, the given covering $\{U_x\}_{x\in B}$ of B can be refined to a covering $\{V_\alpha\}_{\alpha\in\Lambda}$ so that there is a partition of unity f_α subordinate to $\{V_\alpha\}$. (This means that each f_α is a continuous function $B \to [0, 1]$, that $\operatorname{supp}(f_\alpha) \subseteq V_\alpha$, and that $\sum_\alpha f_\alpha \equiv 1$.) Then by Lemma 5.1.2, each p_{V_α} has the section extension property. Next, by Lemma 5.1.3, we can replace each V_α by $f_\alpha^{-1}((0, 1])$, and each p_{V_α} will still have the section extension property. Let $A \subseteq B$, and suppose $A = g^{-1}(1)$, where $g : B \to [0, 1]$ is continuous. Suppose s is a section of p_V, where $V = g^{-1}((0, 1])$. We have to show that $s|_A$ has an extension to a global section of p. Let $\mathcal{F}$ be the set of pairs (T, s_T), where T is a subset of the index set Λ and s_T is a section of $p_{V\cup V_T}$ extending $s|_A$. Here V_T is shorthand for $\bigcup_{\alpha\in T} V_\alpha$. Order $\mathcal{F}$ by

$$(T, s_T) \leq (T', s_{T'}) \Leftrightarrow T \subseteq T' \text{ and } s_{T'} \text{ extends } s_T.$$

Then $\mathcal{F}$ is non-empty, since it contains $(\emptyset, s)$, and every chain in $\mathcal{F}$ clearly has an upper bound, so by Zorn's Lemma, $\mathcal{F}$ has a maximal element (S, s_S). We claim $V \cup V_S = B$. If not, choose β with V_β not contained in

$V \cup V_S$. Define $h : V_\beta \to [0, 1]$ by

$$h(x) = \min\left(1, \frac{g(x) + \sum_{\alpha \in S} f_\alpha(x)}{f_\beta(x)}\right).$$

Then $h > 0$ on $(V \cup V_S) \cap V_\beta$, where s_S is defined. So by the section extension property for V_β, there is a section s' of p_{V_β} extending the restriction of s_S to $h^{-1}(1)$, a set containing $A \cap V_\beta$. Now define a section s'' of p over $V \cup V_S \cup V_\beta$ by letting $s(x) = s_S(x)$ off V_β and on the part of V_β where $h(x) = 1$, and letting $s(x) = s'(x)$ on the part of V_β where $h(x) < 1$. Then $(S \cup \{\beta\}, s'') \in \mathcal{F}$, contradicting maximality of (S, s_S). Hence s_S is a global section extending $s|_A$ and B has the section extension property. In particular, p has a global section s.

To complete the proof, we will show there is a vertical homotopy from id_E to $s \circ p$. Since $p \circ s = id_B$, this will show in particular that p is a homotopy equivalence. Note that the argument we just gave also shows that $p_1 : E \times_B E \times [0, 1] \to E \times [0, 1]$, where $E \times_B E = \{(x, y) \in E \times E : p(x) = p(y)\}$ and p_1 is the projection $p_1(x, y, t) = (x, t)$, has the section extension property. (E is not necessarily paracompact, but the covering of B used above pulls back to a covering of $E \times [0, 1]$ with a partition of unity.) Let

$$A = E \times \{0, 1\} \subseteq V = E \times \left([0, \tfrac{1}{4}) \cup (\tfrac{3}{4}, 1]\right) \subseteq E \times [0, 1].$$

Then V is a halo around A and

$$f(x, t) = \begin{cases} (x, x, t), & t < \frac{1}{4}, \\ (x, s \circ p(x), t), & t > \frac{3}{4} \end{cases}$$

is a section of $(p_1)_V$. By the section extension property, the restriction of this section to A has a global extension. Composing with projection on the second copy of E, we get the desired vertical homotopy. □

Our reason for going through the proof of Theorem 5.1.4 will be to prove the next result.

5.1.5. Theorem. *Let G be any group. Suppose X_1 and X_2 are contractible Hausdorff G-spaces, that is, contractible Hausdorff topological spaces equipped with a actions of G by homeomorphisms, such that G acts freely and properly discontinuously on X_j (i.e., such that for all $x \in X_j$, there exists a neighborhood U_x of x such that $g \cdot U_x \cap U_x \neq \emptyset$ only for $g = 1$), and such that the quotient spaces X_j/G are paracompact, $j = 1, 2$. (Recall that any of the following is automatically paracompact: any compact Hausdorff space, any second-countable locally compact Hausdorff space, or any CW-complex.) Then the quotient spaces X_1/G and X_2/G are homotopy-equivalent.*

Proof. If X is Hausdorff and G acts freely and properly discontinuously on X, then the quotient map $q : X \to X/G$ is easily seen to be a local

homeomorphism and a covering map with G as group of covering transformations. (If U_x is a neighborhood as in the statement of the theorem, then $q : U_x \to q(U_x)$ is a homeomorphism and $q(U_x)$ is evenly covered, with $q^{-1}(q(U_x)) = \coprod_{g \in G} g \cdot U_x$.) In particular, X/G is Hausdorff. Incidentally, if X is also a nice enough space, say locally contractible, X/G is locally contractible with X as a simply connected, and thus universal, covering space, and with G as its fundamental group.

Now suppose one has two G-spaces X_1 and X_2 as in the theorem, for which the quotient spaces are paracompact. Let $q_1 : X_1 \to X_1/G$ and $q_2 : X_2 \to X_2/G$ be the quotient maps. Let $X = X_1 \times X_2$, equipped with the product topology and the diagonal action $g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$. Then X is again Hausdorff and contractible, and the action of G on X is also free and properly discontinuous. It will be enough to show that the projection map $p_1 : X \to X_1$ induces a homotopy equivalence $p_{1*} : X/G \to X_1/G$. (Then by symmetry, p_2 induces a homotopy equivalence $X/G \to X_2/G$, and X_1/G and X_2/G are each homotopy-equivalent to X/G, hence homotopy-equivalent to each other.) Since p_1 commutes with the G-actions on X and X_1, it induces a map $p_{1*} : X/G \to X_1/G$. This map satisfies the hypotheses of Theorem 5.1.4, with $F = X_2$, since X_2 is contractible, X_1/G is paracompact, and X/G is locally a product. Hence p_{1*} is a homotopy equivalence by Theorem 5.1.4. □

Our next objective will be to prove existence of spaces X as in Theorem 5.1.5, but from this point on in the chapter, we will need to use some basic algebraic topology of CW-complexes. While we won't give a course on the subject here, we will at least for the reader's convenience summarize some of the main theorems and give references and some sketches of proofs.

5.1.6. Definition. Let (X, A) be a pair of topological spaces, so that X is a topological space and $A \subseteq X$. Fix a point x_0 in A, called the **basepoint**. The nth **(relative) homotopy group (or set)** of (X, A), denoted $\pi_n(X, A, x_0)$ or $\pi_n(X, A)$ if the basepoint is understood or irrelevant, is the set of homotopy classes of maps $B^n \to X$ with restriction mapping $\partial B^n = S^{n-1}$ to A and some fixed basepoint in S^{n-1} to x_0. When $A = \{x_0\}$, this is the same thing as $\pi_n(X) = \pi_n(X, x_0)$, the set of homotopy classes of maps of $S^n = B^n/S^{n-1}$ to X which send the basepoint of S^n to the basepoint x_0 of X. When the basepoint of X is understood or irrelevant, the notation $\pi_n(X)$ is usually used. For $n \geq 1$, $\pi_n(X, x_0)$ is a group, and for $n \geq 2$ it is an abelian group, where the group structure comes from thinking of B^n as the n-cube and "stacking" two copies of B^n on top of one another, then mapping the larger cube which is the union of the original cubes to X by mapping each half separately. Similarly, the relative groups $\pi_n(X, A)$ are groups if $n \geq 2$ and abelian groups if $n \geq 3$. There is an exact sequence, called the **long exact homotopy sequence** of the pair (X, A), of the form

$$\cdots \to \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \to \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots,$$

where i_* is induced by the inclusion, ∂ comes from restricting a map $(B^n, S^{n-1}) \to (X, A)$ to S^{n-1}, and the map $\pi_n(X) \to \pi_n(X, A)$ comes from thinking of $\pi_n(X)$ as $\pi_n(X, \{x_0\})$ and taking the map induced by the inclusion of $\{x_0\}$ into A. For further details, see [Spanier, Ch. 7, §2] or [Whitehead, Ch. IV, §§1-3].

If X is path-connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for any two base-points x_0 and x_1. However, one gets an isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_1)$$

for each homotopy class of paths from x_0 to x_1, and the isomorphisms obtained this way may differ, so if X is not simply connected, the isomorphism is not canonical in general. In this way the group $\pi_1(X, x_0)$ operates on $\pi_n(X, x_0)$ for all n. A space is said to be **simple** if this action is trivial (for any choice of basepoint).

A space X is called **0-connected** if it is path-connected, **1-connected** if it is path-connected and simply connected (*i.e.*, path-connected and $\pi_1(X, x_0) = 0$ for any choice of basepoint), **n-connected** if it is path-connected and $\pi_j(X) = 0$ for all $j \leq n$, or equivalently, if every map $S^j \to X$, $j \leq n$, is homotopic to a constant map. Similarly, a pair (X, A) is called **relatively n-connected** if every map $(B^j, S^{j-1}) \to (X, A)$, $j \leq n$, is homotopic relative to S^{j-1} to a map into A. This is equivalent to assuming A meets every path-component of X and that $\pi_j(X, A, x_0) = 0$ for every $j \leq n$ and for every choice of basepoint. (Again, see [Spanier, Ch. 7, §2].)

5.1.7. Definition. A pair of spaces (X, A) is called a **relative CW-complex** if A is closed in X and there is a filtration of (X, A) by closed subspace pairs (X^k, A), called **skeleta**, where X^0 is obtained from A by adding 0-cells (*i.e.*, taking the disjoint union with a discrete space), X^k is obtained from X^{k-1} by attaching k-cells, the skeleta exhaust X, and X has the weak topology determined by the skeleta. If A is empty, we just say X is a **CW-complex**; this means X^0 is discrete, X^1 is obtained from X^0 by attaching 1-cells, etc., and a set in X is closed if and only if its intersection with each closed cell is closed.

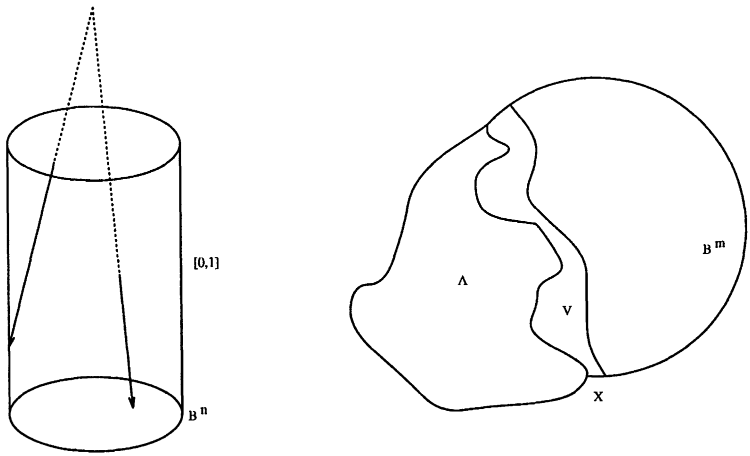
There is an obvious notion of CW-subcomplex, which is required to be closed in the original complex. The skeleta (X^k, A) are (relative) CW-subcomplexes of (X, A). (X, A) is said to have **dimension** n if it has at least one n-cell but $X^k = X$ for $k \geq n$; it is said to be **finite** if there are only finitely many cells or **countable** if there are only countably many cells. A map $(X, A) \to (Y, B)$ between relative CW-complexes is called **cellular** if it sends X^k to Y^k for each k.

Now we summarize the main homotopy-theoretic facts about CW-complexes.

5.1.8. Theorem ("Homotopy Extension Theorem"). *If (X, A) is a relative CW-complex, then the inclusion of A into X is a cofibration, i.e., given a homotopy $h : A \times [0, 1] \to Y$, where Y is any space, and given any*

extension H_0 of h_0 to a (continuous) map $X \to Y$, there is an extension $H : X \times [0, 1] \to Y$ of h which coincides with H_0 on $X \times \{0\}$.

Sketch of proof. This is proved (as is almost any theorem about CW-complexes) by induction over the skeleta. The key step is to handle the case where X is obtained from A by attaching an n-cell, *i.e.*, $X = A \cup_f B^n$, where $f : S^{n-1} \to A$. Then the theorem reduces to being able to define a map $H : B^n \times [0, 1] \to Y$ when H is prescribed on $(B^n \times \{0\}) \cup (S^{n-1} \times [0, 1])$. One visualizes $B^n \times [0, 1]$ as a cylinder, and H is already prescribed on the side and base of this cylinder. So we define H by first retracting the cylinder to the side and base and then using the already-defined map there. The retraction may be defined by projecting down from some point above (see Figure 5.1.9(a)). □



5.1.9. Figure:

(a) A retraction to the base and sides of a cylinder

(b) Attaching a single m-cell to A

5.1.10. Theorem. *If (X, A) is a relative CW-complex with cells only of dimension $\geq n$ (where $n \geq 1$), then (X, A) is $(n-1)$-connected.*

Sketch of proof. Again, this is basically reduced by induction to the case where $X = A \cup_f B^m$ for some $m \geq n$, where $f : S^{m-1} \to A$ is an attaching map. Then A has a neighborhood V in X which has a deformation retraction down to A, and so that $X \setminus V$ is homeomorphic to B^m (see Figure 5.1.9(b)). Suppose $k \leq n-1$ and one is given a map $g : (B^k, S^{k-1}) \to (X, A)$. One can prove using the Simplicial Approximation Theorem that (B^m, S^{m-1}) is $(m-1)$-connected. Then using this fact one can compress f (rel S^{k-1}) into V. Finally, compress f down to A using the deformation retraction from V to A. □

5.1.11. Corollary. *If (X, A) is a relative CW-complex, then the map $\pi_n(X^k, A) \to \pi_n(X, A)$ is surjective for $k = n$ and an isomorphism for $k > n$. (Here we suppose $n \geq 2$ unless A is a single point, so that the homotopy sets are groups.)*

Proof. Since X is obtained from X^k by attaching cells of dimension $\geq k+1$, (X, X^k) is k-connected by the theorem, and thus (for any choice of basepoint in A), $\pi_n(X, X^k) = 0$ for $n \leq k$. Splicing the long exact homotopy sequences of the pairs (X, A), (X, X^k), and (X^k, A) gives the long exact sequence of the triple (X, X^k, A):

$$\pi_{n+1}(X, X^k) \xrightarrow{\partial} \pi_n(X^k, A) \to \pi_n(X, A) \to \pi_n(X, X^k),$$

from which the result follows. □

5.1.12. Theorem ("Cellular Approximation Theorem"). *Any map $(X, A) \to (Y, B)$ between relative CW-complexes is homotopic* (rel *A) to a cellular map, and existence of a homotopy between two cellular maps between relative CW-complexes implies existence of a cellular homotopy between them.*

Sketch of proof. Again this is done by induction on skeleta, using the Homotopy Extension Theorem (Theorem 5.1.8) and Theorem 5.1.10. For details, see [Whitehead, Ch. II, Theorem 4.6] or [Spanier, Ch. 7, §6]. □

5.1.13. Theorem ("Whitehead's Theorem"). *A CW-complex X is contractible if and only if it is connected and $\pi_n(X, x_0) = 0$ for all n (for some choice of a basepoint x_0). A relative CW-complex (X, A) is relatively contractible if and only if A meets every path component of X and $\pi_n(X, A, x_0) = 0$ for any choice of a basepoint and for any n. A map $f : X \to Y$ between connected CW-complexes X and Y is a homotopy equivalence if and only if it induces an isomorphism $\pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, f(x_0))$ for some (hence for all) $x_0 \in X$.*

Sketch of proof. First we give the idea of the contractibility result in the absolute case. Suppose X is a connected CW-complex and we choose

the basepoint x_0 in the 0-skeleton. Assume $\pi_n(X, x_0) = 0$ for all n. First contract the 0-skeleton to x_0, which is possible since X is path-connected. One can extend the homotopy to all of X using the Homotopy Extension Theorem (Theorem 5.1.8). Then contract the 1-skeleton down to the 0-skeleton, using the fact that X is simply connected, and again extend the homotopy to all of X. Doing the homotopies in sequence, one has a homotopy from id_X to a map sending the 1-skeleton to x_0. Continue this process one skeleton at a time, and pass to the limit. The limiting map is continuous by definition of the weak topology, and gives a contraction of X. The relative case is quite similar but just technically a little more complicated.

Next consider the case of a map $f : X \to Y$ between connected CW-complexes X and Y. Let $Z_f = (X \times [0, 1]) \cup_f Y$ be its mapping cylinder. Then f is a homotopy equivalence if and only if the inclusion of X into Z_f (as $X \times \{0\}$) is a homotopy equivalence. Fix a basepoint x_0 in the 0-skeleton of X. If f is a homotopy equivalence, then certainly $f_* : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, f(x_0))$ is an isomorphism for all n. Conversely, if this condition is satisfied, first homotope f to a cellular map f' using Theorem 5.1.12; then f'_* is also an isomorphism for all n. This is equivalent to saying that the pair $(Z_{f'}, X)$ is n-connected for all n. But the pair can be given a relative CW-structure so f' is a homotopy equivalence by the first part of the theorem. Then f is a homotopy equivalence since it is in the same homotopy class as f'.

For more details, see [Spanier, Ch. 7, §6] or [Whitehead, Chs. II, IV, and V]. □

To do any sort of calculations of homotopy groups, one needs to have a few basic cases with which to get started, and then one can try to apply exact sequences or other homological machinery. The key to this is to relate homotopy groups to homology groups, which are much easier to calculate. Recall that if (X, A) is a relative CW-complex, the singular homology groups $H_\bullet(X, A; R)$ coincide with the **cellular homology groups**, which are the homology groups of a chain complex $C_\bullet(X, A; R)$ with $C_k(X, A; R)$ the free R-module on the relative k-cells of (X, A), and with boundary maps determined by the attaching data of the cells. The link between homotopy and homology is then given by the following.

5.1.14. Theorem (The "Hurewicz Theorem"). *Let (X, A) be a pair of spaces with basepoint $x_0 \in A$. There are natural transformations (each usually called the "Hurewicz map") $\pi_\bullet(X, A) \to H_\bullet(X, A; \mathbb{Z})$, $\pi_\bullet(X) \to H_\bullet(X; \mathbb{Z})$, defined by sending the homotopy class of a map $f : (B^n, S^{n-1}) \to (X, A)$ to $f_*([B^n, S^{n-1}])$, where $[B^n, S^{n-1}]$ is the standard generator of the infinite cyclic group $H_n(B^n, S^{n-1}; \mathbb{Z})$, or by sending the homotopy class of a map $f : S^n \to X$ to $f_*([S^n])$, where $[S^n]$ is the standard generator of the infinite cyclic group $H_n(S^n; \mathbb{Z})$. The Hurewicz map factors through $H_0(\pi_1(X), \pi_\bullet(X))$ (group homology here!) in the absolute case, or through $H_0(\pi_1(A), \pi_\bullet(X, A))$ in the relative case.*

If X is n-connected, $n \geq 0$, then $\tilde{H}_j(X; \mathbb{Z}) = 0$ for $j \leq n$, and the

Hurewicz map induces an isomorphism from $\pi_1(X, x_0)_{\mathrm{ab}}$ *to* $H_1(X; \mathbb{Z})$ *(if* $n = 0$*) or from* $\pi_{n+1}(X, x_0)$ *to* $H_{n+1}(X; \mathbb{Z})$ *(if* $n > 0$*). Conversely, if* X *is 1-connected and* $\tilde{H}_j(X; \mathbb{Z}) = 0$ *for* $j \leq n$, $n > 1$, *then* X *is in fact* n*-connected.*

Similarly, if X *and* A *are path-connected and* (X, A) *is* n*-connected,* $n \geq 1$, *then* $\tilde{H}_j(X, A; \mathbb{Z}) = 0$ *for* $j \leq n$, *and the Hurewicz map induces an isomorphism from* $H_0(\pi_1(A, x_0), \pi_{n+1}(X, A))$ *to* $H_{n+1}(X, A; \mathbb{Z})$.

Sketch of proof. It is clear that the Hurewicz map is well defined and natural, and also easy to see that it takes the same value on $[f]$ and on $[\gamma] \cdot [f]$, where $[\gamma]$ is the class of some loop in $\pi_1(X, x_0)$ (absolute case) or $\pi_1(A, x_0)$ (relative case). Thus the Hurewicz map kills everything of the form $[f] - [\gamma] \cdot [f]$, which means it factors through $H_0(\pi_1(X), \pi_\bullet(X))$ or $H_0(\pi_1(A), \pi_\bullet(X, A))$.

The hard parts are therefore the isomorphism theorems. The complete proof is quite tricky and requires proving the absolute and relative theorems simultaneously by induction on n. (See [Spanier, Ch. 7, §§4–5] or [Whitehead, Ch. IV, §§6–7].) However, we can at least give a hint of how to proceed. The fact that $\pi_1(X, x_0)_{\mathrm{ab}} \to H_1(X; \mathbb{Z})$ is an isomorphism for path-connected spaces is relatively elementary and can be done directly from the definition of singular homology. This is used to start the induction. Then one possible strategy is to use the fact that taking the loop space of a (pointed) space shifts all the homotopy groups down by 1, so that X is n-connected if and only if ΩX is $(n-1)$-connected. Thus if X is n-connected, $n \geq 1$, $\tilde{H}_j(\Omega X; \mathbb{Z}) = 0$ for $j \leq n-1$ and the Hurewicz map $\pi_{n+1}(X) = \pi_n(\Omega X) \to H_n(\Omega X; \mathbb{Z})$ is an isomorphism. Thus if one can relate the homology of X to that of ΩX (which can be done using the so-called Serre spectral sequence), one can hope to show that the Hurewicz map $\pi_{n+1}(X) \to H_{n+1}(X; \mathbb{Z})$ is an isomorphism.

Let us sketch another sort of proof for the absolute case when X is a CW-complex. Suppose X is an n-connected CW-complex with $n \geq 1$, and choose a basepoint x_0 in the 0-skeleton X^0 of X. By the proof given above of Theorem 5.1.13, there is a homotopy from id_X to a map which collapses the n-skeleton X^n to x_0. Thus X is homotopy-equivalent to the complex $X' = X/X^n$ with one-point n-skeleton and thus with no (reduced) cellular j chains for $j \leq n$, and $\tilde{H}_j(X; \mathbb{Z}) \cong \tilde{H}_j(X'; \mathbb{Z})$ vanishes for $j \leq n$. Furthermore, the $(n+1)$-skeleton X'^{n+1} of X' is a wedge of S^{n+1}'s. Suppose we know the absolute Hurewicz Theorem for such a space (surjectivity is obvious, and injectivity can be proved by a direct geometrical argument, as in [Whitehead, Ch. I, §3]). Now consider the

commutative diagram

$$\begin{array}{ccc}
\pi_{n+2}(X'^{n+2}, X'^{n+1}) & \longrightarrow & H_{n+2}(X'^{n+2}, X'^{n+1}; \mathbb{Z}) \\
\partial\downarrow & & \partial\downarrow \\
\pi_{n+1}(X'^{n+1}) & \longrightarrow & H_{n+1}(X'^{n+1}; \mathbb{Z}) \\
\downarrow & & \downarrow \\
\pi_{n+1}(X'^{n+2}) & \longrightarrow & H_{n+1}(X'^{n+2}; \mathbb{Z}) \\
\downarrow & & \downarrow \\
\pi_{n+1}(X') & \longrightarrow & H_{n+1}(X'; \mathbb{Z}).
\end{array}$$

Then the first horizontal map is surjective, since each generator of the relative H_{n+2} comes from an $(n+2)$-cell which gives a class in the relative π_{n+2}-group, and the second horizontal map is an isomorphism, since X'^{n+1} is a wedge of spheres. As for the vertical maps, the image of the first in each column is the kernel of the second, and the last is an isomorphism. (In the case of homology, this is because all cellular $(n+1)$-chains come from the $(n+1)$-skeleton, and all cellular $(n+1)$-boundaries must be boundaries of chains from the $(n+2)$-skeleton. In the case of homotopy, this follows from Corollary 5.1.11.) So the Hurewicz map is certainly surjective in degree $n+1$ for X' and thus for X. To prove injectivity, suppose $g : (S^{n+1}, *) \to (X'^{n+1}, x_0)$ maps to 0 in $H_{n+1}(X'; \mathbb{Z})$, and thus in $H_{n+1}(X'^{n+2}; \mathbb{Z})$. Then the homology class $[g]$ corresponding to g in $H_{n+1}(X'^{n+1}; \mathbb{Z})$ is the boundary of some cellular $(n+2)$-chain in X'^{n+2}, *i.e.*, of some linear combination of $(n+2)$-cells in X'. Then using surjectivity of the relative Hurewicz map for (X'^{n+2}, X'^{n+1}), we see there is a class in $\pi_{n+2}(X'^{n+2}, X'^{n+1})$ mapping to the class of g in $\pi_{n+1}(X'^{n+1})$, and the image of the class of g vanishes in $\pi_{n+1}(X'^{n+2})$ and thus in $\pi_{n+1}(X')$. □

We return now to the subject of "classifying spaces."

5.1.15. Theorem. *Let G be a group. Then there exists a contractible CW-complex X on which G acts freely and cellularly (hence properly discontinuously) with a CW-complex as quotient space.*

Proof. The construction follows Eilenberg and Mac Lane [EilMacL, p. 369]. In their terminology, X/G is called a $K(G, 1)$-space.[1] Let $X_0 = G$ (with the discrete topology; this is of course a 0-dimensional CW-complex) and inductively define $X_n = X_{n-1} * G$ for $n \geq 1$, where $*$ denotes the "join" of spaces. Intuitively, the join $A * B$ of two non-empty spaces A and B is constructed by taking disjoint copies of A and B which are in "general position" in some big Euclidean space and taking the union of all line segments joining a point in A to a point in B. More precisely,

[1]The construction was later generalized by Milnor [MilnorUB] to the case where G is a topological group.

$A * B$ is, as a set, the quotient space $A \times B \times [0, 1]/\sim$, where $\sim$ is trivial on $A \times B \times (0, 1)$ and $(a, b, 0) \sim (a, b', 0)$, $(a, b, 1) \sim (a', b, 1)$, for all $a, a' \in A$ and for all $b, b' \in B$. (The equivalence classes of the (a, b, t), $t \in [0, 1]$ should be viewed as a line segment from $a \in A$ to $b \in B$. Note that if B consists of a single point, $A * B$ is just the cone on A with B as the cone point.) G acts on $X_0 = G$ by left translation and inductively on $X_n = X_{n-1} * G$ by $g \cdot (x, g', t) = (g \cdot x, gg', t)$ (this action on $X_{n-1} \times G \times [0, 1]$ preserves $\sim$ and so descends to the quotient space $X_{n-1} * G$). Note that the action is obviously free. There is a G-equivariant embedding of X_{n-1} in X_n sending $x \in X_{n-1}$ to the equivalence class of the $(x, g, 0)$, $g \in G$, and we let $X = \varinjlim X_n$ with the obvious free G-action. Then we can make the X_n's, and $X = X_\infty$, into CW-complexes for which the G-action preserves the cellular structure, with the additional properties that X_n has dimension n and is a subcomplex of X_m for $m > n$. The closed j-cells, $j \leq n$, of X_n will be the closed j-cells of X_{n-1} (provided $j \leq n - 1$), together with the joins $E^{j-1} * g$ of a closed $(j - 1)$-cell E^{j-1} of X_{n-1} with a point in G, together with a copy of the points of G if $j = 0$. The cells of X are just the union of the cells of the X_n's. It is easy to see that there is a unique CW-topology on X_n or on X compatible with the cellular structure. (Recall that in this topology, a set is closed if and only if its intersection with each closed cell is closed.) The CW-structure is countable if and only if G is a countable group. Since the action of G sends each cell homeomorphically onto another, the action of G is continuous and cellular, and so is properly discontinuous. Thus the quotient spaces X_n/G and X/G are also CW-complexes. So we only need to show that X is contractible. While it's possible to construct an explicit contraction of X (cf. [Dold, pp. 252–253], which uses a slightly different topology on the join), for our purposes it will be enough to show by induction on n that for $n \geq 1$, X_n is $(n - 1)$-connected and the CW-pair (X_n, X_{n-1}) is relatively $(n - 1)$-connected. Since $X = \varinjlim X_n$, it will follow that X is n-connected for all n, hence contractible by Theorem 5.1.13. To start the induction, X_1 is path-connected, that is, 0-connected, since its 0-cells are two copies of G, and there is a 1-cell joining any 0-cell in the first copy of G to any 0-cell in the second copy of G. (To join two 0-cells in the same copy of G by an arc, join them each to the same 0-cell in the other copy of G.) Similarly it is clear that X_n is path-connected for each $n \geq 1$, hence that X is path-connected. Let's also observe that X_n is simply connected for $n \geq 2$—since X_n is the union of the $X_{n-1} * g$ for $g \in G$, joined along X_{n-1}, and since X_{n-1} is path-connected and each $X_{n-1} * g$ is a cone, hence contractible, this follows from Van Kampen's Theorem. For the inductive step, suppose $n \geq 3$ and suppose we know X_{n-1} is $(n - 2)$-connected. Because of the Hurewicz Theorem (Theorem 5.1.14), to show X_n is $(n - 1)$-connected, it is enough to show that $\tilde{H}_j(X_n; \mathbb{Z})$ vanishes for $j \leq n-1$. This follows from

the Mayer-Vietoris sequence

$$H_{j+1}(X_n;\,\mathbb{Z}) \to H_j(X_{n-1};\,\mathbb{Z}) \to \bigoplus_{g\in G} H_j(cX_{n-1};\,\mathbb{Z}) = 0$$
$$\to H_j(X_n;\,\mathbb{Z}) \to H_{j-1}(X_{n-1};\,\mathbb{Z})$$

and the inductive hypothesis. Then from the exact sequence

$$\pi_j(X_n) \to \pi_j(X_n,\, X_{n-1}) \to \pi_{j-1}(X_{n-1}),$$

the pair $(X_n,\, X_{n-1})$ is relatively $(n-1)$-connected. Passing to the limit, X is n-connected for all n and thus contractible. □

5.1.16. Definition. If G is a group and X is a G-space as in Theorem 5.1.5, so that X is contractible, G acts freely and properly discontinuously on X, and X/G is paracompact, we write EG for X and BG for X/G, and call $BG = X/G$ a **classifying space** for G. The existence of such a space is guaranteed by Theorem 5.1.15. Note that there is a slight abuse of notation here, since BG is not uniquely defined (up to homeomorphism); however, by Theorem 5.1.5, it is well defined up to homotopy equivalence.

5.1.17. Examples. If G is the trivial group, any contractible paracompact space, in particular, a point, or any Euclidean space, or any contractible CW-complex, is a classifying space for G. The infinite join construction in the proof of Theorem 5.1.15 gives $X_0 = pt$, $X_1 = c(X_{n-1})$, and thus one can see by induction on n that $X_n = \Delta^n$, the n-simplex. Thus this construction yields an "infinite simplex" for $EG = BG$. Note that this is **not** the simplest choice for BG, which of course is a one-point space.

If $G = \mathbb{Z}$, then G acts freely and properly discontinuously on $EG = \mathbb{R}$, with quotient space BG homeomorphic to S^1. Once again, the construction the proof of Theorem 5.1.15 gives a much more complicated model for BG. And there are still other models for BG which have a different "look," for instance, the Möbius band.

This example may be generalized: if $G = F_n$, the free group on n generators, then there is a model for BG which is a wedge of n circles. The universal cover EG of this space is a tree on which G acts freely. Conversely, if EG can be taken to be a tree (*i.e.*, a contractible one-dimensional CW-complex), then G is a free group. For related facts and ideas, see [SerreTrees].

If G is a two-element cyclic group, then the construction the proof of Theorem 5.1.15 does in fact give the simplest possible CW-model for BG. Namely, we can identify G with S^0, and then $X_n = X_{n-1} * S^0$. For any space A, $A * S^0$ is the union of two copies of the cone on A joined along A, or in other words the suspension ΣA of A. So by induction one sees that $X_n = \Sigma S^{n-1} = S^n$, and $X_n/G = S^n/(\text{antipodal map}) = \mathbb{P}^n(\mathbb{R})$. Thus EG is the "infinite sphere" and BG the "infinite real projective space." There is another model for BG which is also interesting: if $\mathcal{H}$ is an infinite-dimensional real Hilbert space, then the orthogonal group $O(\mathcal{H})$ is known

to be contractible with respect to two different topologies that make it into a topological group: the weak operator topology (for which this fact is due to Dixmier and Douady) and the norm topology (for which this fact is due to Kuiper). The center of $O(\mathcal{H})$ consists of the scalar operators ± 1, and the quotient group $PO(\mathcal{H}) = O(\mathcal{H})/\{\pm 1\}$ (using either of the topologies on $O(\mathcal{H})$) is a classifying space for $\{\pm 1\}$.

5.1.18. Proposition. *The classifying space construction gives a functor B from the category of groups and group homomorphisms to the category of topological spaces (or the full subcategory of CW-complexes) and homotopy classes of continuous maps.*

Proof. Suppose $\alpha : G \to H$ is a homomorphism of groups. Then α clearly induces a map α_* from $G * G * \cdots * G$ (n times) to $H * H * \cdots * H$ (n times). Since $\alpha_*(g \cdot x) = \alpha(g) \cdot \alpha_*(x)$, we obtain an induced map from $(G * G * \cdots * G)/G$ to $(H * H * \cdots * H)/H$. Letting $n \to \infty$, we get a map $B\alpha : BG \to BH$, and it is clear from the construction that $B(\alpha \circ \beta) = B(\alpha) \circ B(\beta)$, $B(id) = id$. □

Before proceeding further with the theory of classifying spaces we need to return to our review of basic algebraic topology.

5.1.19. Definition. A continuous map of topological spaces $p : E \to B$ is called a **fibration** if it has the **homotopy lifting property**, that is, if for any homotopy $h : X \times [0, 1] \to B$ and any continuous map $H_0 : X \to E$ such that $h_0 = p \circ H_0$, there is a continuous map $H : X \times [0, 1] \to E$ with $H(x, 0) = H_0(x)$ such that $p \circ H = h$. We call H_0 and H **lifts** of h_0 and of h, respectively. For each $x \in B$, $p^{-1}(x)$ is called the **fiber** of p over x. The space B is called the **base space** of the fibration and the space E is called the **total space**.

5.1.20. Proposition. *The following are fibrations:*

(1) *The projection $p : E \times F \to B$ onto the first factor in a product space (with the product topology). The fiber over any point is homeomorphic to F in this case.*
(2) *Any covering space $E \to B$. The fiber over any point is discrete in this case.*
(3) *The map $p : PB \to B$, where PB is the* **path space** *of a compactly generated (Hausdorff) space B relative to a basepoint b_0, i.e., the set of continuous maps $\gamma : [0, 1] \to B$ with $\gamma(0) = b_0$, and $p(\gamma) = \gamma(1)$. In this case, the fiber over b_0 is ΩB, the loop space of B (relative to b_0).*

Proof. (1) Given a homotopy $h : X \times [0, 1] \to B$ and a continuous map $H_0 : X \to B \times F$ such that $h_0 = p \circ H_0$, we can write $H_0(x) = (h_0(x), f(x))$ with $f : X \to F$. Then define the continuous map $H : X \times [0, 1] \to B \times F$ by $H(x, t) = (h_t(x), f(x))$, and $H(x, 0) = H_0(x)$.

(2) The homotopy lifting property for a covering is obtained by chopping $[0, 1]$ into subintervals. Given $h : X \times [0, 1] \to B$, the image of $h(X \times [t_j, t_{j-1}])$ will be contained in a set over which p is of the form (1) with

F discrete, provided $t_j - t_{j-1}$ is small enough. So we choose a suitable partition of [0, 1] and first lift h over $X \times [0, t_1]$, then lift h over $X \times [t_1, t_2]$, etc. See [Spanier, Ch. 2, §2] or [Whitehead, Ch. I, §7] for more details.

(3) Given a homotopy $h : X \times [0, 1] \to B$ and a continuous map $H_0 : X \to PB$ such that $h_0 = p \circ H_0$, $H_0(x) : [0, 1] \to B$ with $H_0(x)(0) = b_0$ and with $H_0(x)(1) = h_0(x)$. We need to define $H : X \times [0, 1] \times [0, 1] \to B$ with $H(x, t, 0) = H_0(x)(t)$ and with $H(x, 1, s) = h(x, s)$, $H(x, 0, s) = b_0$. The desired H can be constructed as in the proof of the Homotopy Extension Theorem (Theorem 5.1.8). □

5.1.21. Proposition. *Let X and Y be compactly generated (Hausdorff) spaces, and let $f : X \to Y$ be a continuous map. Then there is a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

with p_X and p_Y homotopy equivalences and with $p_Y \circ f'$ homotopic to a fibration. (Thus any map in the category of compactly generated spaces is homotopic to a composite of homotopy equivalences and fibrations.)

Proof. Let Y' be the space of continuous maps $[0, 1] \to Y$, let

$$X' = \{(\gamma, x) : \gamma \in Y', x \in Y, \gamma(1) = f(x)\},$$

and define p_Y and p_X by $p_Y(\gamma) = \gamma(1)$, $p_X(\gamma, x) = x$. Then clearly we have a commutative diagram as indicated, if we let $f'(\gamma, x) = (\gamma)$. The map p_Y is a homotopy equivalence with homotopy inverse c_Y sending any $y \in Y$ to the constant path at y. (Clearly $p_Y \circ c_Y = id_Y$. On the other hand, there is a homotopy h from $id_{Y'}$ to $c_Y \circ p_Y$ given by $h_t(\gamma)(s) = \gamma(t)$, $s \le t$, and $\gamma(s)$, $s \ge t$.) Similarly, the map p_X is a homotopy equivalence with homotopy inverse $g : x \mapsto (c_Y(f(x)), x)$. We only need to show that $p : (\gamma, x) \mapsto \gamma(0)$ is a fibration $X' \to Y$, since this map is homotopic to $p_Y \circ f'$ as in the previous proposition. Suppose Z is some other space and one has a homotopy $h : Z \times [0, 1] \to Y$ as well as a lift $H_0 : Z \to X'$ of h_0. We need to construct a lift $H : Z \times [0, 1] \to X'$ of h extending H_0. Let's write $H(z, t) = (\gamma(z, t), x(z, t))$, $H_0(z) = (\gamma_0(z), x_0(z))$. Here $\gamma_0(z)(1) = f(x_0(z))$, $h(z, 0) = \gamma_0(z)(0)$. Then we need to arrange to have $\gamma(z, t)(1) = f(x(z, t))$, $h(z, t) = \gamma(z, t)(0)$, $\gamma(z, 0) = \gamma_0(z)$, $x(z, 0) = x_0(z)$. Let $x(z, t) = x_0(z)$, so that we want $\gamma(z, t)(1) = f \circ x_0(z)$. Once again, the desired map γ can be constructed by the method of proof of Theorem 5.1.8. □

The simplest examples of fibrations are those listed in Proposition 5.1.20, so in order to have more examples it is important to know that with mild conditions on the base space, a map that is locally a fibration is globally a fibration.

5.1.22. Theorem (Hurewicz). *Let B be a paracompact Hausdorff space, let $p : E \to B$ be a continuous map, and suppose that B can be covered by open sets U such that the restriction of p to $p^{-1}(U)$ is a fibration for each U. Then p is a fibration.*

Sketch of proof. This result is technically difficult, but the ideas are quite similar to those in the proof of Theorem 5.1.4. We will not go through the details since we will not need this theorem in what follows; instead see [Spanier, Ch. 2, §7] or [Dold]. □

We conclude our review of algebraic topology with a discussion of the homotopy properties of fibrations, since we will need to make use of these in Sections 2 and 3 of this chapter.

5.1.23. Proposition. *Let $p : E \to B$ be a fibration, and suppose B is path-connected. Then for any two points b_0 and b_1 in B, the corresponding fibers $F_0 = p^{-1}(b_0)$ and $F_1 = p^{-1}(b_1)$ have the same homotopy type. (This explains why it is customary to refer to "the" fiber of a fibration.)*

Proof. Choose a path $\gamma : [0, 1] \to B$ with $\gamma(0) = b_0$, $\gamma(1) = b_1$. Replacing p by $\gamma^*(p)$, that is, the map

$$\{(t, x) \in [0, 1] \times B : p(x) = \gamma(t)\} \to [0, 1] : (t, x) \mapsto t,$$

which it's easy to see is again a fibration, we may assume that $B = [0, 1]$ and $b_0 = 0$, $b_1 = 1$. Now define $h : E \times [0, 1] \to B = [0, 1]$ by $h(x, t) = \min(p(x) + t, 1)$, and define $k : E \times [0, 1] \to B = [0, 1]$ by $h(x, t) = \max(p(x) - t, 0)$. Define $H_0 : E \to E$ and $K_0 : E \to E$ both to be the identity map. Then H_0 lifts h_0 and K_0 lifts k_0, so by the homotopy lifting property, there are lifts $H : E \times [0, 1] \to E$ with $H(x, 0) = x$ and with $p \circ H(x, t) = \min(p(x)+t, 1)$ and $K : E \times [0, 1] \to E$ with $K(x, 0) = x$ and with $p \circ K(x, t) = \max(p(x) - t, 0)$. Then $H_1 : F_0 \to F_1$, $K_1 : F_1 \to F_0$, and these maps are homotopy inverses of each other since $H_t \circ K_t$, $K_t \circ H_t$ are homotopies to the identity maps (on F_1 and on F_0, respectively). □

5.1.24. Theorem. *Let $p : E \to B$ be a fibration with B path-connected, and let $b_0 \in B$, $F = p^{-1}(b_0)$, $x_0 \in F$. There is an exact sequence, called the* **long exact homotopy sequence of the fibration** *p:*

$$\cdots \to \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \\ \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \to \cdots .$$

Here i_* is induced by the inclusion $i : F \hookrightarrow E$ and p_* is the map on homotopy groups induced by p. The sequence is natural (with respect to maps of fibrations).

Proof. The desired sequence is obtained from the long exact homotopy sequence of the pair (E, F); all we need to do is to show that $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all n. To prove surjectivity, suppose one is given a class in $\pi_n(B)$. View it as being given by

a map $h_1 : (B^n, S^{n-1}) \to (B, b_0)$. Since B^n is contractible, there is a homotopy h from h_1 to the constant map $h_0 : B^n \to \{b_0\}$. Choose any lifting of h_0 and apply the homotopy lifting property; this gives a class in $\pi_n(E, F, x_0)$ mapping to the given class in $\pi_n(B, b_0)$. To prove injectivity when $n \geq 2$ (so that $\pi_n(E, F, x_0)$ is a group), suppose one is given $G_0 : (B^n, S^{n-1}, *) \to (E, F, x_0)$, and suppose its image under p_* is null-homotopic. Let g be a homotopy from $g_0 = p \circ G_0$ to a constant map $B^n \to x_0$. By the homotopy lifting property, there is a lift G of g extending G_0, and so G_0 is the trivial class in $\pi_n(E, F, x_0)$. The reader can easily provide the special arguments needed to show that $\pi_0(E, F, x_0) = 0 = \pi_0(B, b_0)$ and to show that p_* is injective in dimension 1. □

5.1.25. Corollary. *If $p : E \to B$ is a covering map, then $\pi_n(E) \xrightarrow{\cong} \pi_n(B)$ for all $n > 1$ (with respect to any choice x_0 of a basepoint in E and with respect to the basepoint $b_0 = p(x_0)$ of B). Furthermore, if E and B are path-connected and $F = p^{-1}(b_0)$, then F is discrete and there is an exact sequence of sets $0 \to \pi_1(E, x_0) \to \pi_1(B, b_0) \to F \to 0$.*

Proof. In any covering, the fiber over any point is discrete. The rest follows as a special case of Theorem 5.1.24, using the fact from Proposition 5.1.20(2) that p is a fibration. □

5.1.26. Examples. A key example of the situation of Corollary 5.1.25 comes from the case $E = EG$, $B = BG$, p the quotient map. Since p is a covering and E is contractible, we see that $\pi_n(BG) = 0$ for $n > 1$. Conversely, given a connected CW-complex Y with $\pi_1(Y, y_0) = G$ for some choice of basepoint and with $\pi_n(Y, y_0) = 0$ for $n > 1$, the universal covering of Y has all homotopy groups equal to 0, hence is contractible by Theorem 5.1.13. So Y is a classifying space for G.

Let us give a few other examples of homotopy exact sequences of fibrations. Let X be a (compactly generated Hausdorff) space with basepoint x_0 and let ΩX be the loop space of X relative to this basepoint. Without loss of generality, we may assume X is path-connected. Let PX be the path space of X relative to the basepoint x_0. By Proposition 5.1.20(3), the projection $p : PX \to X$ is a fibration with fiber ΩX. But PX is contractible, since any path can be shrunk to its initial point x_0. Thus from Theorem 5.1.24, we recover the fact that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ for all n.

As another example, consider the Hopf fibration $p : S^3 \to S^2$, which comes from viewing S^3 as the group $SU(2)$ and S^2 as the homogeneous space G/H, where H is a maximal torus in G (a circle group). Since the quotient map p is **locally** the projection onto one factor of a product, p is a fibration by Theorem 5.1.22. The fiber over the identity coset is $H \cong S^1$, which has contractible universal cover $\mathbb{R}$. So by our first example $\pi_n(S^1) = 0$ for $n > 1$ (and of course $\pi_1(S^1) = \mathbb{Z}$). So the long exact sequence has the form

$$\cdots \to 0 \to \pi_n(S^3) \to \pi_n(S^2) \to 0 \to \cdots \to 0 \to \pi_3(S^3) \xrightarrow{p_*} \pi_3(S^2)$$
$$\to 0 \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) = \mathbb{Z} \to 0.$$

Since the n-sphere is $(n-1)$-connected, $\pi_2(S^3) = 0$, and by the Hurewicz Theorem (Theorem 5.1.14), $\pi_3(S^3) \cong \mathbb{Z}$, $\pi_2(S^2) \cong \mathbb{Z}$. It follows that $\pi_3(S^2) \cong \mathbb{Z}$, with generator $p_*(id_{S^3}) = p$, and that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n > 3$. In particular, this example makes it clear that $\pi_n(X)$ need not vanish when X is a CW-complex of dimension less than n.

Finally, we return to one more aspect of classifying spaces, namely their homology. (We have already computed their homotopy groups in Example 5.1.26.)

5.1.27. Theorem. *Let G be any group and let BG be a classifying space for G. If M is any abelian group with trivial G-action, then there is a natural isomorphism between $H_\bullet(G, M)$ (group homology) and $H_\bullet(BG; M)$ (singular homology). The same holds for any G-module if we interpret $H_\bullet(BG; M)$ as homology with local coefficients (for the definition, see* [Spanier, Ch. 5, Exercise Set I] *or* [Whitehead, Ch. VI, §§1–4]*).*

Proof. We use the CW-model for BG constructed in the proof of Theorem 5.1.15. Then over each cell of BG, there is a family of cells of EG which is permuted simply transitively by G. So the cellular chain complex of BG with coefficients in $\mathbb{Z}$, $C_\bullet(BG)$, may be identified with $C_\bullet(EG) \otimes_{\mathbb{Z}G} \mathbb{Z}$, where we think of the cellular chain complex of EG as being a complex of free $\mathbb{Z}G$-modules. Similarly, if we use any G-module M as (local) coefficients, $C_\bullet(BG; M)$ may be identified with $C_\bullet(EG) \otimes_{\mathbb{Z}G} M$. However, since EG is contractible, $C_\bullet(EG)$ is acyclic (except in dimension 0), and so gives a resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules. Recall that $H_\bullet(G, M)$ was defined in 4.1.7 to be the homology of the complex $P_\bullet \otimes_{\mathbb{Z}G} M$, where $P_\bullet$ was another specific resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules. However, any two resolutions of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules must be chain homotopy-equivalent by Theorem 1.7.7 and the method of proof of Lemma 3.1.11, so $P_\bullet \otimes_{\mathbb{Z}G} M$ and $C_\bullet(EG) \otimes_{\mathbb{Z}G} M$ are also chain homotopy-equivalent and have the same homology.

See also [Whitehead, Ch. VI, §3] for more details. □

5.1.28. Exercise. Let

$$1 \to N \xrightarrow{i} G \xrightarrow{q} G/N \to 1$$

be a short exact sequence of groups (so that N is a normal subgroup of G and i is the inclusion map). Show that there is a corresponding fibration of classifying spaces $q_* : BG \to B(G/N)$ with fiber BN, and that the long exact homotopy sequence of this fibration recovers the original exact sequence in dimension 1 and is trivial in all other dimensions.

5.1.29. Exercise. Let X be any path-connected space which is nice enough for covering-space theory to apply, for instance a CW-complex. Show from the Hurewicz Theorem (Theorem 5.1.14) that if $\tilde{X}$ is the universal cover of X, $\pi_2(X) \cong H_2(\tilde{X}; \mathbb{Z})$. Also deduce that if $\tilde{H}_j(\tilde{X}; \mathbb{Z})$ vanishes for $j \le n$, $n \ge 2$, then $\pi_j(X)$ vanishes for $2 \le j \le n$. Conclude that if X is an n-dimensional connected CW-complex and if $\tilde{H}_j(\tilde{X}; \mathbb{Z})$ vanishes for $j \le n$, then X is a classifying space for $\pi_1(X)$.

5.1.30. Exercise. Let G be the group with presentation

$$\langle a, b \mid aba^{-1} = b^2 \rangle.$$

Construct a connected 2-dimensional CW-complex X with fundamental group G having one 0-cell, two 1-cells, and one 2-cell. (Attach a 2-cell to $S^1 \vee S^1$ to kill the element $aba^{-1}b^{-2}$ in $\pi_1(S^1 \vee S^1)$ = free group on generators a, b.) Apply the criterion in Exercise 5.1.29 to show that X is a model for BG. (You only need to show that $\tilde{X}$ has no cellular 2-cycles.) Incidentally, the commutator subgroup of G is isomorphic to $\mathbb{Z}[\frac{1}{2}]$, so this example shows that the commutator subgroup of the fundamental group of a finite CW-complex need not be finitely generated.

5.1.31. Exercise. This exercise will show that sometimes one can get information on the Hurewicz map past the dimension in which it must be an isomorphism by the Hurewicz Theorem (Theorem 5.1.14).

(1) Let X be any CW-complex with skeleta X^j. Show that the relative Hurewicz map $\pi_{j+1}(X^{j+1}, X^j) \to H_{j+1}(X^{j+1}, X^j; \mathbb{Z})$ is split surjective for all $j \geq 1$, with a natural splitting. (Hint:

$$H_{j+1}(X^{j+1}, X^j; \mathbb{Z})$$

is the free abelian group on the $(j+1)$-cells of X.)

(2) Let X be an n-connected CW-complex, with $n \geq 1$. Show that the Hurewicz map $\pi_{n+2}(X) \to H_{n+2}(X; \mathbb{Z})$ is surjective. Here is an outline of how to proceed. As in the proof of Theorem 5.1.14, one can assume X^{n+1} is a wedge of S^{n+1}'s. First prove the result for X^{n+2}, by looking at the commutative diagram

$$\begin{array}{ccccccc} \cdots \to & \pi_{n+2}(X^{n+2}) & \to & \pi_{n+2}(X^{n+2}, X^{n+1}) & \to & \pi_{n+1}(X^{n+1}) \\ & \downarrow & & \downarrow & & \downarrow \cong \\ 0 \to & H_{n+2}(X^{n+2}; \mathbb{Z}) & \to & H_{n+2}(X^{n+2}, X^{n+1}; \mathbb{Z}) & \to & H_{n+1}(X^{n+1}; \mathbb{Z}) \end{array}$$

and using (1).

(3) Now deduce the result for X from the result for X^{n+2}.

5.1.32. Exercise (A universal property of classifying spaces). Let G be a group and let $BG = EG/G$ be a classifying space for G, with a fixed basepoint $*$. Let X be a paracompact path-connected space with basepoint x_0 which is nice enough for covering space theory to apply, for instance a CW-complex. There is obviously a map from $[X, BG]$, the set of homotopy classes (rel x_0) of basepoint-preserving maps $f : X \to BG$, to $\mathrm{Hom}(\pi_1(X, x_0), G)$, defined by sending any map f to the induced map on fundamental groups. (This induced map depends only on the homotopy class of f.) By covering space theory, the latter can be identified with the set $\mathrm{Cov}_G(X)$ of normal coverings of X with covering group G. Show that in this way one obtains isomorphisms

$$[X, BG] \leftrightarrow \mathrm{Hom}(\pi_1(X, x_0), G) \leftrightarrow \mathrm{Cov}_G(X).$$

To prove surjectivity, suppose one is given a homomorphism $\pi_1(X, x_0) \to G$ or equivalently a covering space $\tilde{X} \to X$ with G as the group of covering transformations. Form the fiber product $X' = \tilde{X} \times_G EG$. Using Theorem 5.1.4, show that X' is homotopy-equivalent to X. Projection onto the second factor gives a map $X' \to BG$, and composition with the homotopy equivalence $X \to X'$ gives the desired map $X \to BG$. To prove injectivity, suppose $f_0, f_1 : X \to BG$, both sending x_0 to $*$, induce isomorphic coverings of X. When X is a CW-complex, one can construct a homotopy f between them by viewing f_0 and f_1 as defining a map

$$(X \times \{0\}) \cup_{\{x_0\}\times\{0\}} (\{x_0\} \times [0, 1]) \cup_{\{x_0\}\times\{1\}} (X \times \{1\}) \to BG$$

and extending over the rest of $X \times [0, 1]$ one cell at a time using obstruction theory. Alternatively, one can do this with more general spaces X using the section extension property.

2. Quillen's +-construction and its basic properties

In this section, we will use the topological machinery developed in the last section to construct functors $R \rightsquigarrow \mathbf{K}(R)$, $(R, I) \rightsquigarrow \mathbf{K}(R, I)$ into the homotopy category of CW-complexes with basepoint (in which the morphisms are homotopy classes of based continuous maps). Then we will define $K_i(R)$ to be $\pi_i(\mathbf{K}(R))$, $K_i(R, I)$ to be $\pi_i(\mathbf{K}(R, I))$. By convention, we can take $\mathbf{K}(R, R) = \mathbf{K}(R)$, $\mathbf{K}(R, 0) = *$, a space reduced to its basepoint. We want to do this in such a way as to guarantee the following basic properties:

5.2.1. Requirements.

(1) *$K_i(R)$ defined this way agrees with our previous definitions for $0 \le i \le 2$.*

(2) *$K_i(R, I)$ defined this way agrees with our previous definitions for $i = 0, 1$.*

(3) *The maps of K-groups induced by a homomorphism agree with our previous ones (in the cases where (1) and (2) apply).*

(4) *The inclusion $i : I \hookrightarrow R$ and the quotient map $q : R \twoheadrightarrow R/I$ give rise to a fibration*

$$\mathbf{K}(R, I) \xrightarrow{i_*} \mathbf{K}(R) \xrightarrow{q_*} \mathbf{K}(R/I),$$

where this notation means that q_* is a fibration with fiber $\mathbf{K}(R, I)$.

The effect of these requirements will be that we will obtain higher K-groups $K_i(R)$ and $K_i(R, I)$ for all i, with an exact sequence extending that of Theorem 4.3.1.

The basic tool for doing this is the following construction of Quillen.

5.2.2. Theorem (Quillen). *Let X be a connected CW-complex with basepoint x_0, say, chosen from the 0-skeleton, and let π be a perfect normal subgroup of $\pi_1 = \pi_1(X, x_0)$ (thus $\pi = [\pi, \pi] = [\pi_1, \pi]$). Then one may obtain a new CW-complex X^+ by attaching only 2-cells and 3-cells to X, so that the pair (X^+, X) satisfies the following conditions:*

(1) *The map $\pi_1(X, x_0) \to \pi_1(X^+, x_0)$ induced by the inclusion is just the quotient map $\pi_1 \to \pi_1/\pi$.*

(2) *The pair (X^+, X) is homologically acyclic; that is, for any π_1/π-module M (viewed as a local coefficient system on X and on X^+), $H_\bullet(X^+, X; M) = 0$.*

Note that (2) implies:

(2′) *For any covering space $\tilde{X}^+$ of X^+, if $\tilde{X}$ is the corresponding covering space of X, then*

$$H_\bullet(\tilde{X}^+, \tilde{X}; \mathbb{Z}) = 0.$$

Furthermore, X^+ is unique in the following sense: given any other CW-complex X_1^+ containing X as a subcomplex and satisfying these same conditions, even if (X_1^+, X) is allowed to contain relative cells of arbitrary dimensions, there is a homotopy equivalence $X^+ \to X_1^+$ which is homotopic to the identity on X.

5.2.3. Remark. In the situation of Theorem 5.2.2, if $\tilde{X}$ is the covering space of X with fundamental group π and covering group π_1/π, then $\pi_2(X^+) \cong H_2(\tilde{X}; \mathbb{Z})$. Indeed, $\tilde{X}$ is a subcomplex of $\tilde{X}^+$, the universal cover of X^+. By the corollary of the Hurewicz Theorem in Exercise 5.1.29, $\pi_2(X^+) \cong \pi_2(\tilde{X}^+) \cong H_2(\tilde{X}^+; \mathbb{Z})$. However, the inclusion $\tilde{X} \hookrightarrow \tilde{X}^+$ induces an isomorphism on homology by property (2′) of the +-construction.

Note also that it is not quite essential for X in this Theorem to be a CW-complex, since in any event (X^+, X) will be a relative CW-complex and we can use the relative form of Whitehead's Theorem. However, in all cases where we'll use the +-construction, X will at least have the homotopy type of a CW-complex.

Proof of Theorem 5.2.2. Choose generators for π. Then each generator defines a homotopy class of a map $g_i : (S^1, *) \to (X, x_0)$ which is trivial on homology, since the Hurewicz map in dimension 1 kills all commutators and $\pi = [\pi, \pi]$. Let Y be the CW-complex obtained from X by attaching one 2-cell e_i^2 for each $i \in \Lambda$, using the attaching map $g_i : \partial B^2 = S^1 \to X$. Clearly the inclusion $X \hookrightarrow Y$ has property (1); that is, the induced map on fundamental groups is the quotient map $\pi_1 \to \pi_1/\pi$. Let $\tilde{X} \hookrightarrow \tilde{Y}$ be the covering spaces with covering group π_1/π, so that $\tilde{Y}$ is the universal covering of Y and $\tilde{X}$ has fundamental group π. Thus $H_1(\tilde{X}; \mathbb{Z}) \cong \pi_{\text{ab}} = 0$. Since Y is obtained from X and $\tilde{Y}$ is obtained from $\tilde{X}$ by attaching 2-cells, $H_\bullet(Y, X; \mathbb{Z})$ is concentrated in degree 2, where it is the free abelian group on the $[e_i^2]$, and similarly $H_\bullet(\tilde{Y}, \tilde{X}; \mathbb{Z})$ is concentrated in degree 2, where it is the free $\mathbb{Z}(\pi_1/\pi)$-module on the $[e_i^2]$. Since the connecting map

$$\partial : H_2(\tilde{Y}, \tilde{X}; \mathbb{Z}) \to H_1(\tilde{X}; \mathbb{Z}) = 0$$

is trivial, $H_\bullet(\tilde{Y};\mathbb{Z})$ differs from $H_\bullet(\tilde{X};\mathbb{Z})$ by taking the direct sum with $\bigoplus_{i\in\Lambda}\mathbb{Z}(\pi_1/\pi)[e_i^2]$ in degree 2. Furthermore, since $\tilde{Y}$ is simply connected, by Theorem 5.1.14, the generator $[e_i^2]$ is in the image of the Hurewicz map for $\tilde{Y}$ and thus for Y (by pushing back down). Choose $h_i : (S^2, *) \to (Y, x_0)$ mapping under the Hurewicz map to $[e_i^2] \in H_2(Y)$. Now construct X^+ from Y by attaching one 3-cell e_i^3 for each $i \in \Lambda$, using the attaching map $h_i : \partial B^3 = S^2 \to Y$. We claim it has the desired properties. (1) is clear, since it was already true for Y and attaching 3-cells has no effect on the fundamental group. To check (2), choose any π_1/π-module M (viewed as a local coefficient system on X and on X^+). Then $H_\bullet(Y; M)$ differs from $H_\bullet(X; M)$ by taking the direct sum with $\bigoplus_{i\in\Lambda} M[e_i^2]$ in degree 2. Since $H_\bullet(X^+, Y; M)$ is concentrated in degree 3, where it is given by $\bigoplus_{i\in\Lambda} M[e_i^3]$, and since by construction $\partial[e_i^3] = [e_i^2]$, $H_\bullet(X^+, X; M)$ vanishes as desired.

It remains to prove the uniqueness statement. Suppose the CW-pair (X_1^+, X) also satisfies (1) and (2) of the statement of the Theorem. Let $i_1 : X \to X_1^+$ be the inclusion. We will extend i_1 over the 2-cells and 3-cells added to X to construct X^+, and show that we can do this to get a homotopy equivalence $h : X^+ \to X_1^+$ which is the identity on X. First we need to extend i_1 to a map $g : Y \to X_1^+$. Since Y was obtained from X by attaching 2-cells using the attaching maps $g_i : S^1 \to X$, i_1 can be extended to a map $g : Y \to X_1^+$ provided each $i_1 \circ g_i$ is null-homotopic. Since $[g_i] \in \pi$ and $(i_1)_*$ kills π, this condition is satisfied and we can choose an extension $g : Y \to X_1^+$ of i_1. Note that g induces an isomorphism on fundamental groups and so there is an induced map $\tilde{g} : \tilde{Y} \to \tilde{X}_1^+$ of universal covers. Then since X^+ was obtained from X by attaching 3-cells using the attaching maps $h_i : S^2 \to Y$, g can be extended to a map $h : X^+ \to X_1^+$ provided each $g \circ h_i$ is null-homotopic. Let us apply Remark 5.2.3 to X_1^+; it says that

$$\pi_2(X_1^+) \cong \pi_2(\tilde{X}_1^+) \cong H_2(\tilde{X}_1^+;\mathbb{Z}) \cong H_2(\tilde{X};\mathbb{Z}).$$

Thus $g \circ h_i$ will be null-homotopic provided it corresponds to the trivial homology class in $H_2(\tilde{X};\mathbb{Z})$. But the image of h_i in homology is $[e_i^2]$ by construction, which is $\partial[e_i^3]$ and thus a boundary, so $g\circ h_i$ is null-homotopic and we can extend g to a map $h : X^+ \to X_1^+$. Let $\tilde{h}$ be the lifted map on universal covers (extending $\tilde{g}$). To show h is a homotopy equivalence, it suffices by Theorem 5.1.13 to show that h induces isomorphisms on homotopy groups. Since we already know h is an isomorphism on fundamental groups, it is enough to show that $\tilde{h}$ induces isomorphisms on π_j, $j \geq 2$, or is a homotopy equivalence. Now in our construction we can assume g and h are cellular maps, so the mapping cylinder of $\tilde{h}$ is a CW-complex Z. To show $\tilde{h}$ is a homotopy equivalence, we need to show that the CW-pair $(Z, \tilde{X}^+)$ is relatively j-connected for all j. By the relative Hurewicz Theorem (Theorem 5.1.14), since everything is simply connected, it's enough to show $H_j(Z, \tilde{X}^+;\mathbb{Z}) = 0$ for all j, which in turn is equivalent to saying that $\tilde{h}$ is a **homology** equivalence. But this is guaranteed by the fact that

h is the identity on X together with the acyclicity of the pairs $(\tilde{X}^+, \tilde{X})$, $(\tilde{X}_1^+, \tilde{X})$. □

For future purposes we need to know a bit more about the +-construction, in particular the following.

5.2.4. Proposition. *The +-construction of Theorem 5.2.2 is functorial. In other words, given a map of connected CW-complexes $f : X \to Y$, a perfect normal subgroup π_X of $\pi_1(X)$, and a perfect normal subgroup π_Y of $\pi_1(Y)$ with $\pi_Y \supseteq f_*(\pi_X)$, f induces a map $f_* : X^+ \to Y^+$, where X^+ is constructed as in Theorem 5.2.2 using π_X, and Y^+ is similarly constructed from Y using π_Y. Furthermore, f_* is uniquely determined up to homotopy, $f_* = id$ if $X = Y$ and $\pi_X = \pi_Y$, and $(g \circ f)_* = g_* \circ f_*$ when this makes sense.*

Proof. The proof of this is similar to the proof of uniqueness in Theorem 5.2.2. Namely, we need to show that $i_Y \circ f$ extends over the 2-cells and 3-cells added to X to form X^+. Here i_Y is the inclusion of Y into Y^+. First we note that $i_Y \circ f$ induces on fundamental groups the composite $\pi_1(X) \xrightarrow{f_*} \pi_1(Y) \to \pi_1(Y)/\pi_Y$. Since $\pi_Y \supseteq f_*(\pi_X)$, π_X lies in the kernel of this map, which means exactly that for each 2-cell e_i^2 added to X with attaching map g_i, the composite $i_Y \circ f \circ g_i$ is null-homotopic. Thus we may extend over the 2-cells of X^+ to get a map

$$X \bigcup_{g_i : i \in \Lambda} e_i^2 \xrightarrow{f'} Y^+$$

extending $i_Y \circ f$. Next we need to extend f' over the 3-cells of X^+. As in the proof of Theorem 5.2.2, the condition for being able to do this is for each $f' \circ h_i$ to be null-homotopic, where h_i is the attaching map of e_i^3. And as before, we use the fact that $\pi_2(Y^+) \cong H_2(\tilde{Y}; \mathbb{Z})$, where $\tilde{Y}$ is the cover of Y with fundamental group π_Y and covering group $\pi_1(Y)/\pi_Y$. Since each h_i bounds in homology, so does $f' \circ h_i$, and thus we can extend over the 3-cells to get our desired map $f_* : X^+ \to Y^+$.

Since there seems to be some arbitrariness in this construction, we still need to show that f_* is well defined up to homotopy, which will imply that f_* is the identity when $X = Y$ and $\pi_X = \pi_Y$. Suppose one has two maps f_*, $f_*^1 : X^+ \to Y^+$ extending f. To construct a homotopy between them, let $p_1 : X \times [0, 1] \to X$ be projection on the first factor view f_*, f_*^1 and $f \circ p_1$ as defining a map

$$(X^+ \times \{0\}) \cup_{X \times \{0\}} (X \times [0, 1]) \cup_{X \times \{1\}} (X^+ \times \{1\}) \to Y^+.$$

If we can extend this to a map $X^+ \times [0, 1] \to Y^+$, we will have the desired homotopy from f_* to f_*^1. But the proof that such an extension exists is the same as before; we merely need to extend over the cells $e_i^2 \times (0, 1)$ and $e_i^3 \times (0, 1)$, and the homotopy-theoretic obstruction to be able to do this is the same as before. □

5.2.5. Example. Suppose X is the Poincaré homology 3-sphere of Exercise 4.1.27. Recall that this is the quotient of S^3 by a certain finite perfect group π of homeomorphisms. (In fact, π is a central extension of A_5 by $\mathbb{Z}/2$.) Applying the +-construction of Theorem 5.2.2 to the pair (X, π), we obtain a 3-dimensional complex X^+ which is simply connected (since π was all of $\pi_1(X)$) and which has the same homology as X, thus the same homology as S^3. Since X^+ is simply connected and $\tilde{H}_j(X^+; \mathbb{Z}) = 0$ for $j < 3$, the Hurewicz Theorem (Theorem 5.1.14) implies that X^+ is 2-connected and that the Hurewicz map $\pi_3(X^+) \to H_3(X^+; \mathbb{Z}) = \mathbb{Z}$ is an isomorphism. Choose a map $g : S^3 \to X^+$ which corresponds to a generator of $\pi_3(X^+)$. Then by definition of the Hurewicz map, g induces an isomorphism on homology. Since S^3 and X^+ are simply connected, this implies (as in the proof of Theorem 5.2.2) that g also induces an isomorphism on homotopy groups, and so g is a homotopy equivalence by Theorem 5.1.13. Thus from the homotopy point of view, X^+, as characterized by properties (1) and (2) of Theorem 5.2.2, "is" S^3. Of course, the particular model for X^+ constructed in the proof of Theorem 5.2.2 need not be a manifold, and is not necessarily homeomorphic to S^3. (Even if it were a manifold, it is not known if every 3-manifold homotopy-equivalent to S^3 is homeomorphic to S^3. This is the famous Poincaré Conjecture.)

Exactly the same proof in higher dimensions shows that if X is a **homology n-sphere**, that is, a path-connected space (say with the homotopy type of a CW-complex) with the same homology groups as S^n ($n \geq 3$), then X^+ is homotopy-equivalent to S^n. Here to start the construction, we observe that since $H_1(X, \mathbb{Z}) = 0$, $\pi = \pi_1(X)$ can have no abelian quotients, and so is perfect, so that we can apply the +-construction to the pair (X, π).

Now we are ready to use the +-construction to define Quillen's higher K-theory.

5.2.6. Definition. Let R be any ring with unit. Define $\mathbf{K}(R)$ to be the product $BGL(R)^+ \times K_0(R)$, where $BGL(R)$ is defined as in Definition 5.1.16, the +-construction on $BGL(R)$ is taken relative to the perfect subgroup $E(R)$ of $GL(R)$, and $K_0(R)$ is given the discrete topology. If $\varphi : R \to S$ is a homomorphism of rings, there is an induced homomorphism of groups $\varphi_* : GL(R) \to GL(S)$ and thus (by Proposition 5.1.18) an induced map $\varphi_* : BGL(R) \to BGL(S)$ and thus (by Proposition 5.2.4) an induced map $\varphi_* : BGL(R)^+ \to BGL(S)^+$. We define the induced map $\varphi_* : \mathbf{K}(R) \to \mathbf{K}(S)$ to be the product of the map $\varphi_* : BGL(R)^+ \to BGL(S)^+$ with the map $\varphi_* : K_0(R) \to K_0(S)$. We define the K-groups of R to be $K_i(R) = \pi_i(\mathbf{K}(R))$, where all homotopy groups are computed relative to a basepoint in $BGL(R)^+ \times \{0\}$. (If we use the construction for $BGL(R)$ given in the proof of Theorem 5.1.15, then there is a natural choice for a basepoint of $BGL(R)$, namely the image of $1_{GL(R)} \in GL(R) = X_0$. This then gives a natural choice for a basepoint in $BGL(R)^+ \times \{0\}$.) Worries about choices of basepoint will be lessened by Theorem 5.2.12 below. Note that since $BGL(R)^+$ is path-

connected, $\pi_0(\mathbf{K}(R)) = K_0(R)$ by construction. Furthermore, $\pi_1(\mathbf{K}(R)) = \pi_1(BGL(R)^+) = \pi_1(BGL(R))/E(R) = GL(R)/E(R) = K_1(R)$, so this definition of K-groups is at least consistent with our previous definition of K_0 and K_1. In fact, the basic features of this situation are captured in the following Theorem.

5.2.7. Theorem. *Let G be a group with perfect commutator subgroup π, and let $B\pi^+$ and BG^+ be constructed as in Theorem 5.2.2 relative to the perfect subgroup π of $\pi = \pi_1(B\pi)$ and of $G = \pi_1(BG)$. Then $B\pi^+$ is a normal covering space of BG^+ with covering group $G/\pi = G_{ab}$.*

Furthermore, let $\hat{\pi}$ be the universal central extension of π. Then there are natural isomorphisms

$$\pi_1(BG^+) \cong G/\pi = G_{ab}, \quad \pi_2(BG^+) \cong H_2(\pi, \mathbb{Z}), \quad \pi_3(BG^+) \cong H_3(\hat{\pi}, \mathbb{Z}).$$

Also, $\pi_j(BG^+) \cong \pi_j(B\hat{\pi}^+)$ for $j \geq 3$.

Proof. The isomorphism $\pi_1(BG) \cong G/\pi = G_{ab}$ comes immediately from property (1) in the statement of Theorem 5.2.2. Note also that $B\pi$ is a normal covering of BG with covering group $G/\pi = G_{ab}$. (This is a restatement of Exercise 5.1.28; alternatively, $\pi \subseteq G$ acts freely and properly discontinuously on EG, so EG/π is a model for $B\pi$, and then $EG/\pi \to BG = EG/G$ is clearly a normal covering of BG with covering group G/π.) When we attach 2-cells and 3-cells to BG to construct BG^+, lifting to the covering space $B\pi$ results in adding 2-cells and 3-cells to $B\pi$ to kill the fundamental group and preserve the homology (because of property (2′) in the statement of Theorem 5.2.2). Thus the construction of BG^+ automatically induces a construction of $B\pi^+$ and hence $B\pi^+$ is a normal covering space of BG^+ with covering group $G/\pi = G_{ab}$. In particular, $\pi_j(B\pi^+) \cong \pi_j(BG)$ for $j \geq 2$, by Corollary 5.1.25.

Now $B\pi^+$ is simply connected, so by the Hurewicz Theorem, $\pi_2(B\pi^+) \cong H_2(B\pi^+; \mathbb{Z}) \cong H_2(B\pi; \mathbb{Z})$ (using property (2) of the +-construction). By Theorem 5.1.27, this may be identified with $H_2(\pi, \mathbb{Z})$, and so $\pi_2(BG^+) \cong H_2(\pi, \mathbb{Z})$.

Finally, consider the universal central extension

$$1 \to H_2(\pi, \mathbb{Z}) \to \hat{\pi} \xrightarrow{q} \pi \to 1,$$

where we have used Theorems 4.1.3 and 4.1.19. This gives rise by Exercise 5.1.28 to a fibration

$$BH_2(\pi, \mathbb{Z}) \to B\hat{\pi} \xrightarrow{Bq} B\pi.$$

The group $\hat{\pi}$ is also perfect by Theorem 4.1.3, so we can perform the +-construction to $B\hat{\pi}$ as well as to $B\pi$. By Proposition 5.2.4, there is a map $B\hat{\pi}^+ \xrightarrow{q_*} B\pi^+$ which extends the map $B\hat{\pi} \xrightarrow{Bq} B\pi$, and it is unique up to homotopy. Just as in the case of $B\pi^+$, $B\hat{\pi}^+$ is simply connected. However, since $H_2(B\hat{\pi}^+; \mathbb{Z}) \cong H_2(\hat{\pi}, \mathbb{Z}) = 0$ by Corollary

4.1.18, the Hurewicz Theorem imples that $B\hat{\pi}^+$ is 2-connected and that $\pi_3(B\hat{\pi}^+) \cong H_3(B\hat{\pi}^+; \mathbb{Z}) \cong H_3(\hat{\pi}, \mathbb{Z})$. To finish the proof, we need only show that $B\hat{\pi}^+ \xrightarrow{q_*} B\pi^+$ induces an isomorphism on π_3. Since our spaces and maps are only defined up to homotopy anyway, for this purpose we may first convert $q_* : B\hat{\pi}^+ \longrightarrow B\pi^+$ into a fibration by using Proposition 5.1.21. Let F be the (homotopy) fiber of q_*; it fits into a long exact sequence

$$\cdots \longrightarrow \pi_j(F) \longrightarrow \pi_j(B\hat{\pi}^+) \xrightarrow{q_*} \pi_j(B\pi^+) \longrightarrow \cdots .$$

From this we see that F is path-connected and that

$$H_2(\pi, \mathbb{Z}) \cong \pi_2(B\pi^+) \xrightarrow{\partial} \pi_1(F)$$

is an isomorphism. Consider the commutative diagram of fibrations

$$\begin{array}{ccccc} BH_2(\pi, \mathbb{Z}) & \longrightarrow & B\hat{\pi} & \xrightarrow{Bq} & B\pi \\ & & \downarrow & & \downarrow \\ F & \longrightarrow & B\hat{\pi}^+ & \xrightarrow{q_*} & B\pi^+, \end{array}$$

where the two vertical maps are homology isomorphisms killing the perfect fundamental groups $\hat{\pi}$ and π, by the basic properties of the +-construction. The composite

$$BH_2(\pi, \mathbb{Z}) \rightarrow B\hat{\pi} \xrightarrow{Bq} B\pi$$

is null-homotopic, hence the map of $BH_2(\pi, \mathbb{Z})$ into $B\hat{\pi}^+$ becomes null-homotopic after mapping into $B\pi^+$, and by the homotopy lifting property we obtain a map $BH_2(\pi, \mathbb{Z}) \rightarrow F$ compatible with the other maps of the diagram. It is also easy to see that this map is an isomorphism on π_1. On the other hand, by Exercise 5.1.32 (the universal property of classifying spaces), there is a map $F \rightarrow BH_2(\pi, \mathbb{Z})$ in the other direction inducing the inverse isomorphism on π_1, and the composite $BH_2(\pi, \mathbb{Z}) \rightarrow F \rightarrow BH_2(\pi, \mathbb{Z})$ is an isomorphism on π_1, hence on all homotopy groups by Corollary 5.1.25, hence is a homotopy equivalence by Theorem 5.1.13. So the homotopy and homology groups of F contain those of $BH_2(\pi, \mathbb{Z})$ as direct summands, and the other summands vanish at least in dimension 1 by our π_1 calculation. We claim the other summands are trivial in all dimensions, or basically that F can be taken to be $BH_2(\pi, \mathbb{Z})$. If this is so, then from the the long exact homotopy sequence of the fibration, we will have $\pi_j(B\hat{\pi}^+) \cong \pi_j(B\pi^+) \cong \pi_j(BG^+)$ for $j \geq 3$, and thus in particular $\pi_3(B\pi^+) \cong \pi_3(B\hat{\pi}^+) \cong H_3(\hat{\pi}, \mathbb{Z})$. (For this last fact, it is only necessary to show that $\pi_j(F)$ vanishes for $j = 2, 3$.)

Let $E = B\hat{\pi}^+$, the total space of the fibration q_*, let $B = B\pi^+$ be the base of the fibration, and let $B_0 = B\pi$, $E_0 = q_*^{-1}(B_0)$. Recall that the CW-pair (B, B_0) has (relative) cells only in dimensions 2 and 3 and is acyclic. We claim this implies the pair (E, E_0) is acyclic. Indeed, if B_1 is the space made from B by attaching only the 2-cells e_i^2 (*i.e.*, the

2-skeleton of the pair) and $E_1 = q_*^{-1}(B_1)$, then since $B_1 \smallsetminus B$ is a disjoint union of open 2-cells which are contractible, the fibration $E_1 \to B_1$ must be homotopically the same as the projection $(B_1 \smallsetminus B) \times F \to B_1 \smallsetminus B$ over this set. So using excision and the Künneth formula,

$$H_\bullet(E_1,\, E_0;\, \mathbb{Z}) \cong H_\bullet(B_1 \times F,\, B_0 \times F;\, \mathbb{Z}) \cong \bigoplus_i \mathbb{Z}[e_i^2] \otimes_{\mathbb{Z}} H_\bullet(F;\, \mathbb{Z}).$$

Similarly,

$$H_\bullet(E,\, E_1;\, \mathbb{Z}) \cong H_\bullet(B \times F,\, B_1 \times F;\, \mathbb{Z}) \cong \bigoplus_i \mathbb{Z}[e_i^3] \otimes_{\mathbb{Z}} H_\bullet(F;\, \mathbb{Z}).$$

The exact homology sequence of the triple $(E,\, E_1,\, E_0)$ now reduces to

$$\cdots \to H_{k+3}(E,\, E_0;\, \mathbb{Z}) \to \bigoplus_i \mathbb{Z}[e_i^3] \otimes_{\mathbb{Z}} H_k(F;\, \mathbb{Z}) \xrightarrow{\partial} \bigoplus_i \mathbb{Z}[e_i^2] \otimes_{\mathbb{Z}} H_k(F;\, \mathbb{Z}) \to H_{k+2}(E,\, E_0;\, \mathbb{Z}) \to \cdots .$$

The boundary map ∂ comes from gluing two product fibrations where the 3-cells are attached to B_1, and since $\partial[e_i^3] = [e_i^2]$, it is easy to see that this map is an isomorphism. Hence $H_\bullet(E,\, E_0;\, \mathbb{Z})$ is trivial. But since we may assume $B\hat{\pi} \subseteq E_0 \subseteq E = B\hat{\pi}^+$ and the pair $(B\hat{\pi}^+,\, B\hat{\pi})$ is acyclic by the basic property of the +-construction, the exact sequence of the triple $(E,\, E_0,\, B\hat{\pi})$ imples that $(E_0,\, B\hat{\pi})$ is also acyclic.

We're now reduced to the following commutative diagram of fibrations

$$\begin{array}{ccccc} BH_2(\pi,\, \mathbb{Z}) & \longrightarrow & B\hat{\pi} & \xrightarrow{Bq} & B\pi \\ \downarrow & & \downarrow & & \| \\ F & \longrightarrow & E_0 & \xrightarrow{q_*} & B\pi. \end{array}$$

This gives the commutative diagram of fundamental groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_2(\pi,\, \mathbb{Z}) & \longrightarrow & \hat{\pi} & \xrightarrow{q} & \pi & \longrightarrow & 1 \\ \| & & \cong\downarrow & & \downarrow & & \| & & \| \\ 1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(E_0) & \longrightarrow & \pi & \longrightarrow & 1, \end{array}$$

from which we see that the map $B\hat{\pi} \to E_0$ induces an isomorphism on fundamental groups. By Exercise 5.1.32 (the universal property of classifying spaces) again, there is a map $E_0 \to B\hat{\pi}$ in the other direction inducing the inverse isomorphism on π_1, and as before, by Theorem 5.1.13 the composite $B\hat{\pi} \to E_0 \to B\hat{\pi}$ is a homotopy equivalence. So the homotopy and homology groups of E_0 contain those of $B\hat{\pi}$ as direct summands.

To prove that $\pi_j(F)$ vanishes for $j \geq 2$ we proceed by contradiction. Let $\tilde{F}$ and $\tilde{E}_0$ be the universal covers of F and of E_0. By the homotopy

sequence of the fibration, the maps $\pi_j(\tilde{F}) \to \pi_j(\tilde{E}_0)$ are isomorphisms. By the Hurewicz Theorem, if some $\pi_j(\tilde{F})$ is non-zero and we choose the smallest j for which this happens, then $H_k(\tilde{E}_0; \mathbb{Z}) = 0$ for $k < j$ but $H_j(\tilde{E}_0; \mathbb{Z}) \neq 0$. It's not hard to see from the fact that $B\hat{\pi}$ is homotopically a retract of E_0 that then the map $B\hat{\pi} \to E_0$ is homology isomorphism in degrees $k < j$ but not in degree k. This contradicts acyclicity of $(E_0, B\hat{\pi})$, so $\pi_j(F)$ must vanish for $j \geq 2$, so that $\pi_j(B\pi^+) \cong \pi_j(B\pi^+) \cong \pi_j(BG^+)$ for $j \geq 3$. □

5.2.8. Corollary (Quillen, Gersten [Gersten1]**).** *The Quillen K-groups $K_i(R)$ of Definition 5.2.6 satisfy properties (1) and (3) of 5.2.1. Furthermore, there is a natural isomorphism $K_3(R) \cong H_3(\mathrm{St}(R), \mathbb{Z})$, and $K_i(R) \cong \pi_i(BE(R)^+)$ for $i \geq 2$, $K_i(R) \cong \pi_i(B\,\mathrm{St}(R)^+)$ for $i \geq 3$.*

Proof. Let $G = GL(R)$, $\pi = E(R)$, and $\hat{\pi} = \mathrm{St}(R)$. These groups satisfy the conditions of Theorem 5.2.7, so we see that there are natural isomorphisms $K_2(R) \cong H_2(E(R), \mathbb{Z})$, $K_3(R) \cong H_3(\mathrm{St}(R), \mathbb{Z})$, $K_i(R) \cong \pi_i(BE(R)^+)$ for $i \geq 2$, $K_i(R) \cong \pi_i(B\,\mathrm{St}(R)^+)$ for $i \geq 3$. □

5.2.9. Remark. The hard part of the proof of Theorem 5.2.7, which unfortunately is needed to prove that $K_3(R) \cong H_3(\mathrm{St}(R), \mathbb{Z})$, was to show that there is a fibration

$$BH_2(\pi, \mathbb{Z}) \to B\hat{\pi}^+ \xrightarrow{q_*} B\pi^+.$$

If we had known this from the beginning, the theorem would be fairly easy. There is a faster way to prove this, if one is willing to use more obstruction theory (as developed in [Spanier, Ch. 8] or [Whitehead, Ch. V, VI, and IX]). Namely, it is a fact from obstruction theory that for any abelian group A, fibrations

$$BA \to E \to X$$

over any reasonable path-connected space X, with the extra condition of "simplicity," that $\pi_1(X)$ should act trivially on $\pi_1(BA) = A$, are classified by $H^2(X; A)$. (When $X = B\pi$ and $E = B\hat{\pi}$, such fibrations correspond to group extensions

$$1 \to A \to \hat{\pi} \to \pi \to 1,$$

and the simplicity condition amounts to assuming that the extension is central. As we saw in Theorem 4.1.16, central extensions of π by A are classified by $H^2(\pi, A)$, and if $X = B\pi$, $H^2(X; A) \cong H^2(\pi, A)$ by Theorem 5.1.27.) Since the inclusion $B\pi \to B\pi^+$ is a homology isomorphism, the class of the fibration

$$BH_2(\pi, \mathbb{Z}) \to B\hat{\pi} \xrightarrow{Bq} B\pi$$

in $H^2(B\pi; H_2(\pi, \mathbb{Z})) \cong H^2(\pi, H_2(\pi, \mathbb{Z}))$ has a unique extension to a class in $H^2(B\pi^+; H_2(\pi, \mathbb{Z}))$, which corresponds to a fibration

$$BH_2(\pi, \mathbb{Z}) \longrightarrow E \xrightarrow{p} B\pi^+ \tag{5.2.10}$$

with $E \supseteq B\hat{\pi} = p^{-1}(B\pi)$. As in the proof above, the fact that $(B\pi^+, B\pi)$ is acyclic implies that $(E, B\hat{\pi})$ is acyclic. But the fact that the characteristic class of (5.2.10) in $H^2(B\pi^+; H_2(\pi, \mathbb{Z}))$ corresponds to the identity map on $H_2(\pi, \mathbb{Z})$ (cf. Theorem 4.1.19) implies that in the long exact homotopy sequence

$$H_2(\pi, \mathbb{Z}) \cong \pi_2(B\pi^+) \xrightarrow{\partial} \pi_1(BH_2(\pi, \mathbb{Z})) = H_2(\pi, \mathbb{Z}) \\ \longrightarrow \pi_1(E) \longrightarrow \pi_1(B\pi^+) = 0,$$

the connecting map ∂ is an isomorphism and thus $\pi_1(E) = 0$. This together with the fact that $(E, B\hat{\pi})$ is acyclic implies by the uniqueness part of Theorem 5.2.2 that E is homotopy-equivalent to $B\hat{\pi}^+$, which is what we needed to prove. □

One would like to know that the choice of basepoint in $BGL(R)^+$ is inconsequential, and that one can move the basepoint around at will, which means that one would like to show that $BGL(R)^+$ is a simple space (in other words, that its fundamental group $K_1(R)$ acts trivially on the higher homotopy groups, which are the higher K-groups). We will take care of all of these things at the same time.

5.2.11 Definition. Let X be a space with basepoint x_0. Then X is called an **H-space** if there is a "multiplication" map $\mu : X \times X \to X$ with $\mu(x_0, x_0) = x_0$ for which x_0 acts as an identity up to homotopy, in other words if the maps $x \mapsto \mu(x_0, x)$ and $x \mapsto \mu(x_0, x)$ are homotopic to the identity. X is called an **H-group** if in addition the multiplication is associative up to homotopy ($\mu \circ (\mu \times id_X) \simeq \mu \circ (id_X \times \mu)$ rel (x_0, x_0, x_0), *i.e.*, X is an **H-monoid**) and if there is an "inversion" map $\eta : X \to X$ with $\eta(x_0) = x_0$, $\mu \circ (id_X \times \eta) \simeq id_X$ and $\mu \circ (\eta \times id_X) \simeq id_X$ rel (x_0, x_0).

Of course, the prototype of such a space is a topological group, but there are many H-spaces and H-groups that are not homotopy-equivalent to topological groups, at least in any obvious way. The multiplication in the Cayley numbers makes S^7 into an H-space which is not an H-group (homotopy associativity fails). Certain facts about topological groups persist for H-spaces and H-groups: for instance, the fundamental group (computed at the basepoint x_0) is abelian, and acts trivially on higher homotopy groups. The proofs of these facts are easy and will be omitted here; for details, see [Spanier, Ch. 1, §5] and [Whitehead, Ch. III, §§3–4]. Thus a space with non-abelian fundamental group, or a non-simple space, cannot have an H-space structure.

5.2.12. Theorem. *Let R be a ring, and define a group homomorphism*

$$\mu : GL(R) \times GL(R) \to GL(R)$$

by sending $A = (a_{ij})$ and $B = (b_{ij})$ to the matrix $\mu(A, B)$ with (i, j)-entry

$$\mu(A, B)_{ij} = \begin{cases} 0, & \text{if } i \not\equiv j \mod 2, \\ a_{\frac{i+1}{2}\frac{j+1}{2}}, & \text{if } i \equiv j \equiv 1 \mod 2, \\ b_{\frac{i}{2}\frac{j}{2}}, & \text{if } i \equiv j \equiv 0 \mod 2. \end{cases}$$

(Note that $\mu(A, B)$ is conjugate to the block direct sum $A \oplus B$ by a permutation matrix.) Let $B\mu$ be the induced map on classifying spaces

$$B(GL(R) \times GL(R)) = BGL(R) \times BGL(R) \to BGL(R)$$

and let μ_* be the induced map on the +-constructions

$$B(GL(R) \times GL(R))^+ \cong BGL(R)^+ \times BGL(R)^+ \to BGL(R)^+.$$

Then μ_* is a homotopy-associative homotopy-commutative H-space structure on $BGL(R)^+$. In particular, $BGL(R)^+$ is a simple space.

Proof. First of all, $E(R) \times E(R)$ is a perfect normal subgroup of the fundamental group of

$$B(GL(R) \times GL(R)) = BGL(R) \times BGL(R),$$

so $B(GL(R) \times GL(R))^+$ is defined, and it coincides with $BGL(R)^+ \times BGL(R)^+$ by the uniqueness part of Theorem 5.2.2. As we indicated before, if we use the model for classifying spaces constructed in the proof of Theorem 5.1.15, $BGL(R)$ and $BGL(R)^+$ come with a canonical basepoint x_0. It satisfies $\mu(x_0, x_0) = x_0$, but $BGL(R)$ cannot have an H-space structure since its fundamental group is non-abelian. Now if $A = (a_{ij})$,

$$\mu(A, 1)_{ij} = \begin{cases} a_{\frac{i+1}{2}\frac{j+1}{2}}, & \text{if } i \equiv j \equiv 1 \mod 2, \\ \delta_{ij}, & \text{otherwise}, \end{cases}$$

and similarly

$$\mu(1, A)_{ij} = \begin{cases} a_{\frac{i}{2}\frac{j}{2}}, & \text{if } i \equiv j \equiv 0 \mod 2, \\ \delta_{ij}, & \text{otherwise}. \end{cases}$$

To show μ is an H-space structure, we have to show that the maps induced by $l : A \mapsto \mu(A, 1)$ and $r : A \mapsto \mu(1, A)$ are homotopic to the identity rel basepoint, as self-maps of $BGL(R)^+$. Since the two cases are almost identical, we do only the first. The map induced by $l : A \mapsto \mu(A, 1)$ on $\pi_1(BGL(R))$ is clearly not the identity (it is the funny map we have just written down), but it induces the identity on $H_1(BGL(R); \mathbb{Z}) = K_1(R)$ since it sends any $A \in \pi_1(BGL(R))$ to one of its conjugates. In fact, suppose $A \in GL(n, R)$. Then with respect to the usual embedding of $GL(n, R)$ into $GL(2n, R)$, $l(A) = P_n A P_n^{-1}$, where P_n is the permutation matrix in $GL(2n, R)$ corresponding to the permutation

$$i \in \{1, \ldots, 2n\} \mapsto \begin{cases} 2i-1, & \text{if } i \le n, \\ 2(i-n), & \text{if } n+1 \le i \le 2n. \end{cases}$$

Since inner automorphisms act trivially on group homology, it follows that l_* acts trivially on any homology class in

$$H_\bullet(GL(R), \mathbb{Z}) \cong H_\bullet(BGL(R); \mathbb{Z}) \cong H_\bullet(BGL(R)^+; \mathbb{Z})$$

in the image of $H_\bullet(GL(R, n), \mathbb{Z})$. Since group homology commutes with direct limits (Exercise 4.1.29), $H_\bullet(GL(R), \mathbb{Z}) = \varinjlim H_\bullet(GL(R, n), \mathbb{Z})$, and thus l_* acts trivially on homology. From the characterization of l_* in Proposition 5.2.4, together with the characterization of the +-construction, we see l_* is a homotopy equivalence. We also see from Theorem 5.2.2 and Proposition 5.2.4 that for $n \ge 3$ (so that $E(n, R)$ is perfect and one may apply the +-construction to $BGL(n, R)$) that $l_* : BGL(n, R)^+ \to BGL(2n, R)^+$ coincides with the map induced by the usual inclusion of $GL(n, R)$ into $GL(2n, R)$. Hence l_* is homotopic to the identity.

Finally, we show μ_* is homotopy-commutative and homotopy-associative. To show homotopy commutativity, we have to show the maps induced by $(A, B) \mapsto \mu(A, B)$ and by $(A, B) \mapsto \mu(B, A)$ on the +-construction are homotopic to one another. The proof is almost the same as above since again there is a permutation matrix conjugating one to the other for $A, B \in GL(n, R)$. And the proof of homotopy associativity is also almost the same, except that one has to compare the maps induced by $(A, B, C) \mapsto \mu(\mu(A, B), C)$ and by $(A, B, C) \mapsto \mu(A, \mu(B, C))$. □

5.2.13. Remarks. For some purposes, we need the basepoint x_0 of $BGL(R)^+$ to be a "strict" unit, that is, for $\mu_*(x, x_0) = \mu_*(x_0, x) = x_0$ for any x. However, this can always be achieved by changing μ_* within its homotopy class. Secondly, it is also useful to extend the H-space structure on $BGL(R)^+$ to such a structure on $\mathbf{K}(R) = BGL(R)^+ \times K_0(R)$. For this purpose we merely use the product of the multiplication μ_* on $BGL(R)^+$ with the usual addition on the abelian group $K_0(R)$.

Finally, we need to explain how to define relative groups in higher K-theory in order to get a natural exact sequence

$$\cdots \to K_{i+1}(R, I) \xrightarrow{\partial} K_i(R, I) \to K_i(R) \to K_i(R/I) \xrightarrow{\partial} K_{i-1}(R, I) \to \cdots.$$

5.2.14. Definition. Let R be a ring and let I be a two-sided ideal in R. We define $\mathbf{K}(R, I)$ to be the homotopy fiber of the map $q_* : \mathbf{K}(R) \to q_*(\mathbf{K}(R)) \subseteq \mathbf{K}(R/I)$ induced by the quotient map $q : R \to R/I$. (By "homotopy fiber," we mean we convert the map into a fibration using Proposition 5.1.21 and then take "the" fiber of the fibration, which is well defined up to homotopy equivalence by Proposition 5.1.23.) Then the homotopy exact sequence of the fibration q_*, spliced together with the exact sequence of Theorem 2.5.4, gives a natural exact sequence

$$\cdots \to K_{i+1}(R,\, I) \xrightarrow{\partial} K_i(R,\, I) \to K_i(R) \\ \to K_i(R/I) \xrightarrow{\partial} K_{i-1}(R,\, I) \to \cdots .$$

It is also clear that all the conditions (5.2.1) are now satisfied for the higher K-groups.

5.2.15. Exercise. Let R be any ring. Show using Theorem 5.2.7 and Corollary 5.2.8 that the Hurewicz maps for $BGL(R)^+$, $BE(R)^+$, and $B\operatorname{St}(R)^+$ give homomorphisms (also usually called Hurewicz maps) $h_{GL} : K_i(R) \to H_i(GL(R),\, \mathbb{Z})$ for $i \geq 1$, $h_E : K_i(R) \to H_i(E(R),\, \mathbb{Z})$ for $i \geq 2$, $h_{\mathrm{St}} : K_i(R) \to H_i(\operatorname{St}(R),\, \mathbb{Z})$ for $i \geq 3$. Here h_{GL} is an isomorphism in degree 1, h_E is an isomorphism in degree 2, and h_{St} is an isomorphism in degree 3. Show that $h_{GL} = \operatorname{cores} \circ h_E$ and that $h_E = \varphi_* \circ h_{\mathrm{St}}$, where $\varphi : \operatorname{St}(R) \to E(R)$. Also show using Exercise 5.1.31 that h_E is surjective in degree 3 and that h_{St} is surjective in degree 4.

5.2.16. Exercise. Let $G = A_5$, the alternating group on 5 letters, which by Exercise 4.1.27(4) is isomorphic to the symmetry group of a regular icosahedron and by Exercise 4.1.28(3) is also isomorphic to $SL(2,\, \mathbb{F}_4)$. Let $\hat{G}$ be its universal central extension, which by Exercise 4.1.27(7), is an extension of G by a group of order 2 and by Exercise 4.1.28(4) is also isomorphic to $SL(2,\, \mathbb{F}_5)$. Since G and $\hat{G}$ are perfect, one may apply the +-construction to BG and to $B\hat{G}$.

(1) As we observed in Exercise 4.1.27(7), $\hat{G}$ naturally sits inside of $SU(2)$, which in turn may be identified with the group $Sp(1)$ of quaternions of modulus 1, which topologically is the same as S^3. Let $S(\mathbb{H}^\infty)$ denote the inductive limit $\varinjlim S^{4k-1}$ with the obvious CW-structure, where we identify S^{4k-1} with the unit sphere in $\mathbb{H}^k$. Since S^{4k-1} is $(4k-2)$-connected, taking the limit as $k \to \infty$, we see by Theorem 4.1.13 that $S(\mathbb{H}^\infty)$ is contractible. Note that $Sp(1)$ acts freely on S^{4k-1} (by left quaternionic multiplication), with quotient $\mathbb{P}^{k-1}(\mathbb{H})$. Show that by dividing out by $\hat{G} \subseteq Sp(1)$, one gets a fibration

$$Sp(1)/\hat{G} \to S^{4k-1}/\hat{G} \to \mathbb{P}^{k-1}(\mathbb{H}),$$

or by passage to the limit as $k \to \infty$,

$$Sp(1)/\hat{G} \to B\hat{G} \to \mathbb{P}^\infty(\mathbb{H}). \tag{5.2.17}$$

Here $Sp(1)/\hat{G}$ is the Poincaré homology 3-sphere of Exercise 4.1.27. Show that the universal cover of the fibration (5.2.17) is a fibration

$$Sp(1) \to S(\mathbb{H}^\infty) \to \mathbb{P}^\infty(\mathbb{H}). \tag{5.2.18}$$

Deduce that $\pi_n(\mathbb{P}^\infty(\mathbb{H})) \cong \pi_{n-1}(S^3)$ for all n.

(2) Observe that $\mathbb{P}^\infty(\mathbb{H})$ has a CW-structure with cells only in dimensions divisible by 4 and with exactly one cell in each such dimension, so that $H_n(\mathbb{P}^\infty(\mathbb{H}); \mathbb{Z})$ is free abelian if n is a multiple of 4 and vanishes otherwise. In fact it is well known that the cohomology ring of $\mathbb{P}^\infty(\mathbb{H})$ is a polynomial ring on a single generator in degree 4. From this and the "Gysin sequences" (see [Spanier, Ch. 5, §7] or [Whitehead, Ch. VII, §5]) of (5.2.17) and of (5.2.18) (since the fiber is a homology sphere), deduce that the homology of $\hat{G}$ is periodic, with

$$H_n(B\hat{G}; \mathbb{Z}) \cong \begin{cases} 0, & n \equiv 0, 1, 2, \mod 4 \\ \mathbb{Z}/120, & n \equiv 3 \mod 4 \end{cases}, \quad n > 0.$$

(The number 120 is $|\hat{G}|$, and comes from comparing the two sequences, using the fact that the projection $Sp(1) \to Sp(1)/\hat{G}$ is a covering map which is 120-to-1.)

(3) Use (1) and (2) to compute π_1, π_2, and π_3 for BG^+ and for $B\hat{G}^+$, hence for $BGL(2, \mathbb{F}_4)^+$ and for $BGL(2, \mathbb{F}_5)^+$.

(4) As mentioned in Example 5.2.5, $(Sp(1)/\hat{G})^+ \simeq S^3$, so that by Proposition 5.2.4, applying the +-construction to the map $Sp(1)/\hat{G} \to B\hat{G}$ of (5.2.17) gives a map $S^3 \to B\hat{G}^+$. Show that this map is still compatible with the fibration of (5.2.17), so that one gets a fibration (up to homotopy equivalence)

$$S^3 \to B\hat{G}^+ \to \mathbb{P}^\infty(\mathbb{H}). \tag{5.2.17+}$$

To see how this works, it is easiest to begin first with the case of the fibration

$$Sp(1)/\hat{G} \to S^7/\hat{G} \to \mathbb{P}^1(\mathbb{H}) = S^4,$$

which comes from taking two copies of $(Sp(1)/\hat{G}) \times B^4$ and gluing them together along S^3 using the covering map $S^3 \to Sp(1)/\hat{G}$. Applying the +-construction "fiberwise" gives a fibration over S^4 with fibers homotopy-equivalent to S^3 where the "clutching map" in $\pi_3(\text{fiber})$ is 120 times the generator of $\pi_3(\text{fiber}) \cong \pi_3(S^3) \cong \mathbb{Z}$. Extend this argument to the case of $\mathbb{P}^{k-1}(\mathbb{H})$ as the base and pass to the limit. Deduce that there is a long exact sequence

$$\cdots \to \pi_n(S^3) \to \pi_n(S^3) \to \pi_n(B\hat{G}^+) \to \pi_{n-1}(S^3) \to \pi_{n-1}(S^3) \to \cdots.$$

Since it is known that $\pi_n(S^3)$ is finite for $n > 3$, and since $\pi_3(B\hat{G}^+)$ was computed in (3) above, it follows that $\pi_n(B\hat{G}^+)$ is finite for all n.

5.2.19. Exercise. Let S_n be the n-th symmetric group and let A_n be the n-th alternating group. We let $S_\infty = \varinjlim S_n$, $A_\infty = \varinjlim A_n$. Since A_n is simple, and thus certainly perfect, for $n \geq 5$, it makes sense to apply the

+-construction to S_n and to A_n (for $n \geq 5$), with respect to the perfect group A_n, and to S_∞ and to A_∞ (with respect to A_∞).

(1) For any ring R, show that the inclusion of the permutation matrices gives homomorphisms $S_n \to GL(n, R)$, $A_n \to E(n, R)$, and thus maps $BS_\infty^+ \to BGL(R)^+$, $BA_\infty^+ \to BE(R)^+$.

(2) Compute as much as you can about the groups $\pi_n(BS_\infty^+)$. In particular, compute them for $n = 1, 2$ and try to get bounds on them for $n = 3, 4$. One relevant fact is a famous (and difficult) theorem of Schur, published in 1911, which implies that the corestriction map $H_2(A_n, \mathbb{Z}) \to H_2(S_n, \mathbb{Z})$ is an isomorphism for $n = 5$ and for $n \geq 8$ (though not for $n = 6$ or 7), and that $H_2(S_n, \mathbb{Z}) \cong \mathbb{Z}/2$ for $n \geq 2$.

5.2.20. Exercise. Let $F = \mathbb{F}_q$, $q = p^d$, p a prime, be a finite field. By Corollary 4.3.13 and Remark 4.3.14, $\mathrm{St}(F) = E(F) = SL(F)$, and thus, by Corollary 4.2.8, the Hurewicz map $K_3(F) \to H_3(SL(F), \mathbb{Z})$ is an isomorphism. By an analogue of Theorem 4.3.12, one can show that this group is finite. Show (following Quillen) that it has order relatively prime to p. Here is an outline: to show that $K_3(F)$ has order relatively prime to p, it is enough to show (why?) that $H_3(GL(F), \mathbb{F}_p) = 0$. Let $N(n, F)$ be as in Corollary 4.3.13, the group of upper-triangular $n \times n$ matrices over F with 1's on the diagonal, and let $D(n, F) \cong (F^\times)^n$ be the diagonal subgroup of $GL(n, F)$. It is enough to show (why?) that $H_0(D(n, F), H_3(N(n, F), \mathbb{F}_p)) = 0$ for sufficiently large n. This can be proved, even for all n, by methods like those of Lemma 4.3.5, except for a few fields of small cardinality ($q = 2, 3, 4$, or 8). In the exceptional cases, one can still show that $H_3(GL(F), \mathbb{F}_p)$ vanishes by embedding F in a suitable extension field F' with $[F' : F]$ relatively prime to p and considering the induced inclusion $GL(F) \to GL(F')$ and the map in the other direction coming from viewing F' as sitting in $M_{[F':F]}(F)$.

3. A survey of higher K-theory

In the last section of this chapter, we survey some of the things that are known about the higher K-groups $K_i(R)$. Since the methods used in deducing some of the major results are quite complicated, we omit most of the proofs. The reader who wants to learn more should consult the survey articles in [LluisP] or a more comprehensive source such as [Srinivas].

Products. The Steinberg symbol map $R^\times \times R^\times \to K_2(R)$ in the case of a commutative ring R may be viewed as a special case of a kind of product operation $K_1(R) \times K_1(R) \to K_2(R)$. To generalize this to the higher K-groups, one would like some sort of product map $BGL(R)^+ \times BGL(R)^+ \to BGL(R)^+$ that would induce a product structure on the homotopy groups. The H-space structure we defined in §2, since it corresponds to the block direct sum of matrices, really corresponds to "addition." (This is slightly confusing, since the group operation on $K_1(R)$ is usually written multi-

plicatively, but when we extend the H-space structure to $\mathbf{K}(R)$, we use addition on $K_0(R)$.) When R is commutative, however, there should be another operation on K-theory, extending the "multiplication" on $K_0(R)$ corresponding to the tensor product. (See Exercise 1.1.9.) It is this structure which is related to the Steinberg symbol map $R^\times \times R^\times \to K_2(R)$. More generally, when R and S are rings which are not necessarily commutative, the tensor product of projective modules gives a "multiplication" $K_0(R) \otimes K_0(S) \to K_0(R \otimes_{\mathbb{Z}} S)$, where $R \otimes_{\mathbb{Z}} S$ is given the obvious multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$. When R is commutative, one has a ring homomorphism $R \otimes_{\mathbb{Z}} R \to R$ given by the multiplication, and this can be used to turn this "external" product into an "internal" one. (N.B.: For any ring, multiplication defines a homomorphism of abelian groups $R \otimes_{\mathbb{Z}} R \to R$, but this is a ring homomorphism only when R is commutative.) Similarly there are products $K_0(R) \otimes K_1(S) \to K_1(R \otimes_{\mathbb{Z}} S)$ and $K_1(R) \otimes K_0(S) \to K_1(R \otimes_{\mathbb{Z}} S)$ which can be turned into "internal" products when $R = S$ is commutative. The product $K_0(R) \otimes K_1(S) \to K_1(R \otimes_{\mathbb{Z}} S)$ is defined on generators $[P] \otimes [B]$, where P is a finitely generated projective R-module and $B \in GL(m, S)$, as follows. Note that $P \otimes S^m$ is a finitely generated projective $R \otimes_{\mathbb{Z}} S$-module, and that if $P \oplus Q \cong R^n$, then $(P \otimes S^m) \oplus (Q \otimes S^m) \cong (P \oplus Q) \otimes S^m \cong (R \otimes_{\mathbb{Z}} S)^{nm}$. Send $[P] \otimes [B]$ to the class of $(1_P \otimes B) \oplus (1_Q \otimes 1_m) \in GL(nm, R \otimes_{\mathbb{Z}} S)$. It is an easy exercise to see that this gives a well-defined product.

5.3.1. Theorem [Loday]. *Let R and S be rings. Then there is a natural bilinear, associative product operation*

$$K_i(R) \times K_j(S) \to K_{i+j}(R \otimes_{\mathbb{Z}} S)$$

(for $i, j \geq 0$) which agrees with the usual products when $i = 0$ or $j = 0$. When R is commutative, this together with the multiplication map $R \otimes_{\mathbb{Z}} R \to R$ makes $\bigoplus_i K_i(R)$ into a graded commutative ring. (Graded commutativity means that $[a] \cdot [b] = (-1)^{ij} [b] \cdot [a]$ for $[a] \in K_i$, $[b] \in K_j$.)

Proof (Sketch). The idea is to define a map

$$BGL(R)^+ \wedge BGL(S)^+ \xrightarrow{\mu} BGL(R \otimes_{\mathbb{Z}} S)^+$$

and to extend it to a map

$$\mathbf{K}(R) \wedge \mathbf{K}(S) \xrightarrow{\mu} \mathbf{K}(R \otimes_{\mathbb{Z}} S).$$

Recall here that the "smash product" $X \wedge Y$ of based spaces (X, x_0) and (Y, y_0) is the based space obtained from $X \times Y$ by collapsing $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ to a point. The desired product structure will then be defined as follows. Given based maps $a : S^i \to \mathbf{K}(R)$, $b : S^j \to \mathbf{K}(S)$ representing classes in $K_i(R)$ and $K_j(R)$, we define the product $[a] \cdot [b]$ as the homotopy class of the composite

$$S^{i+j} = S^i \wedge S^j \xrightarrow{a \wedge b} \mathbf{K}(R) \wedge \mathbf{K}(S) \xrightarrow{\mu} \mathbf{K}(R \otimes_{\mathbb{Z}} S).$$

The definition of μ is based on a construction similar to that in the proof of Theorem 5.2.12. If we fix a bijection $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, it gives an identification of $R^\infty\otimes_{\mathbb{Z}}S^\infty$ with $(R\otimes_{\mathbb{Z}}S)^\infty$ (here R^∞ means a free R-module of countably infinite rank, etc.), and thus a group homomorphism $GL(R)\times GL(S)\to GL(R\otimes_{\mathbb{Z}}S)$. Applying the +-construction via Proposition 5.2.4, we obtain a map

$$BGL(R)^+\wedge BGL(S)^+\xrightarrow{\mu} BGL(R\otimes_{\mathbb{Z}}S)^+.$$

Changing the bijection $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ used in this construction only changes the map by conjugation by a permutation matrix, and one can show this does not change the homotopy class of μ. Naturality is clear, and bilinearity and associativity are easy to check (using the fact that the homotopy class of μ is independent of the bijection chosen $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$).

Suppose now that R is commutative. Then the ring homomorphism $R\otimes_{\mathbb{Z}}R\to R$ coming from multiplication induces a map $m:\mathbf{K}(R\otimes_{\mathbb{Z}}R)\to\mathbf{K}(R)$. We obtain a composite $m\circ\mu:\mathbf{K}(R)\wedge\mathbf{K}(R)\to\mathbf{K}(R)$. Because of associativity and bilinearity of the product, this makes $\bigoplus_i K_i(R)$ into a graded ring. To prove that this ring is graded commutative, let σ be the "flip" automorphism of $R\otimes_{\mathbb{Z}}R$. This induces a map $\sigma^+:\mathbf{K}(R\otimes_{\mathbb{Z}}R)\to\mathbf{K}(R\otimes_{\mathbb{Z}}R)$. Given based maps $a:S^i\to\mathbf{K}(R)$, $b:S^j\to\mathbf{K}(R)$ representing classes in $K_i(R)$ and $K_j(R)$, we consider the diagram

$$\begin{array}{ccccccc}
S^{i+j}=S^i\wedge S^j & \xrightarrow{a\wedge b} & \mathbf{K}(R)\wedge\mathbf{K}(R) & \xrightarrow{\mu} & \mathbf{K}(R\otimes_{\mathbb{Z}}R) & \xrightarrow{m} & \mathbf{K}(R)\\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma^+} & & \|\\
S^{i+j}=S^j\wedge S^i & \xrightarrow{b\wedge a} & \mathbf{K}(R)\wedge\mathbf{K}(R) & \xrightarrow{\mu} & \mathbf{K}(R\otimes_{\mathbb{Z}}R) & \xrightarrow{m} & K(R).
\end{array}$$

Here s and σ are the obvious "flip" maps.

The left-hand square is commutative by definition, and the middle square is homotopy-commutative by naturality of μ. The right-hand square is homotopy-commutative since $\mathrm{mult}\circ\sigma=\mathrm{mult}$ (recall R is commutative). On the other hand, the map s is easily seen to have degree $(-1)^{ij}$, since the flip on $\mathbb{R}^i\times\mathbb{R}^j$ is orientation-preserving if and only one of i and j is even. So the homotopy class of the composite along the top line of the diagram differs from that along the bottom line of the diagram by a factor of $(-1)^{ij}$. □

Remark. A calculation in [Loday] shows that the above product, when specialized to a product $K_1(R)\otimes_{\mathbb{Z}}K_1(R)\to K_2(R)$ and restricted to $R^\times\otimes_{\mathbb{Z}}R^\times$, differs from the Steinberg symbol pairing of the last chapter by a sign.

K-Theory of Fields and of Rings of Integers. Because of their importance in arithmetic, as well as the fact that all rings are algebras over $\mathbb{Z}$ and many rings commonly encountered are algebras over fields, it is natural to begin the study of higher K-theory with the K-theory of fields and of rings of integers in number fields. Even in these "elementary" cases, the calculation of the higher K-groups is quite difficult and in most cases incomplete. Nevertheless, there are a number of important results known, and a number of conjectures (true in many special cases) which would link

the K-theory of fields and of rings of integers to important problems in number theory and algebraic geometry. No discussion of higher K-theory would be complete without some mention of these results and conjectures, which have provided most of the impetus for recent work in the subject. A more detailed survey (again with most proofs omitted) may be found in Soulé's article in [LluisP].

The first major result on the higher K-theory of fields was due to Quillen.

5.3.2. Theorem [QuillenFinFd]. *The K-groups $K_i(\mathbb{F}_q)$ of a finite field $\mathbb{F}_q$ are finite cyclic for all $i > 0$. They vanish for $i \geq 2$ even, and have order $|K_{2k-1}(\mathbb{F}_q)| = q^k - 1$ for $i = 2k - 1 \geq 1$ odd.*

Some ideas from the proof. While it would take too much algebraic topology to explain Quillen's method of proof, we should mention at least some of the ideas involved. Quillen's result was actually somewhat more precise, since he managed to compute the precise homotopy type of $BGL(\mathbb{F}_q)^+$, showing that it is the fiber in a fibration (up to homotopy)

$$BGL(\mathbb{F}_q)^+ \xrightarrow{b_q^+} BU \xrightarrow{g_q} BU, \tag{5.3.3}$$

where BU is the classifying space for (complex) topological K-theory. This is the (path-connected) space, uniquely defined up to homotopy equivalence, with the property that for any paracompact space X, there is natural bijection between $\tilde{K}^0(X)$ and the set $[X, BU]$ of homotopy classes of continuous maps $X \to BU$. From Bott periodicity, one knows that the homotopy groups of BU are $\pi_i(BU) = 0$, $i = 2k-1 \geq 1$ odd, $\pi_i(BU) \cong \mathbb{Z}$, $i = 2k \geq 2$ even. For any connected compact space X, the map g_q induces a map $\tilde{K}^0(X) \to \tilde{K}^0(X)$ which is natural in X. The map g_q is determined by this "cohomology operation" on K-theory, which is given by $x \mapsto \psi^q(x) - x$, where ψ^q is what is known as the **q-th Adams operation** on $K^0(X)$. This in turn is characterized by the properties that ψ^q is a ring homomorphism (recall $K^0(X)$ is a ring with unit, where the product comes from the tensor product of vector bundles), and that $\psi^q(x) = x^q$ if $x \in K^0(X)$ is the class of a line bundle. Note that ψ^q preserves the dimension of a vector bundle, and thus passes to a well-defined self-map of reduced K-theory, which is the kernel of the dimension homomorphism $K^0(X) \to K^0(pt) = \mathbb{Z}$. From these axioms it is easy to see that under the identification of $\tilde{K}^0(S^{2k})$ with $\mathbb{Z}$, ψ^q corresponds to multiplication by q^k. (Choose a map $(S^2)^k \to S^{2k}$ of degree 1, and observe that the generator of $\tilde{K}^0(S^{2k})$ pulls back to a multiple of a virtual bundle of the form $\prod_{i=1}^k (x_i - 1)$, where the x_i are complex line bundles pulled back from the S^2 factors. But

$$\psi^q\left(\prod_{i=1}^k (x_i - 1)\right) = \prod_{i=1}^k (x_i^q - 1) = \prod_{i=1}^k (q(x_i - 1)) = q^k \prod_{i=1}^k (x_i - 1),$$

which shows ψ^q must multiply the generator of $\tilde{K}^0(S^{2k})$ by q^k.) Thus $(g_q)_*$ must be multiplication by $q^k - 1$ on $\pi_{2k}(BU) \cong \mathbb{Z}$, and the calculation of

the homotopy groups of $BGL(\mathbb{F}_q)^+$ follows from the long exact homotopy sequence (Theorem 5.1.24) of the fibration (5.3.3).

While we do not have the tools here to construct the fibration (5.3.3), we can at least indicate where the crucial map $b_q^+ : BGL(\mathbb{F}_q)^+ \longrightarrow BU$ comes from. Since BU is simply connected, it is enough by Proposition 5.2.4 to construct a map $b_q : BGL(\mathbb{F}_q) \longrightarrow BU$, or in other words, an element of the reduced K-theory of $BGL(\mathbb{F}_q)$. This is constructed by passage to the limit from (compatible) classes in $\tilde{K}^0(BGL(n, \mathbb{F}_q))$ for each n. Now if G is a group and one has a finite-dimensional representation $\rho : G \longrightarrow GL(N, \mathbb{C})$, it immediately gives rise to a vector bundle on BG, namely the fiber product $EG \times_G \mathbb{C}^N$, where G acts on $\mathbb{C}^N$ via ρ. (Recall by Definition 5.1.16 that EG is the contractible universal cover of BG.) So we need a suitable virtual representation (that is, a formal difference of two finite-dimensional complex representations) of $GL(n, \mathbb{F}_q)$. This in turn is constructed by the mechanism of **Brauer lifting**. Instead of constructing the virtual representation, we use that fact that representations of a finite group are characterized by their characters, and so construct instead a **virtual character** on G, that is, the difference of two characters. This is a complex-valued function on G, constant on conjugacy classes. The crucial result is the following.

5.3.4. Theorem (J. A. Green). *Let G be a finite group, and let $\rho : G \longrightarrow GL(n, \bar{\mathbb{F}}_p)$ be a finite-dimensional representation of G over the algebraic closure of the field of p elements, p a prime. Fix an isomorphism $\sigma : \bar{\mathbb{F}}_p^\times \hookrightarrow \mathbb{C}^\times$ of the multiplicative group of $\bar{\mathbb{F}}_p$ with the complex roots of unity of order prime to p. Let $\xi_i(g)$, $i = 1, \dots, n$ be the eigenvalues of $\rho(g)$ for $g \in G$, counted with multiplicities, so that the character of ρ is given by $\operatorname{Tr} \rho(g) = \xi_1(g) + \cdots + \xi_n(g)$. Then $g \mapsto \sigma \circ \xi_1(g) + \cdots + \sigma \circ \xi_n(g)$ is a (complex-valued) virtual character of G, called the Brauer lift $l^\sigma(\rho)$ of ρ. The Brauer lift is additive on short exact sequences of representations. In other words, if*

$$0 \longrightarrow \rho' \longrightarrow \rho \longrightarrow \rho'' \longrightarrow 0$$

is a short exact sequence of representations over $\bar{\mathbb{F}}_p$, then $l^\sigma(\rho) = l^\sigma(\rho') + l^\sigma(\rho'')$.

Proof. This depends on a famous theorem of Brauer (see [Jacobson, II, §5.12]), which implies that a complex-valued function on G, constant on conjugacy classes, is a virtual character if and only of its restriction to each nilpotent subgroup is a virtual character. Because of this and the fact that $g \mapsto \sigma \circ \xi_1(g) + \cdots + \sigma \circ \xi_n(g)$ is a class function, we can immediately reduce to the case where G is nilpotent. But a finite nilpotent group is the direct product of its Sylow subgroups, so we may assume $G = H \times K$, where H is nilpotent with order $|H|$ relatively prime to p, and with K a p-group. If $g \in G$, we can write $g = hk$ with $h \in H$ and $k \in K$, and $\rho(k) \in GL(n, \bar{\mathbb{F}}_p)$ is unipotent (since the minimal polynomial of $\rho(k)$ is of the form $x^{p^r} - 1$ and we are in characteristic p) and commutes with $\rho(h)$. Thus $\rho(g)$ has the same eigenvalues as $\rho(h)$, and $l^\sigma(\rho)$ factors through H. So without loss

of generality, we may replace G by H and assume G is nilpotent of order prime to p.

However, if the order of G is relatively prime to p, then representations of G over $\bar{\mathbb{F}}_p$ are completely reducible. If G is abelian, every representation is therefore a direct sum of characters, and the result is obvious. Furthermore, if ρ is induced from a one-dimensional representation χ of a subgroup H, then it is clear that $l^\sigma(\rho)$ as defined in the the Theorem is just the character of the representation of G induced from the character $\sigma \circ \chi$ of H. To conclude the proof, we note that if G is nilpotent, an easy induction shows every irreducible representation is "monomial," *i.e.*, induced from a one-dimensional representation of a subgroup (again see [Jacobson, II, §5.12]), and so applying the last observation, we see $l^\sigma(\rho)$ is the character of a monomial representation over $\mathbb{C}$. The additivity property comes from the fact that given a short exact sequence as in the theorem, the eigenvalues of $\rho(g)$ are the union of the eigenvalues of $\rho'(g)$ and of $\rho''(g)$. □

Some ideas from the proof of Theorem 5.3.2 (continued). Now we can explain the construction of the map b_q. Fix $\sigma : \bar{\mathbb{F}}_p^\times \hookrightarrow \mathbb{C}^\times$ as in Theorem 5.3.4. Start with the identity representation

$$id_n : GL(n, \mathbb{F}_q) \to GL(n, \mathbb{F}_q) \hookrightarrow GL(n, \bar{\mathbb{F}}_p).$$

The Brauer lift $l^\sigma(id_n)$ is a complex-valued virtual character of $GL(n, \mathbb{F}_q)$ by Theorem 5.3.4, and by the formula for l^σ, $l^\sigma(id_n)(1) = n$, *i.e.*, it has dimension n. So $l^\sigma(id_n) - n$ is a virtual character of dimension 0 and thus gives rise to a virtual vector bundle of dimension 0 on $BGL(n, \mathbb{F}_q)$, and so to a map $b_q^n : BGL(n, \mathbb{F}_q) \to BU$. The diagram

$$\begin{array}{ccc} BGL(n, \mathbb{F}_q) & \xrightarrow{b_q^n} & BU \\ \downarrow & & \| \\ BGL(n+1, \mathbb{F}_q) & \xrightarrow{b_q^{n+1}} & BU \end{array}$$

commutes, at least up to homotopy, since for $g \in GL(n, \mathbb{F}_q)$, its image in $GL(n+1, \mathbb{F}_q)$ is $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, and

$$l^\sigma(id_{n+1}) \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} - (n+1) = l^\sigma(id_n)(g) - n.$$

So passing to the limit we obtain the desired map $b_q : BGL(n, \mathbb{F}_q) \to BU$. Furthermore, we can see immediately that the composite $g_q \circ b_q$ is null-homotopic, for if $g \in GL(n, \mathbb{F}_q)$, the set of eigenvalues of g is invariant under $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$, which is generated by the Frobenius automorphism $x \mapsto x^q$, and thus from the defining property of ψ^q, $\psi^q(Bl^\sigma(id_n)) = Bl^\sigma(id_n)$, and $g_q \circ b_q$ is null-homotopic. From this one can see at least that b_q^+ induces a map from $BGL(\mathbb{F}_q)^+$ to the homotopy fiber of g_q. The bulk of Quillen's proof consists of showing that this induced map is a homology isomor-

phism, and thus a homotopy equivalence by the Hurewicz and Whitehead Theorems. □

5.3.5. Remark. A perhaps disappointing corollary of the computation of the K-groups of a finite field is that in this case, the ring structure of Theorem 5.3.1 is quite trivial: the product $K_0(\mathbb{F}_q) \otimes K_i(\mathbb{F}_q) \to K_i(\mathbb{F}_q)$ is just the usual action of $\mathbb{Z}$ on any abelian group, and all other products vanish for dimensional reasons (the non-trivial K-groups are in odd degree, and the product of two elements of odd degree has even degree and thus vanishes).

5.3.6. Corollary [QuillenFinFd]. *If F is any algebraic extension of $\mathbb{F}_p$, then $K_i(F)$ vanishes for $i \geq 2$ even, and $K_i(F) = K_i(\bar{\mathbb{F}}_p)^{\mathrm{Gal}(\bar{\mathbb{F}}_p/F)}$ for $i = 2k-1 \geq 1$ odd. Furthermore,*

$$K_{2k-1}(\bar{\mathbb{F}}_p) \cong \bigoplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l.$$

Proof. Any algebraic extension F of $\mathbb{F}_p$ is an increasing union of finite fields $\mathbb{F}_{q_j}$. Now it is evident that the constructions of Theorem 5.1.15 and of Theorem 5.2.2 are compatible with increasing unions, so that

$$\begin{aligned} K_i(F) = \pi_i(BGL(F)^+) &= \pi_i(\varinjlim BGL(\mathbb{F}_{q_j})^+) \\ &= \varinjlim \pi_i(BGL(\mathbb{F}_{q_j})^+) = \varinjlim K_i(\mathbb{F}_{q_j}), \end{aligned}$$

which gives 0 for $i \geq 2$ even and gives $\lim_{j\to\infty} \mathbb{Z}/(q_j^k - 1)$ for $i = 2k-1 \geq 1$ odd. In the case of $F = \bar{\mathbb{F}}_p$, the q_j are all the powers of p, indexed by the multiplicative ordering on $\mathbb{N}$. Thus the l-primary part of $K_{2k-1}(\bar{\mathbb{F}}_p)$ vanishes for $l = p$ (since the p-primary part of $K_{2k-1}(\mathbb{F}_q)$ vanishes for q any power of p, by Theorem 5.3.2) and for $l \neq p$ is an increasing union of cyclic groups of orders l^{n_j} with $n_j \to \infty$, in other words $\mathbb{Q}_l/\mathbb{Z}_l$. □

The state of knowledge about the K-theory of other fields is quite incomplete. To phrase some results in a reasonable way, it is necessary to pass to **K-theory with finite coefficients**, which is useful in other contexts, anyway. Setting up the foundations of this theory requires more homotopy theory than we have developed, but we will at least state the main properties.

5.3.7. Theorem (cf. [Browder]**).** *For any positive integer $k \geq 2$, there are functors $R \rightsquigarrow K_i(R; \mathbb{Z}/k)$ from rings to abelian groups and functors $(R, I) \rightsquigarrow K_i(R, I; \mathbb{Z}/k)$ giving exact sequences as in Definition 5.2.14. These fit into natural long exact sequences*

$$\begin{aligned} \cdots \to K_{i+1}(R; \mathbb{Z}/k) \xrightarrow{\partial} K_i(R) \xrightarrow{k} K_i(R) \\ \to K_i(R; \mathbb{Z}/k) \xrightarrow{\partial} K_{i-1}(R) \to \cdots \end{aligned}$$

(and similarly for the relative groups), where k here means multiplication by k on $K_i(R)$. For $i \geq 2$, the K-groups with coefficients in $\mathbb{Z}/k$ may be defined by

$$K_i(R;\, \mathbb{Z}/k) = \left[M_k^i =_{\text{def}} S^{i-1} \cup_k e^i,\ BGL(R)^+\right],$$

the set of homotopy classes of maps into $BGL(R)^+$ from the "Moore space" M_k^i obtained by attaching an i-cell onto S^{i-1} by a map $\partial B^i = S^{i-1} \to S^{i-1}$ of degree k. (A similar definition applies to the relative groups.)

For $i \geq 1$, there is a natural Hurewicz map

$$K_i(R;\, \mathbb{Z}/k) \to H_i(GL(R);\, \mathbb{Z}/k)$$

which is the "mod k" analogue of the map h_{GL} of Exercise 5.2.15.

Partial sketch of proof. The groups $K_i(R;\, \mathbb{Z}/k)$ were first introduced by Browder [Browder], who used $[M_k^i,\, BGL(R)^+]$ as the definition for $i \geq 2$ and $K_i(R) \otimes_{\mathbb{Z}} \mathbb{Z}/k$ as the definition for $i \leq 1$. This gives the "correct" groups in degrees ≥ 2 but is somewhat unsatisfactory in low degrees for general rings, since it leads to failure of the long exact sequences below degree 2. (This was not a problem for Browder since he was mostly interested in fields.) However, there are various ways of remedying this by "dimension-shifting." With this approach, the boundary map $K_i(R;\, \mathbb{Z}/k) \xrightarrow{\partial} K_{i-1}(R)$ is induced by the inclusion of S^{i-1} into M_k^i, and the exact sequence comes from the exact sequence dual to that in Definition 5.1.6. For example, a class in $K_{i-1}(R)$ lies in the image of ∂ if and only if the corresponding map $S^{i-1} \to BGL(R)^+$ extends over M_k^i, which by definition of M_k^i happens if and only if its homotopy class is torsion of order dividing k. The "reduction mod k" map $K_i(R) \to K_i(R;\, \mathbb{Z}/k)$ is induced by a map $M_k^i \to S^i$. The Hurewicz map is defined by noting that $H_i(M_k^i;\, \mathbb{Z}/k) \cong \mathbb{Z}/k$, so that any map $f : M_k^i \to BGL(R)^+$ sends the "mod k fundamental class" of M_k^i to a class in $H_i(GL(R);\, \mathbb{Z}/k)$ only depending on the homotopy class of f.

Another method is to show, using the fact that $BGL(R)^+$ is an H-space, that there is a self-map of $BGL(R)^+$ which induces multiplication by k on the homotopy groups. Then one can take the homotopy fiber of this map and use its homotopy groups, shifted in degree by 1, as the K-groups with coefficients in $\mathbb{Z}/k$. The desired long exact sequences

$$\cdots \to K_{i+1}(R;\, \mathbb{Z}/k) \to K_i(R) \xrightarrow{k} K_i(R) \to K_i(R;\, \mathbb{Z}/k) \to \cdots$$

then follow immediately from the long exact sequence of a fibration (Theorem 5.1.24). A "dimension-shifting" technique is needed to deal with negative K-theory to incorporate it into the same framework. Standard techniques of algebraic topology can be used to show that the two approaches yield the same mod-k K-groups. □

Now we can state some of the remarkable results of Suslin about K-theory of certain fields.

5.3.8. Theorem [SuslinAlgCl]. *Any inclusion of algebraically closed fields induces an isomorphism of mod-k K-groups. Thus the mod-k K-groups of an algebraically closed field of characteristic p are the same as for $\bar{\mathbb{F}}_p$, and so can be read off from Corollary 5.3.6 and from the exact sequence of Theorem 5.3.7.*

Remark. From the exact sequence of Theorem 5.3.7, it is immediate that if F is a field, then

$$K_1(F;\ \mathbb{Z}/k) \cong F^\times/(F^\times)^k$$

(this group already appeared in Theorems 4.4.17 and 4.4.18). This vanishes for algebraically closed fields for all k, but will be non-trivial if some element of F does not have a k-th root in F. Thus one cannot expect an inclusion of an algebraically closed field in another field to induce an isomorphism of mod-k K-groups unless the extension field is also algebraically closed.

5.3.9. Theorem [SuslinLoc]. *Let BU and BO be the classifying spaces for (complex and real, respectively) topological K-theory. The vector bundles*

$$EGL(n,\ \mathbb{C}) \times_{GL(n,\mathbb{C})} \mathbb{C}^n \text{ over } BGL(n,\ \mathbb{C})$$

and

$$EGL(n,\ \mathbb{R}) \times_{GL(n,\mathbb{R})} \mathbb{R}^n \text{ over } BGL(n,\ \mathbb{R})$$

give rise to maps $BGL(\mathbb{C}) \to BU$ and $BGL(\mathbb{R}) \to BO$, and thus to maps $BGL(\mathbb{C})^+ \to BU$ and $BGL(\mathbb{R})^+ \to BO$. These maps induce isomorphisms on mod-k homotopy groups, so that for all $k \geq 2$ and $i > 0$,

$$K_i(\mathbb{C};\ \mathbb{Z}/k) \cong \pi_i(BU;\ \mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k, & i \text{ even}, \\ 0, & i \text{ odd} \end{cases}$$

and

$$K_i(\mathbb{R};\ \mathbb{Z}/k) \cong \pi_i(BO;\ \mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k, & i \equiv 0 \mod 4, \\ 0, & i \equiv 1,\ 2,\ 3 \mod 4 \quad (k \text{ odd}); \end{cases}$$

if k is even, there are suitable $\mathbb{Z}/2$'s coming from the groups $\pi_i(BO) \cong \mathbb{Z}/2$ for $i \equiv 1,\ 2 \mod 8$.

The other fields of greatest interest are number fields F, that is, finite extensions of $\mathbb{Q}$. Since such a field is the field of fractions of a Dedekind ring R, namely the ring of algebraic integers in F, it turns out that the K-theory of F can be computed from that of R and of finite fields (see Theorem 5.3.28 and Example 5.3.29 below). So we next focus attention on rings of integers. For the ordinary integers $\mathbb{Z}$, some specific facts are known. We mentioned previously (Exercise 4.3.20) that $K_2(\mathbb{Z})$ is cyclic of order 2, with generator $\{-1,\ -1\}$.

5.3.10. Theorem [LeeSzc]. *$K_3(\mathbb{Z})$ is cyclic of order 48.*

5.3.11. Theorem [Arl]. *The Hurewicz map $K_4(\mathbb{Z}) \to H_4(\mathrm{St}(\mathbb{Z}),\ \mathbb{Z})$ is an isomorphism. The Hurewicz map $K_5(\mathbb{Z}) \to H_5(\mathrm{St}(\mathbb{Z}),\ \mathbb{Z})$ has cokernel of order 2 and finite kernel.*

In addition, there are two important general theorems about the higher K-theory of rings of integers.

5.3.12. Theorem [QuillenFinGen]. *If R is the ring of integers in a finite extension field F over $\mathbb{Q}$, then $K_i(R)$ is finitely generated for all i.*

5.3.13. Theorem [Borel]. *Suppose R is the ring of integers in a finite extension field F over $\mathbb{Q}$. Write $[F:\mathbb{Q}] = n = r_1 + 2r_2$, where r_1 is the number of distinct embeddings of F into $\mathbb{R}$, and r_2 is the number of distinct conjugate pairs of embeddings of F into $\mathbb{C}$ with image not contained in $\mathbb{R}$. For $i \geq 2$,*

$$K_i(R) \otimes_{\mathbb{Z}} \mathbb{R} \cong \begin{cases} 0, & i \text{ even}, \\ \mathbb{R}^{r_1+r_2}, & i \equiv 1 \mod 4, \\ \mathbb{R}^{r_2}, & i \equiv 3 \mod 4, \end{cases}$$

Quillen's Finite Generation Theorem (5.2.12) and Borel's Theorem (5.3.13) taken together should be viewed as a higher-degree version of the Dirichlet Unit Theorem (Theorem 2.3.8), which showed that $K_1(R)$ is finitely generated, with $K_1(R) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{r_1+r_2-1}$. The upshot of these theorems as far as the K-theory of $\mathbb{Z}$ is concerned is that $K_i(\mathbb{Z})$ is finite for all $i \geq 2$ except when $i \cong 1 \mod 4$, in which case $K_i(\mathbb{Z})$ is a product of an infinite cyclic group and a finite abelian group. A bit is known about the finite groups that appear, in terms of both upper and lower bounds on their sizes, but there is no simple pattern, and there is evidence that the orders of these finite groups are at least in part related to the Bernoulli numbers. (A few examples: it is shown in [Browder] that $K_{8k+3}(\mathbb{Z})$ contains a direct summand isomorphic to $\mathbb{Z}/48$, and it is shown in [Soulé] that $K_{22}(\mathbb{Z})$ surjects onto $\mathbb{Z}/691$.)

Even in the case of the ring of integers R in a general number field F, there seems to be a close link between the K-groups of R and arithmetic properties of the field F, in particular the zeta-function of the field $\zeta_F(s)$. This is the meromorphic function of s obtained by analytic continuation of the Dirichlet series

$$\zeta_F(s) = \sum_{\text{prime ideals } \mathfrak{p}} \frac{1}{1 - |\mathfrak{p}|^{-s}} = \sum_{\text{nonzero ideals } \mathfrak{a}} \frac{1}{|\mathfrak{a}|^s}, \quad \operatorname{Re} s > 1,$$

where $|\mathfrak{p}|$ denotes the norm of an ideal as defined in the proof of Theorem 1.4.19, and we sum over non-zero ideals of R. When $F = \mathbb{Q}$, this is the usual Riemann zeta-function. The functional equation of ζ_F, together with Borel's Theorem (5.3.13), gives rise to the following suggestive observation of Lichtenbaum:

5.3.14. Proposition [Licht, Corollary 2.3]. *The rank of $K_{2k+1}(R)$ is equal to the order of the zero of $\zeta_F(s)$ at $s = -k$.*

In fact, many special values and residues of $\zeta_F(s)$ seem closely related to the orders of the finite summands in the K-groups of R; for some details, see [Licht]. For example, in the case of the Riemann zeta-function,

$$\zeta(-1) = -\frac{1}{12} = -2 \cdot \frac{|K_2(\mathbb{Z})|}{|K_3(\mathbb{Z})|}.$$

The connection with the Bernoulli numbers comes from the classical identity

$$\zeta(-2k+1) = -\frac{B_{2k}}{2k}.$$

Pursuing this matter further leads to relationships between K-theory and étale cohomology in algebraic geometry, and to recent conjectures of Beilinson. For a survey of this whole cycle of ideas, see [Rama].

The Q-Construction and Results Proved with It. Along with the +-construction which we have already discussed, Quillen gave another method for constructing a space $\mathbf{K}(R)$ whose homotopy groups are the higher K-groups of R. This method is now usually known as the Q-construction. Quite a number of variants of the method are known; for a survey of some of them, see [Adams, Chs. 2 and 3]. Much of the power of higher K-theory comes from playing off the +-construction against the Q-construction, for on the face of things, they look very different. The proof that the two constructions give the same K-groups is quite difficult and involved; one version of the proof may be found in [Srinivas, Ch. 7], and another proof is sketched in [Adams, §3.2]. While we will not attempt to go into details of these points, we will explain roughly how to construct K-groups via Quillen's original Q-construction, and will mention a few of the key theorems proved with this construction. For a much more extensive treatment, see §§3–7 of [Srinivas].

To define the Q-construction, we need a slightly more general notion of classifying space than that given in Definition 5.1.16.

5.3.15. Definition. Let $\mathcal{C}$ be a small category, that is, a category whose objects form a set. The **classifying space** $B\mathcal{C}$ of $\mathcal{C}$ is defined to be a CW-complex with one 0-cell for each object of $\mathcal{C}$, and for $n \geq 1$, with one n-cell for each diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

in $\mathcal{C}$, where the X_i's are objects of $\mathcal{C}$, the f_i's are morphisms in $\mathcal{C}$, and we exclude the case where two consecutive X_i's are equal and the morphism between them is the identity morphism. The n-cell associated to

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

is attached in the obvious fashion to any cell of smaller dimension that can be obtained by deleting some X_i and, if $i \neq 0$ or n, replacing f_i and f_{i+1} by $f_{i+1} \circ f_i$. (Any time this leads to an identity morphism, one cancels it.) Note that a functor between small categories induces a cellular map between their classifying spaces.

5.3.16. Examples.

(1) If $\mathcal{C}$ has only one object $*$, and $\mathrm{Hom}_{\mathcal{C}}(*, *) = G$ is a group, then $B\mathcal{C}$ has a single 0-cell $*$, one 1-cell for each $g \neq 1$ in G, and one n-cell

for each n-tuple of elements of G none of which are the identity. Thus if $G = \{1, g\}$ is the cyclic group of order 2, BC has exactly one cell of each dimension. In this case, it is not hard to see that BC is $\mathbb{RP}^\infty$, the infinite real projective space. In general, it is not hard to see that BC coincides with the model for BG constructed in Theorem 5.1.15.

(2) If C has exactly two objects, 0 and 1, and if C has exactly one morphism $0 \xrightarrow{f} 1$ other than the identities id_0 and id_1, then BC has two 0-cells and one 1-cell joining them, so that $BC = [0, 1]$.

(3) It is immediate from the definition of the cells of BC that path-components of BC correspond exactly to connected components of the objects of C, where we say two objects lie in the same connected component if there is a morphism between them.

5.3.17. Lemma (G. Segal). *If $\mathcal{C}$ and $\mathcal{D}$ are small categories, and if $F, G : \mathcal{C} \to \mathcal{D}$ are functors, and there is a natural transformation $F \to G$, then $BF, BG : B\mathcal{C} \to B\mathcal{D}$ are homotopic. Thus if $\mathcal{C}$ has either an initial or a final object, then $B\mathcal{C}$ is contractible.*

Proof (Sketch). A natural transformation $F \to G$ may be viewed as a functor $\mathcal{C} \times \{0, 1\} \to \mathcal{D}$, where $\{0, 1\}$ is the category of example 5.3.16(2). It thus induces a map

$$B(\mathcal{C} \times \{0, 1\}) = B\mathcal{C} \times B\{0, 1\} = B\mathcal{C} \times [0, 1] \to B\mathcal{D}$$

restricting to BF on $B\mathcal{C} \times \{0\}$ and restricting to BG on $B\mathcal{C} \times \{1\}$, in other words a homotopy from BF to BG. If $\mathcal{C}$ has a final object $*$, then we may view $*$ as itself being a subcategory of $\mathcal{C}$ with only one object and only one morphism, and $B*$ is a single point. There is a natural transformation from $\mathcal{C} \xrightarrow{id} \mathcal{C}$ to $\mathcal{C} \to * \hookrightarrow \mathcal{C}$, so the identity map on $B\mathcal{C}$ is homotopic to a map factoring through a point, and $B\mathcal{C}$ is contractible. The case where $\mathcal{C}$ has an initial object is analogous. □

5.3.18. Corollary. *An equivalence of categories yields a homotopy equivalence of classifying spaces.*

Proof. This is immediate from the definition of equivalence and from Lemma 5.3.17. □

Now we can give the Q-construction of the higher K-groups.

5.3.19. Definition [Quillen]. Let $\mathcal{P}$ be a category with exact sequences (as defined in Definition 3.1.1). An **admissible monomorphism** in $\mathcal{P}$ is a morphism $P \rightarrowtail P'$ that can be completed (on the right) to a short exact sequence

$$0 \to P \rightarrowtail P' \to P'' \to 0$$

in $\mathcal{P}$, and an **admissible epimorphism** is a morphism $P \twoheadrightarrow P'$ that can be completed (on the left) to a short exact sequence

$$0 \to P'' \to P \twoheadrightarrow P' \to 0$$

in $\mathcal{P}$. Define $\mathcal{Q}(\mathcal{P})$ to be the category with the same objects as $\mathcal{P}$, with morphisms $\mathrm{Hom}_{\mathcal{Q}(\mathcal{P})}(P_1, P_2)$ the set of all equivalence classes of diagrams

$$P_1 \twoheadleftarrow Q \rightarrowtail P_2,$$

where $\twoheadleftarrow$ and $\rightarrowtail$ are, respectively, an admissible epimorphism and an admissible monomorphism. For example, since we may take $Q = P_1$, any split injection gives a morphism in $\mathcal{Q}(\mathcal{P})$. Two such diagrams are to be identified if there is a commutative diagram

$$\begin{array}{ccccc} P_1 & \longleftarrow & Q & \longrightarrow & P_2 \\ \| & & \downarrow{\scriptstyle\cong} & & \| \\ P_1 & \longleftarrow & Q' & \longrightarrow & P_2. \end{array}$$

To compose arrows, given

$$P_1 \twoheadleftarrow Q \rightarrowtail P_2$$

and

$$P_2 \twoheadleftarrow Q' \rightarrowtail P_3,$$

form

$$P_1 \twoheadleftarrow Q \times_{P_2} Q' \rightarrowtail P_3.$$

Note that strictly speaking, $\mathcal{Q}(\mathcal{P})$ is not necessarily a small category as it stands, but we can replace it by an equivalent small subcategory, and by Corollary 5.3.18, the classifying space of this subcategory is well defined up to homotopy equivalence. With this understanding, we write $B\mathcal{Q}(\mathcal{P})$ for this space. The K-groups of $\mathcal{P}$ are defined to be $K_i(\mathcal{P}) =_{\text{def}} \pi_{i+1}(B\mathcal{Q}(\mathcal{P}))$, computed with respect to the natural basepoint corresponding to the 0-object. (Since there is a morphism in $\mathcal{Q}(\mathcal{P})$ from the 0-object to any other object, $B\mathcal{Q}(\mathcal{P})$ is path-connected.)

If R is a ring and $\mathcal{P}$ is the category of finitely generated projective modules over R, we write simply $K_i(R)$ and $\mathcal{Q}(R)$ for $K_i(\mathcal{P})$ and $\mathcal{Q}(\mathcal{P})$, respectively.

Of course, it would be desirable to know that this definition agrees with our earlier definitions of K_0 and K_1 for categories in Definition 3.1.6, and with Definition 5.2.6 for rings. In fact this is the case for K_0 of categories and for general K-groups of rings, though there are known to be some cases of categories for which Definition 3.1.6 and Definition 5.3.19 give different K_1-groups. Fortunately we are really only interested in the case of rings, where the two competing definitions of higher K-groups coincide. We state this result as a theorem, but as the proof is quite difficult and involved, we refer the reader to [Srinivas, Ch. 7] and to [Adams, §3.2].

5.3.20. Theorem (Quillen–see [Srinivas, Theorem 7.7]**).** *For any ring R, there is a natural homotopy equivalence $\Omega B\mathcal{Q}(R) \to \mathbf{K}(R)$, inducing natural isomorphisms between the K-groups of Definition 5.3.19 and those of Definition 5.2.6.*

The one part of this result that is elementary is the proof that

$$\pi_1(B\mathcal{Q}(R)) \cong K_0(R)$$

([Quillen, §2, Theorem 1] and [Srinivas, Example 4.10]), which actually works for categories: $\pi_1(B\mathcal{Q}(\mathcal{P})) \cong K_0(\mathcal{P})$ for any category $\mathcal{P}$ with exact sequences. Each object P of $\mathcal{P}$ defines a loop in $B\mathcal{Q}(\mathcal{P})$, since there are two distinct paths from the basepoint to the 0-cell of $B\mathcal{Q}(\mathcal{P})$ corresponding to P, corresponding to the two morphisms $0 \twoheadleftarrow P \xrightarrow{id} P$ and $0 \twoheadleftarrow 0 \rightarrow P$ in $\mathcal{Q}(\mathcal{P})$. It is easy to see that sending $[P]$ to the class of this loop gives a map $K_0(\mathcal{P}) \rightarrow \pi_1(B\mathcal{Q}(\mathcal{P}))$, and only slightly harder to see that this is an isomorphism.

The principal advantage of the Q-construction over the +-construction is that it can be applied not only to the finitely generated projective modules over a ring but also to more general categories. Even if one is only interested in the K-groups of a ring R, the easiest approach to computing the K-groups is often to find other categories of R-modules for which the K-groups are computable, then to relate these to $K_i(R)$. The reader may find many examples of this technique in [Quillen] and [Srinivas]. The first basic tool is the following (cf. Proposition 3.1.9).

5.3.21. Lemma. *If $F : \mathcal{P} \rightarrow \mathcal{M}$ is an exact functor between categories with exact sequences, then F induces a map $BQF : B\mathcal{Q}(\mathcal{P}) \rightarrow B\mathcal{Q}(\mathcal{M})$, and in particular induces maps $F_* : K_i(\mathcal{P}) \rightarrow K_i(\mathcal{M})$.*

Proof. Since F preserves exact sequences, it induces a functor $QF : \mathcal{Q}(\mathcal{P}) \rightarrow \mathcal{Q}(\mathcal{M})$, and thus a map between classifying spaces by 5.3.15. □

5.3.22. Example. For an example of 5.3.21, note that a homomorphism $\varphi : R \rightarrow S$ of rings induces an exact functor $\varphi_* : \mathbf{Proj}\, R \rightarrow \mathbf{Proj}\, S$ by $P \mapsto S \otimes_R P$. The exactness here depends on the fact that every short exact sequence of projective modules splits. We conclude that $\varphi : R \rightarrow S$ of rings induces maps $\varphi_* : K_i(R) \rightarrow K_i(S)$.

Now we may generalize the G-groups of Chapter 3 as follows.

5.3.23. Definition [Quillen]. Let R be a ring. (Often one wants to assume that R is left Noetherian, which ensures that any submodule of a finitely generated R-module is finitely generated.) Define $G_i(R)$ (or $K_i'(R)$) by $G_i(R) =_{\text{def}} K_i(R\text{-}\mathbf{Mod}_{\mathbf{fg}})$. The obvious inclusion functors $\mathcal{Q}(\mathbf{Proj}\, R) \hookrightarrow \mathcal{Q}(R\text{-}\mathbf{Mod}_{\mathbf{fpr}}) \hookrightarrow \mathcal{Q}(R\text{-}\mathbf{Mod}_{\mathbf{fg}})$ (see Examples 3.1.2(3) and (4)) give natural homomorphisms $K_i(R) \rightarrow K_i(R\text{-}\mathbf{Mod}_{\mathbf{fpr}}) \rightarrow G_i(R)$.

Note that in general, a ring homomorphism $\varphi : R \rightarrow S$ does *not* in general induce an exact functor $R\text{-}\mathbf{Mod}_{\mathbf{fg}} \xrightarrow{\varphi_*} S\text{-}\mathbf{Mod}_{\mathbf{fg}}$. However, $M \mapsto S \otimes_R M$ is exact if S is flat over R, so in this case we obtain a map $\varphi_* : G_i(R) \rightarrow G_i(S)$. If S is finitely generated as an R-module via φ, then each finitely generated S-module may be viewed as a finitely generated R-module, and without any flatness assumption we obtain an exact functor $S\text{-}\mathbf{Mod}_{\mathbf{fg}} \xrightarrow{\varphi_*} R\text{-}\mathbf{Mod}_{\mathbf{fg}}$ and maps $\varphi^* : G_i(S) \rightarrow G_i(R)$, known as the **transfer homomorphisms**.

Now we can state some of the results proved using the Q-construction. These include analogues of all the major results of Chapter 3. In many cases, the method is more powerful than the result may indicate, and with more work can be used to prove something more delicate.

5.3.24. Theorem ("Devissage"—cf. Theorem 3.1.8). *Let $\mathcal{A}$ be an abelian category in which every simple object is isomorphic to one and only one element of some set $S \subseteq \operatorname{Obj}\mathcal{A}$, and let $\mathcal{A}_{\mathrm{ss}}$, $\mathcal{A}_{\mathrm{fl}}$ be the full subcategories of semisimple objects (finite direct sums of simple objects) and objects of finite length, respectively. Then the inclusion $\mathcal{A}_{\mathrm{ss}} \hookrightarrow \mathcal{A}_{\mathrm{fl}}$ induces isomorphisms*

$$\bigoplus_{M\in S} K_i(\operatorname{End}_{\mathcal{A}}(M)^{\mathrm{op}}) \to K_i(\mathcal{A}_{\mathrm{fl}}).$$

Remarks on the proof. See [Quillen, Theorem 4] and [Srinivas, Theorem 4.8]. It is convenient to slightly generalize the notion of category with exact sequences so as to cover $\mathcal{A}_{\mathrm{ss}}$ (which isn't closed under extensions in $\mathcal{A}$); then there are two steps, to show $BQ(\mathcal{A}_{\mathrm{ss}}) \to BQ(\mathcal{A}_{\mathrm{fl}})$ is a homotopy equivalence, and then to observe that

$$K_i(\mathcal{A}_{\mathrm{fl}}) \cong \bigoplus_{M\in S} K_i(\operatorname{End}_{\mathcal{A}}(M)^{\mathrm{op}}). \quad \square$$

5.3.25. Theorem ("Resolution theorem"—cf. Theorem 3.1.13). *Suppose $\mathcal{M}$ and $\mathcal{P}$ are categories with exact sequences, both contained in the same abelian category $\mathcal{A}$, and with $\mathcal{P}$ a full subcategory of $\mathcal{M}$. Also assume:*

(1) *that for each object $M \in \operatorname{Obj}\mathcal{M}$, there is a finite resolution by objects of $\mathcal{P}$, i.e., an exact sequence (3.1.3) in $\mathcal{M}$ of finite length with $P_j \in \operatorname{Obj}\mathcal{P}$;*
(2) *that if*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$ and $M_2, M_3 \in \operatorname{Obj}\mathcal{M}$ (resp., $\operatorname{Obj}\mathcal{P}$), then $M_1 \in \operatorname{Obj}\mathcal{M}$ (resp., $\operatorname{Obj}\mathcal{P}$). (In other words, $\mathcal{M}$ and $\mathcal{P}$ each contain the kernels of each of their morphisms which are epimorphisms in $\mathcal{A}$.)

Then the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}$ induces isomorphisms of K-groups.

Remarks on the proof. See [Quillen, Theorem 4] and [Srinivas, Theorem 4.8]. $\square$

5.3.26. Corollary. *For any ring R, the inclusion functor* $\mathbf{Proj}\, R \hookrightarrow R\text{-}\mathbf{Mod}_{\mathrm{fpr}}$ *induces isomorphisms of K-groups. Thus if R is a left regular ring, the inclusion functor* $\mathbf{Proj}\, R \hookrightarrow R\text{-}\mathbf{Mod}_{\mathrm{fg}}$ *induces isomorphisms $K_i(R) \to G_i(R)$ for all i.*

5.3.27. Theorem ("Localization Theorem"). *Let $\mathcal{A}$ be a small abelian category and let $\mathcal{B}$ be an additive subcategory closed under taking subobjects, quotients, and extensions in $\mathcal{A}$. Under these hypotheses, there is a well-defined quotient abelian category $\mathcal{A}/\mathcal{B}$ with the same objects as $\mathcal{A}$ and with morphisms obtained from those of $\mathcal{A}$ by inverting morphisms with kernels and cokernels in $\mathcal{B}$, and the natural functors*

$$\mathcal{B} \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

induce a fibration (up to homotopy)

$$BQ(\mathcal{B}) \to BQ(\mathcal{A}) \to BQ(\mathcal{A}/\mathcal{B})$$

and thus a long exact sequence of homotopy groups

$$\cdots \to K_{i+1}(\mathcal{A}/\mathcal{B}) \to K_i(\mathcal{B}) \to K_i(\mathcal{A}) \to K_i(\mathcal{A}/\mathcal{B}) \to K_{i-1}(\mathcal{B}) \to \cdots .$$

Remarks on the proof. See [Srinivas, Appendix B] for a description of the construction of $\mathcal{A}/\mathcal{B}$ and [Quillen, Theorem 5] and [Srinivas, Theorem 4.9] for the theorem itself. □

5.3.28. Corollary. *If R is a Dedekind domain with field of fractions F, there is a long exact sequence*

$$\cdots \to K_{i+1}(F) \to \bigoplus_{\mathfrak{p} \triangleleft R \text{ maximal}} K_i(R/\mathfrak{p}) \to K_i(R) \to K_i(F) \to \cdots .$$

Remarks on the proof. This comes from applying the Localization Theorem to the case where $\mathcal{A}$ is the category of finitely generated R-modules and where $\mathcal{B}$ is the category of finitely generated torsion R-modules. The K-groups of $\mathcal{A}$ are $G_i(R) \cong K_i(R)$ by Corollary 5.3.26. The K-groups of $\mathcal{B}$ are the direct sums of the K-groups of the $R/\mathfrak{p}$'s, by Theorem 5.3.24. The category $\mathcal{A}/\mathcal{B}$ can easily be identified with the category of finite-dimensional vector spaces over F, so its K-groups are those of F and the result follows. □

5.3.29. Example. We may apply Corollary 5.3.28 to the case where F is a number field, and R is the ring of integers in F. Then the $R/\mathfrak{p}$'s are all finite fields, so their K-groups are finite and vanish in even degrees (> 0) by Theorem 5.3.2. So applying Theorems 5.3.12 and 5.3.13, we see that the K-groups of F are countably generated as abelian groups, with $K_i(F)\otimes_{\mathbb{Z}}\mathbb{Q}$ a $\mathbb{Q}$-vector space of dimension 1 if $i = 0$, ∞ if $i = 1$, 0 if $i \geq 2$ is even, $r_1 + r_2$ if $i \geq 5$ is $\equiv 1 \mod 4$, and r_2 if $i \geq 3$ is $\equiv 3 \mod 4$.

In fact, since the K-groups of finite fields vanish in even degrees (> 0), the long exact localization sequence splits into shorter exact sequences. For instance we have

$$0 \to K_3(\mathbb{Z}) \cong \mathbb{Z}/48 \to K_3(\mathbb{Q}) \to 0,$$

$$0 \to K_2(\mathbb{Z}) \cong \mathbb{Z}/2 \to K_2(\mathbb{Q}) \to \bigoplus_{p \text{ prime}} \mathbb{F}_p^\times \to \{\pm 1\} \to \mathbb{Q}^\times \to \bigoplus_{p \text{ prime}} \mathbb{Z} \to 0.$$

The last of these sequences contains in it the calculation of $K_2(\mathbb{Q})$ (cf. Theorem 4.4.9) as well as the Fundamental Theorem of Arithmetic, which says in effect that $\mathbb{Q}^\times \cong \{\pm 1\} \times \bigoplus_{p \text{ prime}} \mathbb{Z}$.

5.3.30. Theorem (Fundamental Theorem—cf. Theorem 3.2.22). *Let R be ring and (as in Theorem 3.2.22) let* **Nil** *R be the category whose objects are pairs (P, A) consisting of a finitely generated projective module P over R and a nilpotent R-endomorphism A of P. The K-groups of* **Nil** *R naturally split as $K_i(R) \oplus \mathrm{Nil}_i(R)$. Furthermore, there are natural isomorphisms*

$$K_i(R[t]) \cong K_i(R) \oplus \mathrm{Nil}_{i-1}(R),$$

$$K_i(R[t, t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R) \oplus \mathrm{Nil}_{i-1}(R) \oplus \mathrm{Nil}_{i-1}(R).$$

Traditionally one writes $NK_i(R)$ for $\mathrm{Nil}_{i-1}(R)$.

If R is left regular, all the Nil-*groups and negative K-groups of R vanish, and $K_i(R[t]) \cong K_i(R)$, $K_i(R[t, t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R)$ for all i.*

Remarks on the proof. See [Quillen, Theorem 8] and [Srinivas, Theorem 5.2] for the case where R is left regular, which is considerably easier. In this case, vanishing of the Nil-groups follows from the Resolution and Devissage Theorems, and for the Fundamental Theorem itself, because of Corollary 5.3.6, we can replace K_i by G_i and work with the abelian category of finitely generated modules, then apply Theorem 5.3.27 (Localization). The general case requires a modification of the Localization Theorem for categories with exact sequences which are not abelian, which is harder to state. See [Gersten2] and [Srinivas, Theorem 9.8]. □

Applications. Since the groups $K_i(R)$ for $i \geq 3$ are defined in a somewhat indirect way compared with the groups $K_i(R)$ for $i \leq 2$, it is harder to find direct applications of them. Nevertheless, these groups also appear in algebraic geometry, number theory, topology, and analysis. A quick sketch of some of the presumed applications in number theory and algebraic geometry appeared in the subsection on "K-theory of fields and of rings of integers." Other applications to algebraic geometry have to do with analogues of the Riemann-Roch Theorem for higher-dimensional algebraic varieties (recall Exercise 3.1.25) and with the so-called **Chow ring** of a variety. We will say just a bit about these topics now; for more details on the algebraic geometry background of the subject, see [Hartshorne, Appendix A], and for more details on how K-theory comes in, see [Srinivas, §§5, 8, and 9].

The Chow ring is an analogue for algebraic varieties (or schemes) of the cohomology ring of a topological space, particularly in the case of a manifold. It gathers information about algebraic subvarieties and how they

intersect with one another. More precisely, if X is an irreducible algebraic variety of dimension n over a field k (say $\mathbb{C}$), the Chow ring $CH^\bullet(X)$ is a commutative graded ring, with $CH^k(X)$ constructed as the free abelian group on the (Zariski-)closed irreducible subvarieties of X of codimension k, modulo a certain equivalence relation called **rational equivalence**. The ring structure on $CH^\bullet(X)$ comes from **intersection**: if Y is a closed subvariety of codimension p and if Z is a closed subvariety of codimension q, then one "wiggles" Y and Z within their respective equivalence classes so that they intersect properly (that is, in a finite union of irreducible subvarieties each of codimension $p+q$), and $[Y]\cdot[Z]$ is the class of this intersection, counted with suitable (possibly negative) multiplicities attached to the various intersection components, in $CH^{p+q}(X)$. For example, if $X = \mathbb{P}^n(\mathbb{C})$, $CH^\bullet(X) \cong \mathbb{Z}[y]/(y^{n+1})$, the truncated polynomial ring on a generator y in degree one, corresponding to the subvariety $Y = \mathbb{P}^{n-1}(\mathbb{C}) \hookrightarrow \mathbb{P}^n(\mathbb{C}) = X$ (a hyperplane section). In this case, all hyperplane sections are in the same rational equivalence class, and y^j corresponds to the intersection of j generic linear hyperplanes, which if $j \leq n$ is just the class of the linear subvariety $\mathbb{P}^{n-j}(\mathbb{C}) \hookrightarrow \mathbb{P}^n(\mathbb{C}) = X$.

For a general non-singular variety X, it is clear that $CH^0(X)$ is the free abelian group on $[X]$, hence $\cong \mathbb{Z}$, and that $CH^1(X)$ is a group of equivalence classes of divisors on X, which is known to coincide with $\text{Pic}(X)$, the group of isomorphism classes of algebraic line bundles, via the correspondence $D \mapsto \mathcal{L}_D$ discussed in Exercise 3.1.25. It is rather easy to show that $\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ is the sheaf of germs of invertible algebraic functions. (See [Hartshorne, Ch. III, Exercise 4.5].) For a long time, it was an open problem to give a comparable description of the higher Chow groups $CH^k(X)$ as cohomology groups of some sort.

This problem was solved by Quillen in the case of non-singular varieties, or more precisely, regular schemes X of finite type over a field k, using the higher K-groups of the variety. The answer (known as **Bloch's formula**, see [Quillen, Theorem 5.19] and [Srinivas, Corollary 5.27]) is that $CH^k(X) \cong H^k(X, \mathcal{K}_{k,X})$, where $\mathcal{K}_{k,X}$ is the sheaf given by "sheafifying" the presheaf $U \mapsto K_k(U)$, for U a Zariski-open subset of X. Since it is easy to see that $\mathcal{K}_{k,X}$ is the constant sheaf $\mathbb{Z}$ when $k = 0$ and the sheaf $\mathcal{O}_X^\times$ when $k = 1$, Bloch's formula generalizes the classical formulas $CH^0(X) \cong H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ and $CH^1(X) \cong H^1(X, \mathcal{O}_X^\times) \cong \text{Pic}(X)$. The case of singular varieties is substantially more complicated, but it is now clear that the study of the higher Chow groups $CH^k(X)$ is inextricably linked with higher algebraic K-theory.

For somewhat related reasons, the higher K-groups of a variety are also related to the Riemann-Roch problem, discussed in Exercise 3.1.25, of computing the Euler characteristic map $\chi : K_0(\mathbf{Vect}\, X) \to \mathbb{Z}$, the reason being that $K_0(\mathbf{Vect}\, X)$ is related to the higher K-groups on subvarieties of X via repeated use of the long exact K-theory sequences. Alternatively, one can see this via the fact (see [Hartshorne, Appendix A]) that in the case of a non-singular projective variety, the map χ can be shown to factor through

a "Chern character" map

$$K_0(\mathbf{Vect}\, X) \to CH^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

while the higher higher Chow groups are related to higher algebraic K-theory via Bloch's formula. For more on this topic, see Gillet's survey in [LluisP].

We conclude this section with a discussion of a nice geometric description of higher algebraic K-theory, due to Max Karoubi, which can be used to relate algebraic K-theory to problems about flat vector bundles. This particular description of the +-construction will also be useful in the next chapter—see Exercise 6.2.25.

5.3.31. Definition [KaroubiHomCyc]. Let G be a group whose commutator subgroup is perfect; the main case of interest will be where $G = GL(R)$ for some ring R. Let X be a CW-complex. A **virtual flat G-bundle** over X is a diagram $E \xrightarrow{\pi} Y \xrightarrow{f} X$, where E and Y are CW-complexes, π is a Galois covering map with covering group G, and f is a fibration with acyclic fibers, that is, whose fibers have vanishing reduced integral homology. We will mostly be interested in the case where Y and X are connected, in which case $E \xrightarrow{\pi} Y$ is determined by a map $\pi_1(Y) \to G$ (unique up to conjugacy). Two virtual flat G-bundles over X, $E \xrightarrow{\pi} Y \xrightarrow{f} X$ and $E' \xrightarrow{\pi'} Y' \xrightarrow{f'} X$, are said to be equivalent if there is a virtual flat G-bundle $E_1 \xrightarrow{\pi_1} Y_1 \xrightarrow{f_1} X$ and a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\sigma} & Y_1 & \xleftarrow{\sigma'} & Y' \\ \| & & \downarrow{\scriptstyle f_1} & & \| \\ Y & \xrightarrow{f} & X & \xleftarrow{f'} & Y' \end{array}$$

such that $\sigma^*(E_1) \cong E$, $\sigma'^*(E_1) \cong E'$. This relation is obviously reflexive and symmetric.

5.3.32. Theorem [KaroubiHomCyc, Appendice I]. *Let G be a group whose commutator subgroup is perfect and let X be a CW-complex. Then equivalence of virtual flat G-bundles over X is an equivalence relation, and the equivalence classes are in natural bijection with the set of homotopy classes of maps $X \to BG^+$. In particular, taking $G = GL(R)$, the equivalence classes of virtual flat G-bundles over S^i are in natural bijection with $K_i(R)$ for $i \geq 1$.*

Proof. First we need to prove transitivity. Suppose one has virtual bundles E over Y, E' over Y', E'' over Y'', E_1 over Y_1, and E_2 over Y_2, and commutative diagrams

$$\begin{array}{ccccc} Y & \xrightarrow{\sigma} & Y_1 & \xleftarrow{\sigma'} & Y' \\ \| & & \downarrow{\scriptstyle f_1} & & \| \\ Y & \xrightarrow{f} & X & \xleftarrow{f'} & Y' \end{array} \qquad \begin{array}{ccccc} Y' & \xrightarrow{\rho} & Y_2 & \xleftarrow{\rho'} & Y'' \\ \| & & \downarrow{\scriptstyle f_2} & & \| \\ Y' & \xrightarrow{f'} & X & \xleftarrow{f''} & Y'' \end{array}$$

such that $\sigma^*(E_1) \cong E$, $\sigma'^*(E_1) \cong E'$, $\rho^*(E_2) \cong E'$, $\rho'^*(E_2) \cong E''$. We may assume from the homotopy point of view (using mapping cylinders) that σ' and ρ are the inclusions of Y' as subcomplexes of Y_1 and Y_2, respectively. Then form $Z = Y_1 \cup Y_2$. Since f_1 and f_2 are each extensions of $f' : Y' \to X$, they together define a map $g : Z \to X$. Replacing Z be a homotopy-equivalent complex, we may assume g is a fibration, and it will still have acyclic fibers. Now we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\sigma} & Z & \xleftarrow{\rho'} & Y' \\ \| & & \downarrow{\scriptstyle g} & & \| \\ Y & \xrightarrow{f} & X & \xleftarrow{f''} & Y''. \end{array}$$

We might as well assume X is connected, in which case E_1 and E_2 are determined by maps $\pi_1(Y_1) \to G$ and $\pi_1(Y_2) \to G$ which we can assume coincide on the image of $\pi_1(Y')$. By Van Kampen's Theorem, $\pi_1(Z) = \pi_1(Y_1) \underset{\text{im } \pi_1(Y')}{*} \pi_1(Y_2)$, so we obtain a uniquely determined map $\pi_1(Z) \to G$ and thus a flat G-bundle E_3 over Z. By construction, the pull-backs of E_3 to Y and Y'' are equivalent to E and E'', respectively, and so E and E'' are equivalent. Thus equivalence is indeed an equivalence relation.

Next, we associate to a virtual flat G-bundle $E \xrightarrow{\pi} Y \xrightarrow{f} X$ over X a map $X \to BG^+$ as follows. Without loss of generality we may assume X is connected (otherwise work separately on each component). Since the fiber F of $Y \xrightarrow{f} X$ is acyclic, $\pi_1(F)$ is perfect, and its image in $\pi_1(Y)$ is a perfect normal subgroup. Since $Y \xrightarrow{f} X$ kills this subgroup and is a homology equivalence, by Theorem 5.2.2 we may identify $Y \xrightarrow{f} X$ (up to homotopy equivalence) with the inclusion $Y \hookrightarrow Y^+$, where the +-construction is performed with respect to the image of $\pi_1(F)$. Now $E \xrightarrow{\pi} Y$ corresponds to a map $\pi_1(Y) \to G$ or to a map of spaces $Y \to BG$. By Proposition 5.2.4, there is an induced map $Y^+ = X \to BG^+$.

Next we claim that the homotopy class of the induced map $Y^+ = X \to BG^+$ only depends on the equivalence class of $E \xrightarrow{\pi} Y \xrightarrow{f} X$. Indeed, if E is equivalent to E' via a diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\sigma} & Y_1 & \xleftarrow{\sigma'} & Y' \\ \| & & \downarrow{\scriptstyle f_1} & & \| \\ Y & \xrightarrow{f} & X & \xleftarrow{f'} & Y' \end{array}$$

such that $\sigma^*(E_1) \cong E$, $\sigma'^*(E_1) \cong E'$, then we obtain a homotopy-commuta-

tive diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\sigma} & Y_1 & \xleftarrow{\sigma'} & Y' \\ \| & & \downarrow{\scriptstyle f_1} & & \| \\ Y & \xrightarrow{f} & X & \xleftarrow{f'} & Y' \\ \downarrow & & \downarrow & & \downarrow \\ BG^+ & = & BG^+ & = & BG^+, \end{array}$$

which is exactly what we need. Thus we have a well-defined map from equivalence classes of virtual flat G-bundles over X to $[X, BG^+]$. This map is surjective, since given $g : X \to BG^+$, we can assume from the homotopy point of view that $BG \xrightarrow{\iota} BG^+$ is a fibration, and then form the pull-back fibration $g^*\iota : Y \to X$. Since ι is a homology equivalence, this will have acyclic fibers. It also comes with a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & BG \\ \downarrow{\scriptstyle g^*\iota} & & \downarrow{\scriptstyle \iota} \\ X & \xrightarrow{g} & BG^+ \end{array}$$

defining a G-bundle over Y, and so we get a virtual flat G-bundle over X mapping to the homotopy class of g.

Finally, we need to show that if $E \xrightarrow{\pi} Y \xrightarrow{f} X$ and $E' \xrightarrow{\pi'} Y' \xrightarrow{f'} X$ define homotopic maps $X \to BG^+$, then they are equivalent. We have a homotopy-commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f_\pi} & BG \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \iota} \\ X & \xrightarrow{g} & BG^+ \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle \iota} \\ Y' & \xrightarrow{f_{\pi'}} & BG. \end{array}$$

As before construct the pull-back fibration $g^*\iota : Z \to X$, which has acyclic fibers. From the diagram, $Y \xrightarrow{f_\pi} BG$ and $Y' \xrightarrow{f_{\pi'}} BG$ are both pulled back from $Z \to BG$, so E and E' are equivalent.

If $G = GL(R)$, then $BGL(R)^+$ is simple (Theorem 5.2.12), so in computing homotopy classes of maps $X \to BG^+$ with X a connected CW-complex, it doesn't matter whether one uses based or unbased maps. Thus the last statement follows. □

5.3.33. Remark. In the case where $G = GL(R)$, one can define a flat virtual $GL(R)$-bundle over X from any flat virtual R-bundle over X. One of these is defined the same way, except that we require $E \xrightarrow{\pi} Y$ to have

fibers which are finitely generated projective modules over R, with covering transformations that are R-module automorphisms. If X is a **finite** CW-complex, it is not too hard to show [KaroubiHomCyc, III] that every virtual $GL(R)$-bundle over X comes from a flat virtual R-bundle, basically because every map $X \to BGL(R)^+$ will factor through $BGL(n, R)^+$ for some sufficiently large n. Similarly [KaroubiHomCyc, §3.12], one can show that the product of Theorem 5.3.1 comes from a tensor product operation on flat virtual R-bundles.

Note that Theorem 5.3.32 and Remark 5.3.33, taken together, now give a somewhat more concrete way to visualize classes in $K_i(R)$, namely, as equivalence classes of flat R-bundles on homology spheres. Furthermore, the pairing $K_i(R) \times H^i(GL(R), A) \to A$, with A an abelian group, defined by pairing the image of a K-theory class under the Hurewicz homomorphism with a cohomology class, may be viewed more geometrically: given a virtual flat G-bundle $E \xrightarrow{\pi} X \xrightarrow{f} S^i$ with f a fibration with acyclic fibers, and given $c \in H^i(GL(R), A)$, we obtain $f_\pi^*(c) \in H^i(X, A)$. Since X is a homology sphere, there is a "fundamental class" $[X] \in H_i(X, \mathbb{Z})$ mapping to the fundamental class of S^i, and we merely take the pairing $\langle c, [X]\rangle$.

5.3.34. Exercise (Suslin [SuslinAlgCl]**).** Let F be an algebraically closed field and let $L \supseteq F$ be any extension field of F.

(1) Show that the natural map $K_i(F) \to K_i(L)$ is injective for all i. (Hint: $L = \varinjlim R$, where R ranges over the finitely generated F-subalgebras over L, directed by inclusion. Since K-theory commutes with direct limits, $K_i(L) = \varinjlim K_i(R)$, and any element of the kernel of $K_i(F) \to K_i(L)$ must lie in the kernel of $K_i(F) \to K_i(R)$ for some finitely generated commutative F-algebra R. Using Hilbert's Nullstellensatz [Jacobson, II, §7.11], show that the inclusion map $F \hookrightarrow R$ has a splitting $R \to F$, and thus that the map $K_i(F) \to K_i(R)$ is split injective for all i.)
(2) Show by example that $K_i(F) \to K_i(L)$ need not be injective if F is not algebraically closed, and examine where the above argument breaks down.

5.3.35. Exercise. Use the Localization Theorem (5.3.27) applied to the Euclidean ring $R = \mathbb{Z}[i]$ (the Gaussian integers) to find a short exact sequence relating $K_2(\mathbb{Z}[i])$ to $K_2(\mathbb{Q}[i])$.

5.3.36. Exercise. Let F be an infinite field. Show, using techiniques similar to those in Exercise 5.3.34, that if K is a *purely transcendental* field extension of F, the inclusion $F \hookrightarrow K$ induces an injection on all K-groups. Hint: first reduce to the case of transcendence degree 1, so that $K = F(t)$. View K as an inductive limit of rings R obtained by inverting finitely many irreducible polynomials in $F[t]$. Show, using the assumption that F is infinite, that for such a ring R, the inclusion $F \hookrightarrow R$ has a splitting, and then deduce the result.

5.3.37. Exercise. Let F be a field. By Theorem 5.3.30, $K_i(F[t]) \cong K_i(F)$ for all i.

(1) Let $a \in F$ and consider the ring $R = F[t, (t-a)^{-1}]$ obtained from $F[t]$ by inverting $t-a$. Use the Localization Theorem (5.3.27) to show that $K_i(R) \cong K_i(F) \oplus K_{i-1}(F)$. (Initially you only get an exact sequence, but a splitting for the boundary map $K_i(R) \xrightarrow{\partial} K_{i-1}(F)$ can be obtained via the map $K_{i-1}(F) \to K_{i-1}(R)$ induced by the inclusion, followed by the product (in the sense of Theorem 5.3.1) with the class of $t-a$ in $R^\times \cong K_1(R)$.)

(2) Generalize (1) to show that if $a_1, \ldots, a_n$ are distinct elements of F, then

$$K_i\left(F\left[t, (t-a_1)^{-1}, \ldots, (t-a_n)^{-1}\right]\right) \cong K_i(F) \oplus \bigoplus_{j=1}^{n} K_{i-1}(F).$$

(3) If F is algebraically closed, show that the rational function field $F(t)$ is the direct limit of the rings considered in (2). Deduce a calculation of $K_i(F(t))$ in terms of the K-theory of F.

(4) What would be different in the calculation of $K_i(F(t))$ if F is not algebraically closed? How is Exercise 5.3.36 relevant here?

5.3.38. Exercise. Deduce from Remark 5.3.33 and the surjectivity of the map $K_2(\mathbb{R}) \to K_2^{\text{top}}(\mathbb{R}) \cong \widetilde{KO}(S^2)$ the perhaps surprising fact that there is a CW-complex with the integral homology of S^2 admitting a flat real vector bundle which is topologically stably non-trivial. (For readers who know about characteristic classes, the non-triviality is detected by the second Stiefel-Whitney class w_2.)

5.3.39. Exercise. Compute the groups $K_i(\overline{\mathbb{F}}_p; \mathbb{Z}/k)$ for the algebraic closure $\overline{\mathbb{F}}_p$ of the field of p elements, p a prime, in the two cases where k is a power of p and where k is relatively prime to p. Compare the results with Suslin's calculations (Theorem 5.3.9) in the case of $\mathbb{C}$.

5.3.40. Exercise. Let D be a division algebra of dimension d^2 over its center F. (For instance, if D is a quaternion algebra, $d = 2$.) Show using the "transfer map" (which comes from the forgetful functor from finitely generated left D-modules to finitely generated left F-modules) that the natural maps

$$K_i(F; \mathbb{Z}/k) \to K_i(D; \mathbb{Z}/k)$$

are isomorphisms if k is relatively prime to d. (See for example [GSR] and its review in *Mathematical Reviews* **58**, #852.)

6

Cyclic homology and its relation to K-Theory

1. Basics of cyclic homology

In this chapter, we introduce the reader to the homology theory for algebras known as **cyclic homology**. As we shall see in the next section, cyclic homology may be viewed as the "linearization" of K-theory, in the same sense in which the matrix ring $M_n(R)$ is the "linearization" of the general linear group $GL(n, R)$. For motivation, it is useful to think of the case where the ring R is $\mathbb{R}$ or $\mathbb{C}$. Then $GL(n, R)$ is a Lie group, and the space $BGL(n, R)^+$ giving rise to the higher K-groups is by its construction an H-space whose homology agrees with the homology of this Lie group (never mind for the moment that we are forgetting the topology!). On the other hand, one of the basic principles of Lie theory is that the best way to study Lie groups (which are "non-linear" objects) is often by way of their Lie algebras (which can be studied using linear algebra). For example, it is a famous result that for any compact Lie group G with Lie algebra $\mathfrak{g}$, the topological cohomology $H^\bullet(G; \mathbb{R})$ (here we are just thinking of G as a space and ignoring the group structure) is canonically isomorphic to the Lie algebra cohomology $H^\bullet(\mathfrak{g}; \mathbb{R})$, which can at least in principle be computed using only finite-dimensional linear algebra. This suggests that some construction with the Lie algebra $M_n(R)$ of $GL(n, R)$ ought to yield a reasonable approximation to the K-theory of R, at least in the limit as $n \to \infty$. While the actual construction of cyclic homology and of the Chern character which relates it to K-theory is a bit roundabout, this philosophy turns out to be basically correct.

Hochschild Homology. Cyclic homology is in fact a modification of a better-known homology theory for algebras, known as Hochschild homology. We begin by defining the latter, and then by discussing the definition(s) of cyclic homology and the relationship between the two homology theories. The connection with K-theory will be saved for the next section of this chapter.

6.1.1. Definition. Let k be a commutative ring and let R be a k-algebra. (In practice, k will usually be either $\mathbb{Z}$, in which case R is just an arbitrary ring, or a field. While it is possible to generalize the theory to the non-unital case, we will always assume here that R has an identity.) We write $R^{\otimes n}$ for

$$\underbrace{R \otimes_k R \cdots \otimes_k R}_{n \text{ factors}}.$$

The **Hochshild homology** of R (here k is understood, since it plays a role in the definition) is by definition the homology $HH_\bullet(R)$ of the complex

$$C_\bullet(R): \quad \cdots \xrightarrow{b_{n+2}} R^{\otimes n+2} \xrightarrow{b_{n+1}} R^{\otimes n+1} \xrightarrow{b_n} R^{\otimes n} \xrightarrow{b_{n-1}} \cdots \xrightarrow{b_1} R,$$

where $R^{\otimes n+1}$ occurs in degree n and the boundary map b is the k-linear map defined by the formula

$$b_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = b'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) + (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}),$$
(6.1.2)

where

$$b'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

Since the differential is k-linear, $HH_i(R)$ is a k-module for each i (though usually not an R-module). It is useful to note, however, that if R is commutative, b' and b commute with multiplication by R on the left, so $HH_i(R)$ is an R-module.

The fact that $b^2 = 0$ is fairly easy to check here, since we can rewrite b_n as $\sum_{i=0}^{n} (-1)^i d_i^n$, where

$$d_i^n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, & i < n, \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, & i = n. \end{cases}$$

Then

$$b_{n-1} \circ b_n = \left(\sum_{j=0}^{n-1} (-1)^j d_j^{n-1} \right) \circ \left(\sum_{i=0}^{n} (-1)^i d_i^n \right) = \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} d_j^{n-1} \circ d_i^n,$$

but for $j < i$,

$$d_j^{n-1} \circ d_i^n = d_{i-1}^{n-1} \circ d_j^n, \tag{6.1.3}$$

so each term occurs twice with opposite signs.

This rather *ad hoc* definition has an explanation which makes it seem somewhat more canonical.

6.1.4. Proposition. *Let k be a commutative ring and let R be a k-algebra which is projective as a module over k (this is of course automatic if k is a field). The Hochschild homology $HH_\bullet(R)$ is just $\mathrm{Tor}_\bullet^{R\otimes_k R^{\mathrm{op}}}(R, R)$, where R^{op} denotes R with multiplication reversed, and we identify two-sided R-modules with left or right modules for $R \otimes_k R^{\mathrm{op}}$.*

Proof. Clearly we can think of right R-modules as left R^{op}-modules and of left R-modules as right R^{op}-modules, and thus of two-sided R-modules as left or right modules for $R \otimes_k R^{\mathrm{op}}$. Thus R is both a left and a right module over $R \otimes_k R^{\mathrm{op}}$. To compute $\mathrm{Tor}_\bullet^{R\otimes_k R^{\mathrm{op}}}(R, R)$, we need to choose a projective resolution over $R \otimes_k R^{\mathrm{op}}$ of one copy of R, then tensor this resolution over $R \otimes_k R^{\mathrm{op}}$ with another copy of R, and take the homology of the resulting complex.

First let's verify that

$$B_\bullet(R): \quad \cdots \xrightarrow{b'_{n+2}} R^{\otimes n+2} \xrightarrow{b'_{n+1}} R^{\otimes n+1} \xrightarrow{b'_n} R^{\otimes n} \xrightarrow{b'_{n-1}} \cdots \xrightarrow{b'_1} R,$$

with b' defined as in (6.1.2), is a projective resolution of R in the category of right $R \otimes_k R^{\mathrm{op}}$-modules. Here the right action of $x \otimes y \in R \otimes_k R^{\mathrm{op}}$ on $(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \in R^{\otimes n+1}$ is given by

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \cdot (x \otimes y) = ya_0 \otimes a_1 \otimes \cdots \otimes a_n x.$$

It is obvious that b' commutes with the module action, and $(b')^2 = 0$ by another application of (6.1.3). Furthermore, since R is projective as a module over k, so are its tensor powers, and we see that

$$R^{\otimes n} \cong (1 \otimes R^{\otimes n-2} \otimes 1) \cdot (R \otimes_k R^{\mathrm{op}})$$

is projective over $R \otimes_k R^{\mathrm{op}}$ for $n \geq 2$. So we need only show that $B_\bullet(R)$ is acyclic. For this we show it is chain-contractible as a complex of k-modules (recall Definition 1.7.2). Define $s_{n-1} : R^{\otimes n} \to R^{\otimes n+1}$ by

$$s_{n-1}(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \tag{6.1.5}$$

This is k-linear (though not $R \otimes_k R^{\mathrm{op}}$-linear) and it's easy to check that $sb' + b's = id$, so $B_\bullet(R)$ is acyclic. Thus we have a projective resolution of R.

Now we can compute $\mathrm{Tor}_\bullet^{R\otimes_k R^{\mathrm{op}}}(R, R)$ as the homology of

$$B_\bullet(R) \otimes_{R\otimes_k R^{\mathrm{op}}} R$$

(where we first knock off the final R from $B_\bullet(R)$). The n-th term of this complex is

$$R^{\otimes n+2} \otimes_{R\otimes_k R^{\mathrm{op}}} R = (1 \otimes R^{\otimes n-2} \otimes 1) \cdot (R \otimes_k R^{\mathrm{op}}) \otimes_{R\otimes_k R^{\mathrm{op}}} R = R^{\otimes n+1},$$

and the differential is given by

$$\begin{aligned}a_0 \otimes a_1 \otimes \cdots \otimes a_n &\cong (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes_{R\otimes_k R^{\mathrm{op}}} 1 \\ &\mapsto b'_{n+1}(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes_{R\otimes_k R^{\mathrm{op}}} 1 \\ &= b'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &\qquad + (-1)^n (a_0 \otimes a_1 \otimes \cdots \otimes a_n) \otimes_{R\otimes_k R^{\mathrm{op}}} 1 \\ &= b'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &\qquad + (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \\ &= d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n).\end{aligned}$$

Thus our complex for computing $\mathrm{Tor}_{\bullet}^{R\otimes_k R^{\mathrm{op}}}(R, R)$ is just $C_\bullet(R)$, and the result follows. □

6.1.6. Corollary. *If $R = k$ is a commutative ring (viewed as an algebra over itself), $HH_0(R) = R$ and $HH_i(R) = 0$ for $i > 0$. For a general k-algebra, $HH_0(R) = R/[R, R]$. (Here $[R, R]$ denotes the k-submodule of R generated by commutators.) If R is a commutative k-algebra, then $HH_1(R) \cong \Omega^1_{\mathrm{ab}}(R)$, the universal k-module on elements $a_1\, da_2$, where $a_1, a_2 \in R$, subject to the relations that $a_1\, da_2$ is k-bilinear in a_1 and a_2 and that $a_2 \mapsto da_2$ is a k-linear derivation. (The subscript "ab" here is to distinguish this from a variant of this construction to be introduced in Definition 6.1.38 below.)*

Proof. For any k-algebra R, $HH_0(R)$ is by definition just $R/b_1(R^{\otimes 2})$, and $b_1(a_0 \otimes a_1) = a_0 a_1 - a_1 a_0 = [a_0, a_1]$. So $HH_0(R) = R/[R, R]$. If R is commutative this is simply R. And if $R = k$, $R \otimes_k R^{\mathrm{op}} = R$. Since R is projective as a module over itself, the higher Tor's vanish.

As for the statement about $HH_1(R)$ in the commutative case, we have already noted that $b_1 \equiv 0$ if R is commutative, so in this case

$$\begin{aligned}HH_1(R) &= (R \otimes_k R) \,/\, \mathrm{im}\, b_2 \\ &= (R \otimes_k R) \,/ \mathrm{span}\, \{a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1 \mid \\ &\qquad\qquad a_0, a_1, a_2 \in R\}.\end{aligned}$$

Note that

$$1 \otimes a_1 a_2 \equiv a_1 \otimes a_2 + a_2 \otimes a_1 \mod \mathrm{im}\, b_2.$$

Hence if we write $a_1\, da_2$ for the image in the quotient of $a_1 \otimes a_2$, then the fact that we have divided out by the image of b_2 means exactly that we are requiring $a_2 \mapsto da_2$ to be a k-linear derivation. □

6.1.7. Examples. (a) Let $R = k[t]$ be a polynomial ring in one variable. This is free over k (with basis the monomials t^i), hence certainly k-projective. Also $R = R^{\mathrm{op}}$ (since R is commutative) and $R \otimes_k R^{\mathrm{op}} \cong k[t, s]$. As a $k[t, s]$-module, R is just $k[t, s]/(t - s)$. So

$$k[t, s] \xrightarrow{(t-s)} k[t, s] \twoheadrightarrow R$$

is a $R \otimes_k R^{\text{op}}$-projective resolution of R, and thus $HH_1(R) \cong HH_0(R) \cong R$, $HH_i(R) = 0$ for $i > 1$.

(b) Another perhaps more interesting example is $R = k[t]/(t^2)$, sometimes called the algebra of dual numbers over k. Here R is free of rank 2 over k and $R \otimes_k R^{\text{op}} = S \cong k[t, s]/(t^2, s^2)$. As a module over this ring S, R is $S/(t-s)$, but, this time, multiplication by $t - s$ on S has a non-zero kernel (the ideal generated by $t + s$). For computing the Tor-group, we can use the periodic S-projective resolution

$$\cdots \to S \xrightarrow{(t-s)} S \xrightarrow{(t+s)} S \xrightarrow{(t-s)} S \twoheadrightarrow R.$$

Tensoring this with R gives the complex

$$\cdots \to R \xrightarrow{0} R \xrightarrow{2t} R \xrightarrow{0} R,$$

from which we see that $HH_i(R) \cong R$ for all i if $k \supseteq \mathbb{F}_2$, while $HH_i(R)$ has rank 1 over k for all $i > 0$ if 2 is invertible in k.

Just as with K-theory, one also has relative groups for pairs (R, I), where I is a two-sided ideal in R, and a long exact sequence relating the relative groups to the absolute groups.

6.1.8. Definition. Let k be a commutative ring and let R be a k-algebra, I a two-sided ideal in R. There is an obvious surjective map of chain complexes $C_\bullet(R) \to C_\bullet(R/I)$. We denote the kernel by $C_\bullet(R, I)$ and its homology groups by $HH_\bullet(R, I)$. By Theorem 1.7.6, there is a long exact sequence, called the long exact sequence of the pair (R, I):

$$\cdots \to HH_{n+1}(R/I) \xrightarrow{\partial} HH_n(R, I) \to HH_n(R) \to HH_n(R/I) \xrightarrow{\partial} HH_{n-1}(R, I) \to \cdots.$$

6.1.9. Examples. Let $R = k[t]$ be a polynomial ring in one variable, and let $I = (t)$. Then $R/I \cong k$ and so by the results of Corollary 6.1.6 and Example 6.1.7(a), together with the long exact sequence of 6.1.8, $HH_i(R, I)$ vanishes for $i > 1$ and is isomorphic to R when $i = 1$, to I (the kernel of the map $R \to R/I$) for $i = 0$. Similarly, $HH_0(R, I)$ can always be identified with I when R is commutative.

When $R = k[t]/(t^2)$ and $I = (t)$, again $R/I \cong k$. Now Example 6.1.7(b) together with the long exact sequence of 6.1.8 shows that $HH_i(R, I)$ is non-zero for all i. For instance, if 2 is invertible in k, $HH_i(R, I)$ has rank 1 over k for all $i \geq 0$.

Cyclic Homology. While it will turn out that there are natural maps $K_i(R) \to HH_i(R)$ which can be used in the study of K-theory, these are rarely close to being isomorphisms. (For instance, if R is commutative, then regardless of the choice of the ground ring k, we have $HH_0(R) = R$, which usually bears no resemblance to $K_0(R)$. Furthermore, though k and $k[t]$

have the same K-theory if k is regular, we have seen that the Hochschild homology of these two algebras is different in degree 1.) To get something closer to K-theory, we need to introduce the cyclic homology groups. These were originally defined by Connes (in two different ways, one fairly computational [Connes1] and one more in the spirit of homological algebra [Connes2]), by Loday and Quillen [LodayQuil], and by Feigin and Tsygan (see especially [Tsy]). The theory has since been simplified and reworked, especially by Hood and Jones [HoodJones]. Aside from the original papers, good sources are the short survey by Cartier [Cartier] (quite readable but a little out of date) and the very comprehensive book of Loday [LodayCH].

6.1.10. Definition. Let k be a commutative ring and let R be a k-algebra. We retain the notation of Definition 6.1.1. Let $t_n : R^{\otimes n+1} \to R^{\otimes n+1}$ be defined by

$$t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}).$$

Note that $(t_n)^{n+1} = 1$, so that t_n gives rise to an action of a cyclic group of order $n+1$ on $R^{\otimes n+1}$. (This is the origin of the name "cyclic homology.") The linear operator $N_n = 1 + t_n + (t_n)^2 + \cdots + (t_n)^n$ is called the **norm** operator on $R^{\otimes n+1}$. We also introduce the map $B_n : R^{\otimes n+1} \to R^{\otimes n+2}$ defined by $B_n = (1 - t_{n+1}) \circ s_n \circ N_n$, where s_n is as defined in (6.1.5). The **cyclic double complex** $CC_{\bullet\bullet}(R)$ of R is the diagram

$$\begin{array}{lccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle b_3} & & \downarrow{\scriptstyle -b_3'} & & \downarrow{\scriptstyle b_3} & & \\
j=2 & \cdots \xleftarrow{N_2} & R^{\otimes 3} & \xleftarrow{1-t_2} & R^{\otimes 3} & \xleftarrow{N_2} & R^{\otimes 3} & \xleftarrow{1-t_2} & \cdots \\
 & & \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle -b_2'} & & \downarrow{\scriptstyle b_2} & & \\
j=1 & \cdots \xleftarrow{N_1} & R^{\otimes 2} & \xleftarrow{1-t_1} & R^{\otimes 2} & \xleftarrow{N_1} & R^{\otimes 2} & \xleftarrow{1-t_1} & \cdots \\
 & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle -b_1'} & & \downarrow{\scriptstyle b_1} & & \\
j=0 & \cdots \xleftarrow{N_0} & R & \xleftarrow{1-t_0} & R & \xleftarrow{N_0} & R & \xleftarrow{1-t_0} & \cdots,
\end{array}$$

in which the even-numbered vertical columns are copies of the Hochschild complex $C_\bullet(R)$ of (6.1.1), and the odd-numbered vertical columns are similar but with b replaced by $-b'$ (recall (6.1.2) for the definitions). The horizontal rows consist of alternating copies of N and $1-t$; note from Exercise 4.1.25 that these rows are complexes from which one can compute the homology of the the cyclic group action on $R^{\otimes n+1}$.

6.1.11. Lemma. *The cyclic double complex $CC_{\bullet\bullet}(R)$ of Definition 6.1.10 is a true "double complex," in that the rows and columns are chain complexes (of k-modules) and each square anticommutes.*

Proof. We've already observed that the horizontal rows are complexes (for computing cyclic group homology) and that the even-numbered vertical rows are complexes (for computing Hochschild homology). The odd-numbered vertical rows are not only complexes but in fact exact, since up

to sign, they are copies of the acyclic complex $B_\bullet(R)$ from the proof of Proposition 6.1.4. So we need to check the relations

$$b_n \circ (1 - t_n) = (1 - t_{n-1}) \circ b'_n, \qquad b'_n \circ N_n = N_{n-1} \circ b_n.$$

We go back to the definitions

$$\begin{aligned} b_n &= \sum_{i=0}^{n} (-1)^i d_i^n, \\ b'_n &= \sum_{i=0}^{n-1} (-1)^i d_i^n, \end{aligned}$$

$$d_i^n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, & i < n, \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, & i = n. \end{cases}$$

Thus $d_0^n \circ t_n = (-1)^n d_n^n$, and for $i < n-1$,

$$\begin{aligned} t_{n-1} \circ d_i^n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= t_{n-1}(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &= (-1)^{n-1}(a_n \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots), \\ d_{i+1}^n \circ t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= (-1)^n d_{i+1}^n(a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \\ &= (-1)^n(a_n \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots) \\ &= -t_{n-1} \circ d_i^n(a_0 \otimes a_1 \otimes \cdots \otimes a_n). \end{aligned}$$

Similarly,

$$\begin{aligned} t_{n-1} \circ d_{n-1}^n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= t_{n-1}(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}a_n) \\ &= (-1)^{n-1}(a_{n-1}a_n \otimes a_0 \otimes \cdots \otimes a_{n-2}), \\ d_n^n \circ t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= (-1)^n d_n^n(a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \\ &= (-1)^n(a_{n-1}a_n \otimes a_0 \otimes \cdots \otimes a_{n-2}) \\ &= -t_{n-1} \circ d_{n-1}^n(a_0 \otimes a_1 \otimes \cdots \otimes a_n). \end{aligned}$$

Thus

$$\begin{aligned} b_n \circ (1 - t_n) &= \sum_{i=0}^{n} (-1)^i d_i^n \circ (1 - t_n) \\ &= \left(\sum_{i=0}^{n} (-1)^i d_i^n\right) - \sum_{i=0}^{n} (-1)^i d_i^n \circ t_n \\ &= b'_n + (-1)^n d_n^n - (-1)^n d_n^n - \sum_{i=1}^{n} (-1)^i \left(-t_{n-1} \circ d_{i-1}^n\right) \\ &= b'_n - \sum_{i=0}^{n-1} (-1)^i t_{n-1} \circ d_i^n \\ &= b'_n - t_{n-1} b'_n = (1 - t_{n-1}) \circ b'_n. \end{aligned}$$

This proves the first of the desired relations. As for the other, note first that

$$
\begin{aligned}
d_i^n \circ t_n^j &= (d_i^n \circ t_n) \circ t_n^{j-1} \\
&= \begin{cases} \left(t_{n-1} \circ d_{i-1}^n\right) \circ t_n^{j-1}, & i > 0, \\ (-1)^n \, d_n^n \circ t_n^{j-1}, & i = 0, \end{cases} \\
&= \begin{cases} (-1)^j \, t_{n-1}^j \circ d_{i-j}^n, & j \le i, \\ (-1)^i \, t_{n-1}^i \circ d_0^n \circ t_n^{j-i}, & j > i, \end{cases} \\
&= \begin{cases} (-1)^j \, t_{n-1}^j \circ d_{i-j}^n, & j \le i, \\ (-1)^{i+n} \, t_{n-1}^i \circ d_n^n \circ t_n^{j-i-1}, & j > i, \end{cases} \\
&= \begin{cases} (-1)^j \, t_{n-1}^j \circ d_{i-j}^n, & j \le i, \\ (-1)^{n+j+1} \, t_{n-1}^{j-1} \circ d_{i+n-j+1}^n, & j > i. \end{cases}
\end{aligned}
$$

Thus

$$
\begin{aligned}
b_n' \circ N_n &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-1)^i \, d_i^n \circ t_n^j \\
&= \left(\sum_{0 \le j \le i \le n-1} (-1)^{i+j} \, t_{n-1}^j \circ d_{i-j}^n \right) \\
&\qquad + \sum_{0 \le i < j \le n-1} (-1)^{n+i+j-1} \, t_{n-1}^{j-1} \circ d_{i+n-j+1}^n \\
&= \left(\sum_{j=0}^{n-1} t_{n-1}^j \right) \circ \left(\sum_{i \le j} (-1)^{n+i+j} \, d_{i+n-j}^n + \sum_{i \ge j} (-1)^{i+j} \, d_{i-j}^n \right) \\
&= N_{n-1} \circ b_n. \quad \square
\end{aligned}
$$

Now we're ready to define cyclic homology. There are various versions, depending on whether one chooses the double complex of Definition 6.1.10 to live in the first quadrant, in the upper half-plane, or mostly in the second quadrant.

6.1.12. Definition. Let k be a commutative ring and let R be a k-algebra. The **cyclic homology** $HC_\bullet(R)$ (which implicitly depends on the choice of the ground ring k as well) is the homology of the total complex associated to the cyclic double complex $CC_{\bullet\bullet}(R)$ of R, chosen to live in the first quadrant. (The notation HC, which is starting to become standard, stands for the French "homologie cyclique.") In other words, $HC_\bullet(R)$ is the homology of the chain complex (concentrated in non-negative degrees) whose term in degree n is

$$
\bigoplus_{\substack{i+j=n \\ i,j \ge 0}} CC_{i,j}(R) = \bigoplus_{j=0}^{n} R^{\otimes j+1}, \tag{6.1.13}
$$

and whose boundary map is the sum of all the maps in the diagram of (6.1.10) from something of total degree n to something of total degree $n-1$. (Lemma 6.1.11 guarantees that this boundary map has square $= 0$.) In particular,

$$\begin{aligned} HC_0(R) &= R/\left\{\operatorname{im}\left(b_1\colon R^{\otimes 2} \to R\right) + \operatorname{im}\left(1 - t_0\colon R \to R\right)\right\} \\ &= R/\left[R,\, R\right] \cong HH_0(R). \end{aligned}$$

(Since t_0 is the identity map on R, $\operatorname{im}(1 - t_0) = 0$.) Similarly, because of the fact that the bottom row of the double complex of Definition 6.1.10 is acyclic, $CC_{1,0}$ doesn't contribute to $HC_1(R)$, and

$$HC_1(R) = \\ \ker\left(b_1\colon R^{\otimes 2} \to R\right) / \left\{\operatorname{im}\left(b_2\colon R^{\otimes 3} \to R^{\otimes 2}\right) + \operatorname{im}\left(1 - t_1\colon R^{\otimes 2} \to R^{\otimes 2}\right)\right\},$$

which is the quotient of $HH_1(R)$ by the image of the map which is the quotient of $HH_1(R)$ by the image of the map induced by $1 - t_1$. We will get to the general relationship between $HC_\bullet(R)$ and $HH_\bullet(R)$ shortly.

We will also need certain modifications of $HC_\bullet(R)$ known as $HC_\bullet^-(R)$ and as $HP_\bullet(R)$ (the notation HP stands for the French "homologie [cyclique] périodique"). To define HP, we use the same construction as for HC, but dropping the condition that $i \geq 0$ and replacing $\bigoplus$ by $\prod$ in (6.1.13) (this makes a difference since one now has infinitely many terms with the same total degree). In other words, $HP_\bullet(R)$ is the homology of the chain complex whose term in degree n is

$$\prod_{j=0}^{\infty} CC_{n-j,\,j}(R) = \prod_{j=0}^{\infty} R^{\otimes j+1}. \tag{6.1.14}$$

This complex is obviously periodic with period 2, as only the parity of n matters, and thus $HP_\bullet(R)$ is called the **periodic cyclic homology** of R. Finally, $HC_\bullet^-(R)$ is the homology of the subcomplex of this complex where we only take terms with $i \leq 1$, in other words, the homology of the chain complex whose term in degree n is

$$\prod_{j\geq n-1}^{\infty} CC_{n-j,\,j}(R) = \prod_{j=\max(n-1,\,0)}^{\infty} R^{\otimes j+1}. \tag{6.1.15}$$

6.1.16. Example. Suppose $R = k$, the commutative ground ring. Then $R^{\otimes n+1} \cong k$, generated as a k-module by $1\otimes 1\otimes \cdots \otimes 1$, so t_n is multiplication by $(-1)^n$. Thus $1 - t_n$ is 0 for n even, multiplication by 2 for n odd, and N_n is multiplication by $n + 1$ for n even, 0 for n odd. Similarly, $b'_n = \sum_{j=0}^{n-1}(-1)^n$ and $b_n = \sum_{j=0}^{n}(-1)^n$. Thus $b_n = 0$ for n odd and $b'_n = 0$

for n even. So $CC_{\bullet\bullet}(k)$ collapses to

$$
\begin{array}{lccccccccc}
 & & & \vdots & & \vdots & & \vdots & & \\
 & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle 0} & & \\
j=2 & \cdots & \xleftarrow{3} & k & \xleftarrow{0} & k & \xleftarrow{3} & k & \xleftarrow{0} & \cdots \\
 & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} & & \\
j=1 & \cdots & \xleftarrow{0} & k & \xleftarrow{2} & k & \xleftarrow{0} & k & \xleftarrow{2} & \cdots \\
 & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle 0} & & \\
j=0 & \cdots & \xleftarrow{1} & k & \xleftarrow{0} & k & \xleftarrow{1} & k & \xleftarrow{0} & \cdots.
\end{array}
$$

Here the odd-numbered columns are acyclic and the even-numbered columns are acyclic except at the bottom, so we claim only the even-numbered terms in the bottom row contribute to (any of the three forms of) the cyclic homology. We check this for $HP_\bullet$, which is the hardest case. An element of the associated complex of total degree n is an infinite sequence $a = \{a_j\}_{j\geq 0}$ of elements of k, where a_j is located in the $(n-j)$-th row and j-th column of the double complex. Note that the differentials from terms of odd total degree to terms of even total degree vanish, so if n is odd such an element is a cycle. But then a is the boundary of

$$(0,\ -a_0,\ a_1 + 2a_0,\ -a_2 + 3a_1 + 6a_0,\ \ldots).$$

If n is even, there are no boundaries, but if a is a cycle, we have

$$a_0 - a_1 = 0,\ 2a_1 + a_2 = 0,\ 3a_2 - a_3 = 0,\ \ldots,$$

which enables us to solve inductively for a_1, a_2, ... in terms of a_0. Thus the space of even-dimensional cycles is a free module of rank one. Similar calculations work for $HC_\bullet$ and $HC_\bullet^-$. So $HC_n(k) = HP_n(k) = HC_n^-(k) = 0$ for n odd, $HP_n(k) \cong k$ for all n even, $HC_n(k) \cong k$ for all n even and non-negative, and $HC_n^-(k) \cong k$ for all n even and non-positive. Some of the original motivation for the development of cyclic homology as a "linearized version" of K-theory derives from the observed similarity between the groups $HP_n(k)$ and the topological K-groups $KU^{-n}(pt)$.

6.1.17. Remark. For the same reasons as with Hochschild homology, it is obvious that $R \rightsquigarrow HC_\bullet(R)$ (or $HC_\bullet^-(R)$ or $HP_\bullet(R)$) is a functor on the category of k-algebras. Also, just as in the case of Hochschild homology, if I is a two-sided ideal in R, the complex defining any of the three kinds of cyclic homology for R surjects onto the corresponding complex for R/I. The kernel of the map of complexes can therefore be used to define relative cyclic homology groups $HC_\bullet(R, I)$, $HC_\bullet^-(R, I)$, and $HP_\bullet(R, I)$ that fit into long exact sequences such as that of Definition 6.1.8.

6.1.18. Definition. The **Connes periodicity operator** is the self-map S of the cyclic double complex $CC_{\bullet\bullet}(R)$ that shifts everything two columns to the left. This induces self-maps of the complexes used to define $HC_\bullet(R)$, $HC_\bullet^-(R)$, and $HP_\bullet(R)$, as well as maps on the corresponding homology groups. With slight abuse of notation, we customarily denote all of these maps by S.

It is obvious that $S\colon HP_\bullet(R) \to HP_{\bullet-2}(R)$ is an isomorphism. However, while S is surjective as a self-map on the double complex defining $HC_\bullet(R)$, it has a non-trivial kernel, namely the columns numbered $i = 0$ and $i = 1$. Since the column numbered $i = 1$ is acyclic, the total complex

$$\bigoplus_{\substack{i+j=n\\ i=0 \text{ or } 1}} CC_{i,j}(R)$$

has the same homology as the column numbered $i = 0$, which is a copy of the Hochschild complex. Thus we obtain the following.

6.1.19. Theorem (Connes). *Let k be a commutative ring and let R be a k-algebra. There is a functorial long exact sequence*

$$\cdots \xrightarrow{S} HC_{n-1}(R) \xrightarrow{B} HH_n(R) \xrightarrow{I} HC_n(R)$$
$$\xrightarrow{S} HC_{n-2}(R) \xrightarrow{B} HH_{n-1}(R) \to \cdots .$$

Here the map I comes from the inclusion of the Hochschild complex into the cyclic double complex as the 0-th column. This map is always an isomorphism for $n = 0$ and a surjection for $n = 1$.

Proof. This follows from applying Theorem 1.7.6 to the short exact sequence of chain complexes coming from the map S of double complexes. The map I in the long exact sequence initially comes from the inclusion of the columns numbered $i = 0$ and $i = 1$, but since the column numbered $i = 1$ is acyclic, it doesn't have any effect. Since $HC_n(R) = 0$ and $HH_n(R) = 0$ for $n < 0$ (for any k-algebra R), the exact sequence begins with

$$0 = HC_{-1}(R) \to HH_0(R) \xrightarrow{I} HC_0(R) \xrightarrow{S} HC_{-2}(R) = 0$$

and with

$$HC_0(R) \xrightarrow{B} HH_1(R) \xrightarrow{I} HC_1(R) \xrightarrow{S} HC_{-1}(R) = 0,$$

so that I is always an isomorphism for $n = 0$ and a surjection for $n = 1$. □

6.1.20. Examples. (a) First consider the case where $R = k$ is the ground ring. By Corollary 6.1.6, $HH_0(k) \cong k$ and $HH_n(k) = 0$ for $n > 0$. Since I

is surjective in degree 1 by Theorem 6.1.19, $HC_1(k) = 0$. Then one sees by induction on n that $HC_n(k) \xrightarrow{S} HC_{n-2}(k)$ must be an isomorphism when n is even and $n \geq 2$, and that $HC_n(k)$ must vanish for n odd. This is in keeping with Example 6.1.16.

(b) Now take $R = k[t]$, a polynomial ring in one variable. By Example 6.1.7(a), $HH_1(R) \cong R$ and $HH_n(R) = 0$ for $n > 1$. We have an exact sequence

$$\begin{aligned} 0 = HH_2(R) \xrightarrow{I} HC_2(R) \xrightarrow{S} HC_0(R) \\ \xrightarrow{B} HH_1(R) \xrightarrow{I} HC_1(R) \xrightarrow{S} HC_{-1}(R) = 0, \end{aligned}$$

so a computation of $HC_0(R) \cong R \xrightarrow{B} R \cong HH_1(R)$ will yield a calculation of $HC_1(R)$ and $HC_2(R)$. Then since the Hochschild homology vanishes past degree 1, the higher cyclic homology groups must be periodic with period 2.

To compute the map B, we need to trace through the proof of Theorem 1.7.6 in this context. If $x \in R$, it defines an element of $C_{0,0}(R)$ and thus a class in $HC_0(R)$. To see where this maps under B, note that x is the image under S of the corresponding element of $C_{2,0}(R)$. Under the differential N_0, this maps to element corresponding to x in $C_{1,0}(R) \cong R$. Now the map $-b_1'\colon R^{\otimes 2} \to R$ sends $a_0 \otimes a_1$ to $-a_0 a_1$. Thus $-1 \otimes x \in C_{1,1}(R)$ maps to $x \in C_{1,0}(R)$, but it also maps under $1 - t_1$ to $-(1 \otimes x + x \otimes 1)$. Thus B sends the element corresponding to x to the Hochschild homology class of the 1-cycle $-(1 \otimes x + x \otimes 1)$. So far this discussion is completely general and applies to any R.

If we specialize now to the case where $R = k[t]$, the identification of R with $HH_1(R) \cong \Omega^1_{\text{ab}}(R)$ in Corollary 6.1.6 is via $f(t) \mapsto f(t)\,dt$. On the other hand, B sends $x = t^m$ to the image of $-(1 \otimes x + x \otimes 1)$, in other words, to $-1\,d(t^m) + t^m\,d1 = -d(t^m) = -mt^{m-1}dt$. Thus in this case $B\colon HC_0(R) \to HH_1(R)$ viewed as a map $R \to R$ corresponds to $t^m \mapsto -mt^{m-1}$, the derivative (up to sign). The kernel and cokernel of B now depend on the ground ring. If $k \supseteq \mathbb{Z}$, the only polynomials with derivative $= 0$ are the constants, so $HC_2(R) \cong k$. If $k \supseteq \mathbb{Q}$, then B is surjective and so the odd cyclic homology of $R = k[t]$ vanishes. On the other hand, if $k = \mathbb{Z}$, then $HC_1(R) \cong \bigoplus_{m>0} \mathbb{Z}/m$.

Now that we've seen how $HC_\bullet(R)$ is related to Hochschild homology, we shall study the relations among the three different versions of cyclic homology, and see how the other two also relate to Hochschild homology. Again, the basic tool is the shift map S on the cyclic double complex. If we truncate the double complex on the right instead of on the left (corresponding to the choice giving $HC_\bullet^-$), then S becomes *injective* instead of surjective, with cokernel the complex concentrated in the columns numbered 0 and 1. So Theorem 6.1.19 is replaced by the following.

6.1.21. Theorem. *Let k be a commutative ring and let R be a k-algebra. There is a functorial commutative diagram whose rows are exact sequences:*

$$\begin{array}{ccccc} \cdots \longrightarrow & HC^-_{n-1}(R) & \longrightarrow & HH_{n-1}(R) & \longrightarrow \\ & \downarrow & & \downarrow & \\ \cdots \longrightarrow & HP_{n-1}(R) & \longrightarrow & HC_{n-1}(R) & \longrightarrow \end{array}$$

$$\begin{array}{cccccccc} \longrightarrow & HC^-_n(R) & \xrightarrow{S} & HC^-_{n-2}(R) & \longrightarrow & HH_{n-2}(R) & \longrightarrow & \cdots \\ & \Vert & & \downarrow & & \downarrow & & \\ \longrightarrow & HC^-_n(R) & \xrightarrow{S} & HP_{n-2}(R) & \longrightarrow & HC_{n-2}(R) & \longrightarrow & \cdots . \end{array}$$

Proof. Consider the shift map S restricted to the part of the cyclic double complex with columns numbered ≤ 1, viewed either as a self-map of this truncated double complex, or as a map into the whole double complex (living in both the first and second quadrants). In the first case the cokernel is the complex concentrated in the columns numbered 0 and 1, which gives rise to the Hochschild homology HH, and in the second case, it is the complex concentrated in non-negative degrees, which gives rise to HC. Applying Theorem 1.7.6, we get the two exact sequences of the theorem. The vertical maps between them come from the obvious inclusions of double complexes. □

Another relationship between periodic and non-periodic cyclic homology comes from the fact that the complex giving rise to HP is the *direct* limit (under the shift map S) of copies of the complex giving rise to HC^-, and the *inverse* limit of copies of the complex giving rise to HC (recall that formulas (6.1.14) and (6.1.15) involve (infinite) *products*, whereas (6.1.13) involves (finite) *sums*). Since homology commutes with direct limits but not necessarily with inverse limits, we may deduce the following.

6.1.22. Theorem. *Let k be a commutative ring and let R be a k-algebra. Then $HP_\bullet(R) \cong \varinjlim_S HC^-_\bullet(R)$, and also $HP_\bullet(R)$ surjects onto $\varprojlim_S HC_\bullet(R)$. (However in the second case there may be a non-zero kernel.) The map $HP_\bullet(R) \to \varprojlim_S HC_\bullet(R)$ is an isomorphism if the* **Mittag-Leffler condition** *is satisfied, that is, if for each n, $S^j(HC_{n+2j}(R))$ is independent of j for j sufficiently large ("how large" may depend on n).*

Proof. The first statement is immediate from the fact that homology commutes with direct limits. The second statement comes from the fact that the complex $E_\bullet(R)$ giving rise to $HP_\bullet(R)$ is the inverse limit of truncated complexes $E^i_\bullet(R)$ corresponding to shifts under S of the complex for computing $HC_\bullet(R)$. Furthermore, $E^i_\bullet(R)$ surjects onto $E^{i-1}_\bullet(R)$. A class in $\varprojlim HC_n(R)$ is represented by a sequence $\{z_i\}$, where $z_i \in E^i_n(R)$, $d(z_i) = 0$

in $E^i_{n-1}(R)$, and z_i maps to z_{i-1} modulo boundaries. In other words, there are elements $c_i \in E^i_{n+1}(R)$ such that z_i maps to $z_{i-1} + d(c_{i-1})$. We can inductively lift c_i to an element $c'_{i+1} \in E^{i+1}_{n+1}(R)$. Then if we replace z_i by $z_i + d(c'_{i+1})$, we can, without changing the homology class of z_i, arrange for z_i to map to z_{i-1} (exactly). This shows the homology of $\varprojlim E^i_n(R)$, *i.e.*, $HP_n(R)$, surjects onto $\varprojlim_S HC_n(R)$.

Finally, suppose the Mittag-Leffler condition is satisfied, and suppose a class in $HP_n(R)$ maps to 0 in $\varprojlim_S HC_n(R)$. Let the given class be the homology class of a cycle $\{z_i\}$ in $\varprojlim E^i_n(R)$, where $z_i \in E^i_n(R)$, $d(z_i) = 0$ in $E^i_{n-1}(R)$, and z_i maps to z_{i-1}. Since the class maps to 0 in $\varprojlim_S HC_n(R)$, we may suppose each z_i is a boundary. Without loss of generality (otherwise start the sequence $\{E^i_\bullet(R)\}$ further out), we may assume (by the Mittag-Leffler condition) that the maps $H_{n+1}(E^i(R)) \to H_{n+1}(E^{i-1}(R))$ are all surjective. We need to inductively choose $c_i \in E^i_{k+1}(R)$ such that $d(c_i) = z_i$ and c_i maps to c_{i-1} in $E^{i-1}_{n+1}(R)$; then $d(\{c_i\}) = \{z_i\}$ and so our class is trivial in $HP_n(R)$. To begin the induction, choose any c_1 with $d(c_1) = z_1$. Assuming $c_1, \ldots, c_m$ are constructed so that $d(c_i) = z_i$ and c_i maps to c_{i-1} for $i \le m$, choose any $c \in E^{m+1}_{n+1}(R)$ such that $d(c) = z_{m+1}$. Then c maps to an element $c' \in E^m_{n+1}(R)$ and $d(c') = d(c_m) = z_m$, so $c_m - c'$ is a cycle. Since the map $H_{n+1}(E^{m+1}(R)) \to H_{n+1}(E^m(R))$ is surjective, there is a cycle $x \in E^{m+1}_{n+1}(R)$ mapping to its homology class, and we may assume that in fact x maps to $c_m - c'$. Then if $c_{m+1} = c + x$, $d(c_{m+1}) = z_{m+1}$ and c_{m+1} maps to c_m, completing the inductive step. □

6.1.23. Corollary. *Let k be a commutative ring and let R be a k-algebra. If $HH_n(R) = 0$ for n sufficiently large, then*

$$HP_\bullet(R) \cong \varprojlim_S HC_\bullet(R).$$

Proof. By Theorem 6.1.19, S is an isomorphism from $HC_{n+2}(R)$ to $HC_n(R)$ for n sufficiently large. Hence the Mittag-Leffler condition is satisfied, and the result follows from Theorem 6.1.22. □

Alternatively, we may reformulate things in the form of universal coefficient theorems as stated in [HoodJones].

6.1.24. Theorem. *Let k be a commutative ring and let R be a k-algebra. View the various cyclic homology groups as graded modules over the polynomial ring $k[u]$, where the generator u has degree -2 and corresponds to the Connes S-operator. Then $HP_\bullet(R) \cong k[u, u^{-1}] \otimes_{k[u]} HC^-_\bullet(R)$. If k is a field, there is a natural short exact sequence (which splits, but only non-canonically)*

$$0 \to \left(k[u, u^{-1}]/uk[u]\right) \otimes_{k[u]} HC^-_\bullet(R) \to HC_\bullet(R) \to \operatorname{Tor}_{k[u]}\left(k[u, u^{-1}]/uk[u],\, HC^-_{\bullet-1}(R)\right) \to 0.$$

Proof. The assertion that $HP_\bullet(R) \cong k[u, u^{-1}] \otimes_{k[u]} HC^-_\bullet(R)$ is just a reformulation of the first statement of Theorem 6.1.22. For the second statement, first observe that if $E^-_\bullet(R)$ is the complex for computing $HC^-_\bullet(R)$, then the complex for computing $HC_\bullet(R)$ may be identified with $\big(k[u, u^{-1}]/uk[u]\big) \otimes_{k[u]} E^-_\bullet(R)$. Now we have a short exact sequence of chain complexes

$$0 \to E^-_\bullet(R) \xrightarrow{u} k[u, u^{-1}] \otimes_{k[u]} E^-_\bullet(R) \\ \to \big(k[u, u^{-1}]/uk[u]\big) \otimes_{k[u]} E^-_\bullet(R) \to 0,$$

and applying Theorem 1.7.6 gives an exact sequence (which also appeared in Theorem 6.1.21) which sandwiches $HC_\bullet(R)$ between two copies (one shifted in degree by 1) of

$$HC^-_\bullet(R) \xrightarrow{u} k[u, u^{-1}] \otimes_{k[u]} HC^-_\bullet(R).$$

Assuming k is a field, taking the cokernel and kernel of this map gives the tensor and Tor terms in the statement of the theorem. The sequence splits (non-canonically) since the image of the boundary map on the complex $\big(k[u, u^{-1}]\big) \otimes_{k[u]} E^-_\bullet(R)$ is a free $k[u]$-module. Thus on this image one can choose a splitting to the boundary map, which induces a splitting of the exact sequence. □

Next we discuss a number of simplified ways for computing cyclic homology groups, the first of which was actually used in [Connes1] to define cyclic homology in the first place.

6.1.25. Definition. Let k be a commutative ring and let R be a k-algebra. Let $C^\lambda_\bullet(R)$ be the chain complex with $C^\lambda_n(R) = R^{\otimes n+1}/(1-t_n)R^{\otimes n+1}$, where t_n is defined in Definition 6.1.10, and where the differential is induced by $b_n : R^{\otimes n+1} \to R^{\otimes n}$. To see that this makes sense, recall that by Lemma 6.1.11, $b_n \circ (1-t_n) = (1-t_{n-1}) \circ b'_n$. Thus b_n maps the image of $1-t_n$ into the image of $1-t_{n-1}$ and passes to the quotient. It gives a legitimate differential since we also had $b_{n-1} \circ b_n = 0$. We define $H^\lambda_\bullet(R)$ to be the homology of the complex $C^\lambda_\bullet(R)$; clearly this is a functor of the k-algebra R.

6.1.26. Theorem. *Let k be a commutative ring and let R be a k-algebra. If $k \supseteq \mathbb{Q}$, there is a canonical isomorphism $HC_\bullet(R) \xrightarrow{\cong} H^\lambda_\bullet(R)$ induced by the quotient maps $CC_{0,n}(R) \to C^\lambda_n(R)$.*

Proof. Clearly, mapping $CC_{0,n}(R)$ onto $C^\lambda_n(R)$ and sending $CC_{i,n}(R)$ to 0 for $i \neq 0$ gives a map of chain complexes, from the complex computing $HC_\bullet(R)$ to the one computing $H^\lambda_\bullet(R)$. The kernel of this map of complexes

is the total complex of the double complex

$$
\begin{array}{lccccccc}
 & \vdots & & \vdots & & \vdots & & \\
 & \Big\downarrow{\scriptstyle b_3} & & \Big\downarrow{\scriptstyle -b'_3} & & \Big\downarrow{\scriptstyle b_3} & & \\
j=2 & \operatorname{im}(1-t_2) & \xleftarrow{1-t_2} & R^{\otimes 3} & \xleftarrow{N_2} & R^{\otimes 3} & \xleftarrow{1-t_2} & \cdots \\
 & \Big\downarrow{\scriptstyle b_2} & & \Big\downarrow{\scriptstyle -b'_2} & & \Big\downarrow{\scriptstyle b_2} & & \\
j=1 & \operatorname{im}(1-t_1) & \xleftarrow{1-t_1} & R^{\otimes 2} & \xleftarrow{N_1} & R^{\otimes 2} & \xleftarrow{1-t_1} & \cdots \\
 & \Big\downarrow{\scriptstyle b_1} & & \Big\downarrow{\scriptstyle -b'_1} & & \Big\downarrow{\scriptstyle b_1} & & \\
j=0 & \operatorname{im}(1-t_0) & \xleftarrow{1-t_0} & R & \xleftarrow{N_0} & R & \xleftarrow{1-t_0} & \cdots .
\end{array}
$$

The rows of this double complex are $\mathbb{Q}$-vector spaces with maps alternating between $1 - t_j$ and N_j. Since $(t_n)^{n+1} = 1$ and the polynomial $t^{n+1} - 1$ factors in $\mathbb{Q}[t]$ as $(t-1)(1 + \cdots + t^n)$, $1 - t_j$ and N_j are (up to invertible scalar factors) the projections onto complementary subspaces. Thus each row is acyclic, and so it is easy to see that the whole total complex of the double complex is acyclic. Applying Theorem 1.7.6, we see that our map of complexes induces an isomorphism on homology. □

6.1.27. Examples.

(1) If $R = k$, $R^{\otimes n+1} \cong k$, with generator $1 \otimes 1 \otimes \cdots \otimes 1$. Also, t_n acts on this element by multiplication by $(-1)^n$. So $(1 - t_n)R^{\otimes n+1} = 0$ if n is even, and if 2 is invertible in k, $(1 - t_n)R^{\otimes n+1} = R^{\otimes n+1}$ if n is odd. Thus $C_n^\lambda(R) = R^{\otimes n+1}/(1 - t_n)R^{\otimes n+1}$ is isomorphic to k for n even, 0 for n odd, and $H_n^\lambda(R)$ is isomorphic to k for n even, 0 for n odd. If $k \supseteq \mathbb{Q}$, Theorem 6.1.26 now gives a quicker reconfirmation of the result of (6.1.16).

(2) Suppose $k \supseteq \mathbb{Q}$ and as in Example 6.1.7(b), let $R = k[t]/(t^2)$, the ring of dual numbers over k. Then $R^{\otimes n+1}$ is a free k-module of rank 2^{n+1}, with basis all tensor products of sequences of t's and 1's. Since the map $R \to k$ obtained by sending $t \mapsto 0$ is split, $C_\bullet^\lambda(R)$ contains $C_\bullet^\lambda(k)$ as a direct summand, and $HC_\bullet(R) \cong HC_\bullet(k) \oplus H_\bullet^\lambda(R, I)$, where I is the ideal generated by t and $C_n^\lambda(R, I)$ is spanned by the images of tensor products of sequences of t's and 1's containing at least one t. If such a sequence contains a 1, it can be moved by a cyclic permutation to one of the form $s \otimes 1$, and $b_n(s \otimes 1) = b'_{n-1}(s) \otimes 1$. Thus $C_n^\lambda(R, I)$ splits as a direct sum of the complex generated by the $t \otimes t \otimes \cdots \otimes t$ in even degrees and the complex generated by images of tensor products of sequences of t's and 1's containing at least one t and at least one 1. The latter is acyclic since the b' complex is acyclic, so we find that $HC_n(R)$ has rank 2 for n even, rank 0 for n odd.

(3) More generally, the same calculations as in (2) prove the following. Suppose $k \supseteq \mathbb{Q}$ and R is a k-algebra with an augmentation (surjec-

tive algebra homomorphism) $R \to k$. Then $k \hookrightarrow k \cdot 1$ gives a splitting of R as $k \cdot 1 \oplus I$, where I is the augmentation ideal (the kernel of the augmentation). We have $HC_\bullet(R) \cong HC_\bullet(k) \oplus H_\bullet^\lambda(R, I)$, and $H_\bullet^\lambda(R, I)$ may be obtained as the homology of the complex $C_\bullet^\lambda(R, I)$ obtained by dividing $C_\bullet^\lambda(R)$ by the span of all elementary tensors containing a 1 as one of the factors.

Another convenient approach to cyclic homology is through "mixed complexes," first popularized in [Kassel] (though they appeared earlier in disguise—see the history in [LodayCH, p. 87]). These have the advantage of being applicable even in finite characteristic, yet of being easier to handle than the bicomplexes we have discussed up till now.

6.1.28. Definition. A **mixed complex** $(C_\bullet, b, B)$ is a chain complex $(C_\bullet, b)$ (where b is the differential lowering degree by 1), together with an additional differential B *raising* degree by 1 and anticommuting with b (so that $b^2 = 0$, $B^2 = 0$, and $Bb + bB = 0$). If k is a commutative ring and R is a k-algebra, the **cyclic mixed complex** of the k-algebra R is the Hochschild complex $(C_\bullet(R), b)$ of Definition 6.1.1, together with the additional differential B introduced in Definition 6.1.10. That this terminology is consistent (in other words, that the cyclic mixed complex is indeed a mixed complex) follows from the following lemma.

6.1.29. Lemma. *If k is a commutative ring and R is a k-algebra, the cyclic mixed complex of R is indeed a mixed complex, that is, $B^2 = 0$ and $Bb + bB = 0$.*

Proof. First of all, since $N_{n+1} \circ (1 - t_{n+1}) = 1 - t_{n+1}^{n+2} = 0$, we see that

$$B_{n+1} \circ B_n = (1 - t_{n+2}) \circ s_{n+1} \circ N_{n+1} \circ (1 - t_{n+1}) \circ s_n \circ N_n = 0.$$

Next, because of Lemma 6.1.11 and the relation $sb' + b's = id$, we have

$$\begin{aligned} B_{n-1} \circ b_n &= (1 - t_n) \circ s_{n-1} \circ N_{n-1} \circ b_n \\ &= (1 - t_n) \circ s_{n-1} \circ b'_n \circ N_n \\ &= (1 - t_n) \circ N_n - (1 - t_n) \circ b'_{n+1} \circ s_n \circ N_n \\ &= -b_{n+1} \circ (1 - t_{n+1}) \circ s_n \circ N_n \\ &= -b_{n+1} \circ B_n, \end{aligned}$$

as required. □

6.1.30. Theorem. *If k is a commutative ring and R is a k-algebra, the cyclic homology of R, together with its module structure over the polynomial ring $k[u]$, where the generator u has degree -2 and corresponds to the Connes S-operator, may be computed from the cyclic mixed complex as follows. $HC_\bullet(R)$ is naturally isomorphic to the homology of the complex $k[u, u^{-1}]/uk[u] \otimes_k C_\bullet(R)$ with differential $\partial = 1 \otimes b + u \otimes B$, with grading*

coming from the grading on $C_\bullet(R)$ and the requirement that u have degree -2. In other words, elements of degree n in the complex are given by sums

$$\sum_{j=0}^{[\frac{n}{2}]} u^{-j} \otimes x_j$$

with $x_j \in C_{n-2j}(R)$, and the differential has degree -1 since u lowers degree by 2 but B raises it by 1. Similarly, $HC^-_\bullet(R)$ and $HP_\bullet(R)$ are naturally isomorphic to the homologies of the complexes with the same differential and with elements of the same form, except that the sums are now allowed to be formal infinite sums, with j running from $-\infty$ to 0 in the case of $HC^-_\bullet(R)$, and running from $-\infty$ to ∞ in the case of $HP_\bullet(R)$.

Proof. We give the proof for HC; the arguments for HC^- and HP are similar. First consider the following double complex $MC_{\bullet\bullet}(R)$:

(6.1.31)
$$\begin{array}{ccccccccc}
 & i=0 & & i=1 & & i=2 & & i=3 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & \downarrow b_4 & & \downarrow b_3 & & \downarrow b_2 & & \downarrow b_1 \\
j=3 & R^{\otimes 4} & \xleftarrow{B_2} & R^{\otimes 3} & \xleftarrow{B_1} & R^{\otimes 2} & \xleftarrow{B_0} & R \\
 & \downarrow b_3 & & \downarrow b_2 & & \downarrow b_1 & & \\
j=2 & R^{\otimes 3} & \xleftarrow{B_1} & R^{\otimes 2} & \xleftarrow{B_0} & R & & \\
 & \downarrow b_2 & & \downarrow b_1 & & & & \\
j=1 & R^{\otimes 2} & \xleftarrow{B_0} & R & & & & \\
 & \downarrow b_1 & & & & & & \\
j=0 & R & & & & & &
\end{array}$$

filled out with zeroes whenever $i > j$. By Lemma 6.1.29, this indeed satisfies the requirements for a double complex, and evidently the associated total complex is exactly the complex described in the statement of the theorem. Now consider the map Φ sending $x \in MC_{i,j}(R) = C_{j-i}(R)$ to $x \oplus sNx \in CC_{2i,j-i}(R) \oplus CC_{2i-1,j-i+1}(R)$. This is not a map of double complexes since it doesn't respect the bigrading; however it induces a homomorphism of total complexes, preserving degree and commuting with the differentials, since

$$\partial(x) = Bx \oplus bx \in MC_{i-1,j}(R) \oplus MC_{i,j-1}(R) = C_{j-i+1}(R) \oplus C_{j-i-1}(R),$$

which maps under Φ to $Bx \oplus sNBx \oplus bx \oplus sNbx \in CC_{2i-2,j-i+1}(R) \oplus CC_{2i-3,j-i+2}(R) \oplus CC_{2i,j-i-1}(R) \oplus CC_{2i-1,j-i}(R)$. We can omit the term

$sNBx$ since $N(1-t)=0$ and thus $NB=N(1-t)sN=0$. On the other hand,

$$\begin{aligned}\partial(\Phi(x)) &= \partial(x \oplus sNx)\\ &= bx \oplus (1-t)sNx \oplus Nx - b'sNx\\ &= bx \oplus Bx \oplus Nx - b'sNx\\ &\in CC_{2i,\,j-i-1}(R) \oplus CC_{2i-2,\,j-i+1}(R) \oplus CC_{2i-1,\,j-i}(R).\end{aligned}$$

The Bx terms in $CC_{2i-2,\,j-i+1}(R)$ and the bx terms in $CC_{2i,\,j-i-1}(R)$ clearly agree, so we have only to compare the terms $Nx - b'sNx$ and $sNbx$ in $CC_{2i-1,\,j-i}(R)$. These agree since

$$sNbx = s(Nb)x = s(b'N)x = (sb')Nx = (1-b's)Nx = Nx - b'sNx.$$

Thus we indeed get a chain map.

To finish the proof we only need to show Φ induces an isomorphism on homology. Since Φ is obviously injective, we can do this by showing that its cokernel is acyclic and then applying Theorem 1.7.6. But the formula for Φ shows that the cokernel of Φ may be identified with the subcomplex of the total complex of the double complex $CC_{\bullet\bullet}(R)$ consisting of the odd-numbered columns. Since these columns are acyclic (because of the fact that we proved $B_\bullet(R)$ is acyclic in the course of the proof of Proposition 6.1.4), we are done. □

Remarks. The advantage of the "mixed complex" approach can be seen from comparing the double complexes $CC_{\bullet\bullet}(R)$ and $MC_{\bullet\bullet}(R)$. The latter has only about half as many summands in each total degree, which makes it more efficient for calculation.

The astute reader will no doubt notice that we have used the letter B twice, once for Connes' map $HC_{n-1}(R) \xrightarrow{B} HH_n(R)$ and once for the horizontal boundary maps in the double complex $MC_{\bullet\bullet}(R)$. This is not accidental, since (6.1.31) makes clear that $HC_1(R)$ is the quotient of $HH_1(R)$ by the image of the map induced by B_0 from $HC_0(R)$ to $HH_1(R)$. More generally we have the following.

6.1.32. Lemma. *The boundary map B in the Connes exact sequence 6.1.19 may be realized as follows in terms of the double complex $MC_{\bullet\bullet}(R)$. If a class in $HC_{n-1}(R)$ is represented by a sum x of elements $x_{ij} \in MC_{ij}(R)$, $i+j=n-1$, $j \geq i$, with $b(x_{ij}) + B(x_{i+1,j-1}) = 0$, then its image in $HH_n(R)$ is represented by the class of $Bx_{0,\,n-1}$ in $R^{\otimes n+1}$.*

Proof. Under the isomorphism from $HC_\bullet(R)$ to the homology of the total complex of $MC_{\bullet\bullet}(R)$, the shift operator S corresponds to the self-map of $MC_{\bullet\bullet}(R)$ shifting everything one unit down and one unit to the left. Thus B is the connecting map produced by Theorem 1.7.6 from the short exact sequence of complexes coming from this shift map on $MC_{\bullet\bullet}(R)$. Recall how this map is produced: we lift x to an element of the middle complex, and measure the extent to which this lifted element fails to be

a cycle. Now in our case, we have an obvious lifting of x, obtained by shifting x one unit up and one unit to the right. Since x was a cycle and the diagram is periodic, the obstruction is given by the Hochschild class of $Bx_{0,n-1}$ in $R^{\otimes n+1}$. □

There are cohomology theories dual to Hochschild and cyclic homology, which in fact show up more often in the literature than the corresponding homology theories. Even though we won't be using these as much, for the sake of completeness we record the basic definitions.

6.1.33. Definition. Let k be a commutative ring and let R be a k-algebra. The **Hochshild cohomology** of R (with respect to the ground ring k) is by definition the (co)homology $HH^\bullet(R)$ of the complex $C^\bullet(R)$ dual to $C_\bullet(R)$. In other words, $C^\bullet(R) = \mathrm{Hom}_k(C_\bullet(R), k)$, with the differential of degree $+1$ dual to b. Thus $HH^0(R)$ consists of k-linear maps $\varphi : R \to k$ vanishing on the image of b_1, in other words, satisfying the relation $\varphi(a_0a_1) = \varphi(a_1a_0)$ for $a_0, a_1 \in R$. Such a map is called a **trace**, after the most famous example, the usual trace $M_n(k) \to k$.

6.1.34. Definition. Let k be a commutative ring and let R be a k-algebra. The **dual cyclic double complex** $CC^{\bullet\bullet}(R)$ of R is the double cochain complex given by $\mathrm{Hom}_k(CC^{\bullet\bullet}(R), k)$, with the obvious dual differentials. Two cases are now of interest: that where the complex is chosen to live in the first quadrant, in which case the cohomology of the total complex is called $HC^\bullet(R)$, the **cyclic cohomology** of R, and that where the complex is chosen to live in both the first and the second quadrants, in which case the cohomology of the total complex is called $HP^\bullet(R)$, the **periodic cyclic cohomology** of R. Note that in the second of these cases, we use an infinite *sum* and not a product. As before we have an S-operator, dual to the one on homology, but it increases degree by 2. Similarly we have a Connes cyclic cochain complex $(C^\bullet_\lambda(R), b^*)$ obtained by taking $C^n_\lambda(R)$ to be the k-submodule of the Hochschild cochains $C^n(R)$ which are invariant under the operator dual to t_n. The cohomology of this complex is denoted $H^\bullet_\lambda(R)$. Cocycles $\varphi : R^{\otimes n+1} \to k$ in $C^n_\lambda(R)$ are sometimes called **n-multitraces.** They satisfy the conditions first written down in [Connes1]:

$$\varphi(a_0, a_1, \ldots, a_n) = (-1)^n\varphi(a_1, \ldots, a_n, a_0),$$

$$\sum_{i=0}^{n}(-1)^i\varphi(a_0, \ldots, a_ia_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1}\varphi(a_{n+1}a_0, a_1, \ldots, a_n) = 0$$

for $a_0, a_1, \ldots, a_n, a_{n+1} \in R$.

6.1.35. Theorem (Connes). *Let k be a commutative ring and let R be a k-algebra. There is a functorial long exact sequence*

$$\cdots \xrightarrow{S} HC^{n+1}(R) \xrightarrow{I} HH^{n+1}(R) \xrightarrow{B} HC^n(R)$$
$$\xrightarrow{S} HC^{n+2}(R) \xrightarrow{I} HH^{n+2}(R) \to \cdots .$$

Furthermore, $HP^n(R) = \varinjlim_S HC^{n+2j}(R)$. *Finally, if* $k \supseteq \mathbb{Q}$, *then there is a natural isomorphism* $H^\bullet_\lambda(R) \to HP^\bullet(R)$.

Proof. The first part is exactly dual to Theorem 6.1.19. The second statement follows from the fact that the complex computing $HP^\bullet(R)$ is naturally isomorphic to the direct limit under S of a sequence of copies of the complex computing $HC^\bullet(R)$. The last statement is precisely dual to Theorem 6.1.26. □

To conclude this subsection, we discuss the Hochschild homology and cyclic homology of algebras of matrices. The results will be vital when we attempt to link Hochschild homology and cyclic homology to K-theory, since studying the K-theory of a k-algebra R requires studying the algebra $M_n(R)$ of $n \times n$ matrices over R in the limit as $n \to \infty$.

Before working things out systematically in general, let's begin with the case of HH_0. Recall that by Corollary 6.1.6, for a general k-algebra R, $HH_0(R) = R/[R, R]$. Now consider the matrix algebra $M_n(R)$. It is spanned by elements $E_{ij}(a)$, $a \in R$, where this notation denotes the matrix with an a in the (i, j)-slot, and with 0's elsewhere. (We've used a capital "E" to distinguish this from the similar but different matrix $e_{ij}(a)$ of Definition 2.1.1.) Note that $E_{ij}(a)E_{jl}(b) = E_{il}(ab)$ and that $E_{ij}(a)E_{ml}(b) = 0$ if $j \neq m$. Thus we find that

$$[E_{ij}(a), E_{ml}(b)] =_{\text{def}} E_{ij}(a)E_{ml}(b) - E_{ml}(b)E_{ij}(a)$$
$$= \begin{cases} 0, & j \neq m,\ i \neq l, \\ E_{il}(ab), & j = m,\ i \neq l, \\ -E_{mj}(ba), & j \neq m,\ i = l, \\ E_{ii}(ab) - E_{jj}(ba), & j = m,\ i = l. \end{cases}$$

This shows that each $E_{ij}(a)$ with $i \neq j$ is a commutator, and that the images of $E_{ii}(ab)$ and of $E_{jj}(ba)$ coincide in

$$HH_0(M_n(R)) = M_n(R)/\left[M_n(R), M_n(R)\right].$$

Furthermore, we see that the usual trace (the sum of the diagonal entries) of $[E_{ij}(a), E_{ml}(b)]$ is either 0 or $ab - ba \in [R, R]$. So the usual trace $M_n(R) \to R$, followed by the quotient map $R \to HH_0(R) = R/[R, R]$, sends $[M_n(R), M_n(R)]$ to 0 and maps

$$HH_0(M_n(R)) = M_n(R)/\left[M_n(R), M_n(R)\right]$$

isomorphically onto $HH_0(R) = R/[R, R]$. This, together with the fact that the higher Hochschild groups should be viewed in some sense as "derived functors" of HH_0, suggests the following result.

6.1.36. Theorem (Morita invariance of Hochschild homology). *Let* k *be a commutative ring and let* R *be a* k*-algebra. Let* $M_n(R)$ *be the*

k-algebra of $n \times n$ matrices over R. Then there are natural isomorphisms $HH_m(M_n(R)) \to HH_m(R)$ induced by the "generalized trace"

$$\mathrm{Tr} : M_n(R)^{\otimes m+1} \to R^{\otimes m+1}$$

given by (the unique k-linear extension of) the map

$$\begin{aligned}&\mathrm{Tr}\left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right)\\ &\quad = \begin{cases} a_0 \otimes a_1 \otimes \cdots \otimes a_m, & j_0 = i_1,\ j_1 = i_2,\ \ldots,\ j_m = i_0, \\ 0, & \textit{otherwise}. \end{cases}\end{aligned}$$

Proof. First let's show that Tr is a chain map from the Hochschild complex of $M_n(R)$ to the Hochschild complex of R. Since $b_m = \sum_{l=0}^{m}(-1)^l d_l^m$, it's enough to show $\mathrm{Tr} \circ d_l^m = d_l^m \circ \mathrm{Tr}$, and it's enough to check this on elements of the form

$$E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m).$$

The result now follows by considering separately the case where $j_0 = i_1,\ j_1 = i_2,\ \ldots,\ j_m = i_0$, in which case both formulas give the same answer, and the case where these equalities are not all satisfied, in which case both $\mathrm{Tr} \circ d_l^m$ and $d_l^m \circ \mathrm{Tr}$ vanish on this element. So Tr is a chain map.

Secondly, it is obvious that the inclusion of R as a non-unital subalgebra of $M_n(R)$ (via $a \mapsto E_{11}(a)$) induces an injection ι of the Hochschild complex for R into that for $M_n(R)$, and that Tr provides a splitting for this inclusion. So the maps $HH_m(M_n(R)) \to HH_m(R)$ induced by the generalized trace are split surjections. We need to show that the kernel of the generalized trace is an acyclic subcomplex of the Hochschild complex of $M_n(R)$.

For this purpose we introduce a homotopy operator as follows. Define a map $h = \sum_{l=0}^{m}(-1)^l h_l : M_n(R)^{\otimes m+1} \to M_n(R)^{\otimes m+2}$, where

$$\begin{aligned}&h_l\left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right)\\ &\quad = \delta_{j_0 i_1} \cdots \delta_{j_{l-1} i_l}\left(E_{i_0 1}(a_0) \otimes E_{11}(a_1) \otimes \cdots \otimes E_{11}(a_l) \otimes E_{1 j_l}(1)\right.\\ &\qquad\qquad \left.\otimes E_{i_{l+1} j_{l+1}}(a_{l+1}) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right).\end{aligned}$$

(Here δ_{ij} is the usual "Kronecker delta," 1 if $i = j$ and 0 otherwise.) Thus, for example, $h_0\left(E_{i_0 j_0}(a_0)\right) = E_{i_0 1}(a_0) \otimes E_{1 j_0}(1)$ and

$$b_1 \circ h_0\left(E_{i_0 j_0}(a_0)\right) = \begin{cases} E_{i_0 i_0}(a_0) - E_{11}(a_0), & i_0 = j_0, \\ E_{i_0 j_0}(a_0), & i_0 \neq j_0, \end{cases}$$

which is just $id - \iota \circ \mathrm{Tr}$ applied to $E_{i_0 j_0}(a_0)$. More generally, we find that $d_p^{m+1} \circ h_l = h_l \circ d_{p-1}^{m+1}$ if $p > l + 1$ (this is fairly obvious since h_l does not change the last tensor factors) and that

$$\begin{aligned}&d_p^{m+1} \circ h_l\left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right)\\ &= \begin{cases} d_0^{m+1}\left(E_{i_0 1}(a_0) \otimes E_{1 j_0}(1) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right) \\ \qquad = E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m),\ p = l = 0, \\ d_{m+1}^{m+1}\left(E_{i_0 1}(a_0) \otimes E_{11}(a_1) \otimes \cdots \otimes E_{11}(a_m) \otimes E_{1 j_m}(1)\right) \\ \qquad \delta_{i_0 j_m} \delta_{j_0 i_1} \cdots \delta_{j_{m-1} i_m} \\ \qquad = E_{11}(a_0) \otimes E_{11}(a_1) \otimes \cdots \otimes E_{11}(a_m) \\ \qquad \delta_{i_0 j_m} \delta_{j_0 i_1} \cdots \delta_{j_{m-1} i_m}, \qquad p = m+1,\ l = m. \end{cases}\end{aligned}$$

Thus $d_0^{m+1} \circ h_0 = id$ and $d_{m+1}^{m+1} \circ h_m = \iota \circ \mathrm{Tr}$. Finally, we see that

$$\begin{aligned}
&d_l^{m+1} \circ h_l \left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right) \\
&\quad = \delta_{j_0 i_1} \cdots \delta_{j_{l-1} i_l}\, d_l^{m+1} \left(E_{i_0 1}(a_0) \otimes E_{11}(a_1) \otimes \cdots \right.\\
&\quad\quad \left. \otimes E_{11}(a_l) \otimes E_{1 j_l}(1) \otimes E_{i_{l+1} j_{l+1}}(a_{l+1}) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right) \\
&\quad = \delta_{j_0 i_1} \cdots \delta_{j_{l-1} i_l}\, E_{i_0 1}(a_0) \otimes E_{11}(a_1) \otimes \cdots \\
&\quad\quad \otimes E_{1 j_l}(a_l) \otimes E_{i_{l+1} j_{l+1}}(a_{l+1}) \otimes \cdots \otimes E_{i_m j_m}(a_m) \\
&\quad = d_l^{m+1} \circ h_{l-1} \left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right)
\end{aligned}$$

and that

$$\begin{aligned}
&d_p^{m+1} \circ h_l \left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right) \\
&\quad = \delta_{j_0 i_1} \cdots \delta_{j_{l-1} i_l}\, d_p^{m+1} \left(E_{i_0 1}(a_0) \otimes E_{11}(a_1) \otimes \cdots \right.\\
&\quad\quad \left. \otimes E_{11}(a_l) \otimes E_{1 j_l}(1) \otimes E_{i_{l+1} j_{l+1}}(a_{l+1}) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right) \\
&\quad = \delta_{j_0 i_1} \cdots \delta_{j_{l-1} i_l}\, E_{i_0 1}(a_0) \otimes E_{11}(a_1) \otimes \cdots \otimes E_{11}(a_p a_{p+1}) \otimes \cdots \\
&\quad\quad \otimes E_{11}(a_l) \otimes E_{1 j_l}(1) \otimes E_{i_{l+1} j_{l+1}}(a_{l+1}) \otimes \cdots \otimes E_{i_m j_m}(a_m) \\
&\quad = \delta_{j_p i_{p+1}} h_{l-1} \left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \right.\\
&\quad\quad \left. \otimes E_{i_p j_{p+1}}(a_p a_{p+1}) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right) \\
&\quad = h_{l-1} \circ d_p^m \left(E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_m j_m}(a_m)\right)
\end{aligned}$$

if $p < l$. So on $M_n(R)^{\otimes m+1}$,

$$\begin{aligned}
b \circ h + h \circ b &= \sum_{p=0}^{m+1} \sum_{l=0}^{m} (-1)^{p+l} d_p^{m+1} \circ h_l + \sum_{p=0}^{m} \sum_{l=0}^{m-1} (-1)^{p+l} h_l \circ d_p^m \\
&= id - \iota \circ \mathrm{Tr} + \sum_{l=1}^{m+1} \pm \left(d_l^{m+1} \circ h_l d_l^{m+1} \circ h_{l-1}\right) + 0 + 0 \\
&= id - \iota \circ \mathrm{Tr},
\end{aligned}$$

and thus $\iota \circ \mathrm{Tr}$ is chain homotopic to the identity. □

6.1.37. Theorem (Morita invariance of cyclic homology). *Let k be a commutative ring and let R be a k-algebra. Let $M_n(R)$ be the k-algebra of $n \times n$ matrices over R. Then the generalized trace of Theorem 6.1.36 induces natural isomorphisms $HC_m(M_n(R)) \to HC_m(R)$, $HC_m^-(M_n(R)) \to HC_m^-(R)$, $HP_m(M_n(R)) \to HP_m(R)$, $HC^m(M_n(R)) \to HC^m(R)$, and $HP^m(M_n(R)) \to HP^m(R)$.*

Proof. By the proof of Theorem 6.1.36, the generalized trace commutes with each summand d_l^m in the Hochschild differential b, and not only with the full differential. Thus it also commutes with b'. It is also clearly invariant under cyclic permutations and thus commutes with t_m and N_m. So it induces maps of the cyclic double complex and of the dual cyclic

double complex, and thus maps of all the cyclic homology and cohomology groups. Again these maps are splittings for the map the other way induced by the non-unital inclusion of R into $M_n(R)$ via $a \mapsto E_{11}(a)$.

We still need to show that the maps on cyclic homology groups induced by the generalized trace are isomorphisms. For the groups $HC_\bullet$, this follows easily from Theorem 6.1.36, since $HC_m = 0$ for $m < 0$ and $HC_0 = HH_0$ by Theorem 6.1.19. This enables us to prove the result by induction on m, starting with the case $m = 0$. Assuming that $HC_m(M_n(R)) \to HC_m(R)$ is an isomorphism for $m < m_0$, we note that Theorem 6.1.19 gives us a commutative diagram with exact columns

$$\begin{array}{ccc}
HC_{m_0-1}(M_n(R)) & \longrightarrow & HC_{m_0-1}(R) \\
{\scriptstyle B}\big\downarrow & & {\scriptstyle B}\big\downarrow \\
HH_{m_0}(M_n(R)) & \longrightarrow & HH_{m_0}(R) \\
{\scriptstyle I}\big\downarrow & & {\scriptstyle I}\big\downarrow \\
HC_{m_0}(M_n(R)) & \longrightarrow & HC_{m_0}(R) \\
{\scriptstyle S}\big\downarrow & & {\scriptstyle S}\big\downarrow \\
HC_{m_0-2}(M_n(R)) & \longrightarrow & HC_{m_0-2}(R) \\
{\scriptstyle B}\big\downarrow & & {\scriptstyle B}\big\downarrow \\
HH_{m_0-1}(M_n(R)) & \longrightarrow & HH_{m_0-1}(R).
\end{array}$$

Applying Theorem 6.1.36 and the inductive hypothesis, we see that the two horizontal arrows on the top and the two horizontal arrows on the bottom are isomorphisms. Thus by the Five-Lemma, $HC_{m_0}(M_n(R)) \to HC_{m_0}(R)$ is an isomorphism. Similar arguments can be given for $HC^\bullet$ using Theorem 6.1.35, and then one gets an isomorphism $HP^\bullet(M_n(R)) \to HP^\bullet(R)$ by taking limits.

It remains to handle $HC_\bullet^-$ and $HP_\bullet$. One way to do this is to show first that Tr induces isomorphisms on $HP_\bullet$. When the Mittag-Leffler condition is satisfied, this follows from the result for $HC_\bullet$ together with Theorem 6.1.22; even when this is not the case, one can show with a little more homological algebra that the kernel of the map $HP_\bullet \to \varprojlim_S HC_\bullet$ can be identified with $\varprojlim_S{}^1 HC_\bullet$, and thus Tr induces an isomorphism on this as well. Then one gets an isomorphism on $HP_\bullet$ by the Five-Lemma, and from this one gets an isomorphism on $HC_\bullet^-$ by Theorem 6.1.21 and the Five-Lemma. Alternatively, one can check directly as in the proof of Theorem 6.1.36 that Tr is a homotopy equivalence on the relevant cyclic double complexes. □

Connections with "Non-commutative de Rham Theory." Much of the motivation for the study of Hochschild and of cyclic homology comes

from the case where R is commutative and can be viewed as an algebra of smooth functions on a manifold or an algebraic variety X. In this case, the Hochschild or cyclic homology of R is closely related to the de Rham cohomology of X, the closed differential forms modulo the exact forms (those that are exterior derivatives of other forms). This suggests studying a calculus of "non-commutative differential forms" over an arbitrary k-algebra. The following framework for doing so was basically developed in [KaroubiHomCyc]. For a more recent survey, see [Cuntz].

6.1.38. Definition. Let k be a commutative ring and let R be a k-algebra. Let $\bar{R}$ denote the quotient k-module $R/(k \cdot 1)$. (In most cases of interest, R splits as $k \cdot 1 \oplus \tilde{R}$ for some k-submodule $\tilde{R}$ of R, isomorphic to $\bar{R}$. This is of course automatic if either R is an augmented k-algebra, in which case we can take $\tilde{R}$ to be the augmentation ideal, or if k is a field.) The **algebra of non-commutative differential forms** ΩR over R is by definition the universal (associative but not necessarily commutative) k-algebra generated by a copy of R and by symbols $\{da : a \in R\}$, subject to the conditions that the identity 1 of R also be an identity for the whole algebra ΩR and that $a \mapsto da$ be a k-linear derivation (in other words, that it be k-linear and satisfy $d(a_0 \cdot a_1) = da_0 \cdot a_1 + a_0 \cdot da_1$ for $a_0, a_1 \in R$). (These relations imply that $d1 = 0$, since $d1 = d(1 \cdot 1) = d1 \cdot 1 + 1 \cdot d1 = d1 + d1$.) Note that any element of ΩR can be written as a linear combination of elements of the form $a_0\, da_1 \cdots da_n$, since $dx \cdot y = -x \cdot dy + d(xy)$ and thus we can always move factors from R to the left. The algebra ΩR is $\mathbb{N}$-graded, via the grading

$$\deg(a_0\, da_1 \cdots da_n) = n.$$

There is a linear operator $d : \Omega R \to \Omega R$ defined by

$$d(a_0\, da_1 \cdots da_n) = da_0 da_1 \cdots da_n = 1 \cdot da_0 \cdots da_n,$$

and clearly $d^2 = 0$. In fact, d is a graded derivation; if ω_1 is homogeneous of a certain degree, then

$$d(\omega_1 \omega_2) = d(\omega_1)\omega_2 + (-1)^{\deg \omega_1} \omega_1\, d\omega_2.$$

To relate Hochschild homology and cyclic homology to differential forms, we begin by noting that as a graded k-module, ΩR is naturally isomorphic to $\bigoplus_{n=0}^{\infty} R \otimes_k \bar{R}^{\otimes n}$, via the identification

$$x_0\, dx_1 \cdots dx_n \longleftrightarrow x_0 \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_n.$$

(Note that for $x \in R$, dx only depends on the image $\bar{x}$ of x in $\bar{R}$. Thus there is a natural map from $\bigoplus_{n=0}^{\infty} R \otimes_k \bar{R}^{\otimes n}$ onto ΩR. To show that it is an isomorphism, introduce the operator d and a multiplication in the set of tensors in the obvious way; then the map must be injective by universality.) We introduce another operator $b : \Omega R \to \Omega R$, lowering the degree by 1, by the condition

$$b(\omega\, dx) = (-1)^{\deg \omega} [\omega, x], \quad \omega \in \Omega R,\ x \in R.$$

One has to check that b is well defined, in other words, that if $\omega_0\, dx_0 = \omega_1\, dx_1$, then $[\omega_0,\, x_0] = [\omega_1,\, x_1]$. But in fact, under the above identification of ΩR with the tensor algebra, b is just the ordinary Hochschild boundary (for the "reduced Hochschild complex," in which we divide out by the span of all elementary tensors containing a 1 except in the first slot), since

$$\begin{aligned} b\,(x_0\, dx_1 \cdots dx_n) &= (-1)^{n-1}\,[x_0\, dx_1 \cdots dx_{n-1},\, x_n] \\ &= (-1)^{n-1}\,(x_0\, dx_1 \cdots dx_{n-1})\, x_n + (-1)^n x_n x_0\, dx_1 \cdots dx_{n-1}, \end{aligned}$$

and by induction on n we can write

$$\begin{aligned} (dx_1 \cdots dx_{n-1})\, x_n =& (-1)^{n-1} x_1\, dx_2 \cdots dx_n \\ &+ \sum_{j=1}^{n-1} (-1)^{n-1+j} dx_1 \cdots d(x_j x_{j+1}) \cdots dx_n. \end{aligned}$$

Thus

$$\begin{aligned} b\,(x_0\, dx_1 \cdots dx_n) = & (x_0 x_1)\, dx_2 \cdots dx_n \\ &+ \sum_{j=1}^{n-1} (-1)^j x_0\, dx_1 \cdots d(x_j x_{j+1}) \cdots dx_n \\ &+ (-1)^n (x_n x_0)\, dx_1 \cdots dx_{n-1}, \end{aligned}$$

which corresponds exactly to (6.1.2), and the homology of the complex

$$\cdots \xrightarrow{b} \Omega^n R \xrightarrow{b} \Omega^{n-1} R \xrightarrow{b} \cdots$$

is just the Hochschild homology of the k-algebra R.

6.1.39. Definition. Let k be a commutative ring and let R be a k-algebra. The **non-commutative de Rham homology** $H_\bullet^{\mathrm{de\,R}}(R)$ of R (with respect to k) is the (co)homology of the (co)chain complex $(\Omega_{\mathrm{ab}}^\bullet R,\, d)$, where $\Omega_{\mathrm{ab}}^\bullet R$ is the quotient of ΩR by the k-submodule generated by all graded commutators $\omega_1\omega_2 - (-1)^{\deg \omega_1 \deg \omega_2} \omega_2 \omega_1$ with ω_1 and ω_2 homogeneous. The d here is the map induced by d on the quotient; it makes sense since the original d in Definition 6.1.38 is a graded derivation. Note for example that $\Omega_{\mathrm{ab}}^0 R = R/[R,\, R] = HH_0(R)$, that

$$\Omega_{\mathrm{ab}}^1 R = \Omega^1 R / \langle x\, dy - dy \cdot x : x,\, y \in R \rangle\,,$$

and so on. Also note that $\Omega^n(k) = 0$ for $n > 0$, so that $H_n^{\mathrm{de\,R}}(k) = 0$ for $n > 0$. The reader familiar with exterior calculus on manifolds will recall that if $R = C^\infty(X)$ for some manifold X, then the exterior differential forms on X are a graded-commutative algebra generated by R and by dR, and are thus a quotient of $\Omega_{\mathrm{ab}}^\bullet R$ by universality. However, though $H^{\mathrm{de\,R}}$ is *closely related* to ordinary de Rham cohomology for manifolds and algebraic varieties, it is almost always different from the latter.

The connection with cyclic homology is now the following. To avoid worrying about the correct definition of "reduced groups" (see [LodayQuil, §4] for a more complete discussion) we deal for simplicity with the case of an augmented k-algebra.

6.1.40. Theorem (Karoubi [KaroubiHomCyc]). *Let $k \supseteq \mathbb{Q}$ be a commutative ring and let R be an augmented k-algebra with augmentation ideal I, so that $HH_\bullet(R) \cong HH_\bullet(k) \oplus HH_\bullet(R, I)$, and similarly with HC. Then for $n > 0$, there is a natural isomorphism from $H_n^{\text{de R}}(R)$ to the kernel of the map $B : HC_n(R, I) \to HH_{n+1}(R)$ of Theorem 6.1.19. When $n = 0$, $H_0^{\text{de R}}(R)$ is naturally isomorphic to the kernel of the map $B : HC_0(R) \to HH_1(R)$.*

Proof. First of all, we use the hypothesis that $k \supseteq \mathbb{Q}$ to replace $HC_\bullet$ by $H_\bullet^\lambda$. We already saw that the normalized Hochschild complex of R can be identified with $(\Omega R, b)$. Let $\bar{C}_\bullet^\lambda(R)$ be the quotient of $C_\bullet^\lambda(R)$ by the graded k-submodule generated by all elementary tensors $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ with some $a_j = 1$. Since

$$\begin{aligned} b_n(1 \otimes a_1 \otimes \cdots \otimes a_n) &= (a_1 \otimes \cdots \otimes a_n) + (1 \otimes \cdots) \\ &\qquad + (-1)^n (a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}) \\ &= (1 - t_{n-1})(a_1 \otimes \cdots \otimes a_n) + (1 \otimes \cdots), \end{aligned}$$

this submodule is b-invariant and $\bar{C}_\bullet^\lambda(R)$ is also a chain complex. The complex $\bar{C}_\bullet^\lambda(R)$ corresponds to the "cyclicization" of the normalized Hochschild complex and is also evidently isomorphic to the quotient of the Hochschild complex of the non-unital ring I by $1 - t$, so it computes $H_\bullet^\lambda(R, I) \cong HC_\bullet(R, I)$. We map ΩR to $\bar{C}_\bullet^\lambda(R)$ by sending $x_0\, dx_1 \cdots dx_n$ to the image of $x_0 \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_n$. The image of d in ΩR is generated by things of the form $dx_1 \cdots dx_n = 1\, dx_1 \cdots dx_n$, which goes to 0 in the reduced complex $\bar{C}_\bullet^\lambda(R)$. From the calculation in Definition 6.1.38, any commutator $[\omega, x]$ goes to something in the image of b. Similarly, any graded commutator $\omega\, dx - (-1)^{\deg \omega} dx \cdot \omega$, with $\omega = x_0\, dx_1 \cdots dx_n$, can be rewritten as

$$\begin{aligned} &(x_0\, dx_1 \cdots dx_n)\, dx - (-1)^n dx \cdot x_0\, dx_1 \cdots dx_n \\ &\qquad = x_0\, dx_1 \cdots dx_n\, dx + (-1)^{n+1} d(x x_0)\, dx_1 \cdots dx_n + (-1)^n x\, dx_0 \cdots dx_n. \end{aligned}$$

The second term on the right is in the image of d and thus goes to 0, and the sum of the other two terms goes to

$$x_0 \otimes \cdots \otimes x_n \otimes x + (-1)^n x \otimes x_0 \otimes \cdots \otimes x_n = (1 - t_{n+1})(x_0 \otimes \cdots \otimes x_n \otimes x),$$

which goes to 0 in $C_\bullet^\lambda(R)$. So we get an induced map $\Omega_{\text{ab}}^\bullet R/d(\Omega_{\text{ab}}^\bullet R) \to \bar{C}_\bullet^\lambda(R)/b(\bar{C}_\bullet^\lambda(R))$. This map is in fact a k-linear isomorphism, except for an extra factor of $k \cdot 1$ on the left in degree 0, since it has an inverse induced from the obvious map $\bar{C}_\bullet^\lambda(R) \to \Omega R/d(\Omega R)$ together with the observation that the image of b maps into the k-submodule generated by (graded) commutators.

We'll now use our isomorphism $\Omega_{\text{ab}}^\bullet R/d(\Omega_{\text{ab}}^\bullet R) \to \bar{C}_\bullet^\lambda(R)/b(\bar{C}_\bullet^\lambda(R))$ to get the desired isomorphism on homology. By definition, $H_n^{\text{de R}}(R)$ is the kernel of the map induced by d on $\Omega_{\text{ab}}^n R/d(\Omega_{\text{ab}}^{n-1} R)$. This goes over in $\bar{C}_n^\lambda(R)/b(\bar{C}_{n+1}^\lambda(R))$ to the kernel of the map induced by s_n (see (6.1.5)). Now consider the action of t_n on $I^{\otimes n}$. Since $k \supseteq \mathbb{Q}$, $I^{\otimes n}$ splits as the

direct sum of the kernel of $1 - t_n$, which we can identify with $\bar{C}^\lambda_\bullet(R)$, and a complementary space on which $1 - t_n$ is invertible. Thus the Connes operator $B_n = (1 - t_{n+1}) \circ s_n \circ N_n$ differs only by the invertible scalar factor of $n+1$ from the map induced by s_n on $\bar{C}^\lambda_n(R)/b(\bar{C}^\lambda_{n+1}(R))$. It follows that $H^{\text{de R}}_n(R)$ corresponds exactly to the submodule of $\bar{C}^\lambda_n(R)/b(\bar{C}^\lambda_{n+1}(R))$ killed by the map induced by B. We claim this is (for $n > 0$) exactly the kernel of the map $B : HC_n(R, I) \to HH_{n+1}(R)$. For this we represent a class in $HC_n(R, I)$ by a cycle in $\bar{C}^\lambda_n(R)$, viewed as a subcomplex of the 0-th column of $MC_{\bullet\bullet}(R)$. By Lemma 6.1.32, the image of our class under the Connes B-map is given by the map induced by B on our cycle in $\bar{C}^\lambda_n(R)$, and so the theorem follows. The only difference in degree 0 is that $\bar{C}^\lambda_0(R)/b(\bar{C}^\lambda_1(R))$ should be replaced by $C^\lambda_0(R)/b(\bar{C}^\lambda_1(R))$, which explains why we need to replace $HC_0(R, I)$ by $HC_0(R)$. □

6.1.41. Exercise. Let k be a commutative ring and let R be a k-algebra. Show from Theorem 6.1.21 that the maps

$$S: HC^-_n(R) \to HC^-_{n-2}(R)$$

and $HC^-_n(R) \to HP_n(R)$ are isomorphisms for $n \leq 0$.

6.1.42. Exercise. Let k be a field of characteristic 0 and let $R = k[t]$.

(1) Show from Theorem 6.1.22 and Corollary 6.1.23 that the map $HP_n(R) \to HC_n(R)$ is an isomorphism for $n > 0$, so that $HP_n(R)$ vanishes for n odd and is $\cong k$ for n even. Deduce that $HC^-_1(R)$ is infinite-dimensional (as a vector space over k) but that $HC^-_n(R)$ vanishes for all other odd n.
(2) Now compute $HC^-_\bullet(R)$ as a module over $k[u]$ and verify the relations of Theorem 6.1.24.
(3) Compute $H^{\text{de R}}_\bullet(R)$ directly from the definition, and verify the result of Theorem 6.1.40 for this example.

6.1.43. Exercise. Let k be a field of characteristic 0 and let $R = k[v, v^{-1}]$, the Laurent polynomial ring in one variable.

(1) Argue as in Example 6.1.7(a) to compute $HH_\bullet(R)$.
(2) Compute $HC_1(R)$ by dividing $HH_1(R)$ by the image of
$$B : HC_0(R) \cong R \to HH_1(R)$$
as in Example 6.1.20(b). Show that $HC_1(R)$ is one-dimensional and that its generator may be identified with $v^{-1}dv$, the "logarithmic derivative" of v.
(3) Now apply Theorem 6.1.19 to show that $HC_n(R)$ is one-dimensional for all $n \geq 1$. Show from Theorem 6.1.22 and Corollary 6.1.23 that the map $HP_n(R) \to HC_n(R)$ is an isomorphism for $n > 0$, so that $HP_n(R) \cong k$ for all n.
(4) Compute $HC^-_\bullet(R)$ as a module over $k[u]$ and verify the relations of Theorem 6.1.24 for this example.
(5) Compute $H^{\text{de R}}_\bullet(R)$, and verify the result of Theorem 6.1.40 for this example.

6.1.44. Exercise (Additivity of Hochschild and cyclic homology for direct products of rings). Let k be a commutative ring and let $R = R_1 \times R_2$ be the direct product of k-algebras R_1 and R_2. Show that $HH_\bullet(R) \cong HH_\bullet(R_1) \oplus HH_\bullet(R_2)$, and similarly for $HC_\bullet$, $HP_\bullet$, etc.

6.1.45. Exercise. Let k be a commutative ring and let G be a group. Then we can form the k-algebra $R = kG$.

(1) Show that $HH_\bullet(kG)$ can be naturally identified with the group homology $H_\bullet(G, M)$, where M denotes kG viewed as a G-module via the action of G on itself by conjugation.
(2) Then show that M decomposes as a direct sum of submodules supported on the conjugacy classes of G, and thus that $HH_\bullet(kG)$ splits up as such a direct sum as well.
(3) Show that the summand of M associated to the conjugacy class of an element $g \in G$ is isomorphic as a kG-module to $kG \otimes_{kC_G(g)} k$, where $C_G(g)$ is the centralizer of g in G (or the stabilizer of g for the conjugation action). Deduce from Shapiro's Lemma (Corollary 4.1.12) that
$$HH_\bullet(kG) \cong \bigoplus_{\langle g \rangle} H_\bullet(C_G(g),\, k),$$
where $\langle g \rangle$ runs over the conjugacy classes of G. Note that the group homology $H_\bullet(G, k)$ corresponds to the conjugacy class of the identity element.
(4) If $k = \mathbb{C}$ and G is a finite group, then kG splits as a direct sum of matrix algebras over $\mathbb{C}$ corresponding to the various irreducible representations of G, and we can alternatively use this decomposition and Morita invariance (Theorem 6.1.36) to compute that $HH_\bullet(kG)$ vanishes except in degree 0, where it has dimension equal to the number of irreducible representations. (Here you need the result of Exercise 6.1.44.) Show that the two methods for computing Hochschild homology agree in this case (using some of the basics of representations of finite groups as found say in Chapter 5 of [Jacobson, II]).

6.1.46. Exercise. Let k be a commutative ring and let $R = k[e]/(e^2 - e)$, the universal unital k-algebra generated by a single idempotent e. Note that any element of R can be written uniquely in the form $a(1-e) + be$.

(1) Show that ΩR is the universal associative R-algebra on one additional generator de satisfying $e \cdot de = de \cdot (1-e)$.
(2) Assuming that 2 is invertible in k, show that the images of de and of $e \cdot de$ vanish in $\Omega^1_{\text{ab}} R$, hence that $H_0^{\text{de R}}(R) \cong R$ and $H_1^{\text{de R}}(R) = 0$. (Caution: $\Omega^\bullet_{\text{ab}} R$ is obtained by dividing out by the k-submodule generated by graded commutators, not by the *ideal* generated by graded commutators, so the fact that ΩR is generated as an R-algebra by de, plus the fact that the image of de vanishes in $\Omega^1_{\text{ab}} R$, do not imply that $\Omega^\bullet_{\text{ab}} R$ vanishes in degrees > 0.)

(3) For $k \supseteq \mathbb{Q}$, compute $\Omega^\bullet_{\mathrm{ab}} R$ and thus compute $H^{\mathrm{de\,R}}_\bullet$ directly. Verify the conclusions of Theorem 6.1.40 for this example. (You can compute the cyclic homology using Exercise 6.1.44.)

6.1.47. Exercise. Let $k \subseteq F \subseteq L$ be fields, with F algebraically closed. Viewing F and L as k-algebras, show that the induced maps on Hochschild and on cyclic homology $HH_\bullet(F) \to HH_\bullet(L)$ and $HC_\bullet(F) \to HC_\bullet(L)$ are injective. (Duplicate the argument of Suslin in Exercise 5.3.34.) Is this still true if one drops the assumption on F?

6.1.48. Exercise. Let k be a field, so that Proposition 6.1.4 is applicable. Show that if R is any k-algebra, $HH_n(R[u,\, u^{-1}]) \cong HH_n(R) \oplus HH_{n-1}(R)$. (Exercise 6.1.43 gives a special case of this. Hint: tensor together a resolution of R as an $R \otimes_k R^{\mathrm{op}}$-module with a resolution of $k[u,\, u^{-1}]$ as a $k[u,\, u^{-1}] \otimes_k k[u,\, u^{-1}]^{\mathrm{op}}$-module to get a double complex from which you can compute the Tor-groups.)

This suggests that there should be a chain map

$$C_\bullet(R) \to C_{\bullet+1}(R[u,\, u^{-1}])$$

which induces a split injection on Hochschild homology. Such a map is given explicitly by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto -ua_0 \otimes \left(u^{-1} \otimes a_1 \otimes \cdots \otimes a_n - a_1 \otimes u^{-1} \otimes \cdots \otimes a_n \right.$$
$$\left. + \cdots + (-1)^n a_1 \otimes \cdots \otimes a_n \otimes u^{-1}\right).$$

Check that this is a chain map and that when followed by the augmentation map sending $u \mapsto 1$, it induces the 0-map $HH_\bullet(R) \to HH_{\bullet+1}(R)$.

6.1.49. Exercise. Let k be a field and let R be a k-algebra. Using Exercise 6.1.48, show by induction on n that $HC_{n-1}(R[u,\, u^{-1}]) \cong HC_{n-1}(R) \oplus HC_{n-2}(R)$, that the map

$$B : HC_{n-1}(R[u,\, u^{-1}]) \to HH_n(R[u,\, u^{-1}])$$

splits up as a direct sum of $B : HC_{n-1}(R) \to HH_n(R)$ and of $B : HC_{n-2}(R) \to HH_{n-1}(R)$, and that the shift map S for $R[u,\, u^{-1}]$ splits up as a direct sum of two copies of the shift map for R, one shifted in degree by 1. When k has characteristic 0, so that $HC_\bullet \cong H^\lambda_\bullet$, the split injection of $HC_n(R) \cong H^\lambda_n(R)$ into $H^\lambda_{n+1}(R[u,\, u^{-1}])$ is given by the chain map given by the same formula as in Exercise 6.1.48.

2. The Chern character

The aim of this section is to construct homomorphisms from K-groups to cyclic homology groups, which go under the general name of "Chern characters." We begin with the classical theory that motivated these, and

then describe first the theory for K_0 (which is a little more concrete), and then the theory for the higher groups.

The Classical Chern Character. When Chern did his work on the theory of so-called "characteristic classes," neither K-theory nor cyclic homology had been invented yet. Nevertheless, much of what he did to define the "Chern character" will be applicable to our setting, so we start with a sketch of the classical theory. Suppose X is a compact smooth manifold and $p : E \to X$ is a smooth complex vector bundle over X. (This means it is a $\mathbb{C}$-vector bundle in the sense of Definition 1.6.1, with the added requirement that E have the structure of a smooth manifold and that all the structure maps be C^∞.) Chern showed how to use the notion of "curvature" of the bundle to define classes $c_n(E) \in H^{2n}_{\text{de R}}(X, \mathbb{C})$, the de Rham cohomology of X, which can be put together to get an invariant called the Chern character

$$\text{Ch}(E) \in H^{\text{even}}_{\text{de R}}(X, \mathbb{C}) =_{\text{def}} \bigoplus_{n \geq 0} H^{2n}_{\text{de R}}(X, \mathbb{C})$$

which has the useful properties that $\text{Ch}(E_1 \oplus E_2) = \text{Ch}(E_1) + \text{Ch}(E_2)$, $\text{Ch}(E_1 \otimes E_2) = \text{Ch}(E_1) \cup \text{Ch}(E_2)$. It then follows fairly easily that Ch extends to a ring homomorphism from $KU^0(X)$ to $H^{\text{even}}_{\text{de R}}(X, \mathbb{C})$. One can then (after replacing de Rham cohomology by Čech cohomology) extend the definition to arbitrary compact spaces X (since any vector bundle on a compact space is the pull-back under a continuous map of a smooth vector bundle on a manifold), and even show that for general compact spaces, the Chern character Ch induces a ring isomorphism from $KU^0(X) \otimes_{\mathbb{Z}} \mathbb{C}$ to $H^{\text{even}}(X, \mathbb{C})$. Some of the details can be found in [Karoubi, Chapter V.3].

While a complete discussion of the classical Chern character would require too much of a digression into differential geometry, it is worth explaining the main idea in the construction of $\text{Ch}(E)$ in order to motivate the algebraic theory that will follow. Following the method of proof of Theorem 1.6.3, one can show first that smooth vector bundles over X correspond (via the correspondence between E and its module $\Gamma^\infty(X, E)$ of smooth sections) exactly to finitely generated projective modules, not over $C^{\mathbb{C}}(X)$, but over the ring of smooth functions $R = C^\infty_{\mathbb{C}}(X)$. In particular, any smooth vector bundle E is a direct summand in a trivial bundle, say of rank k. This now gives us a way to differentiate smooth sections of E, though it depends on a choice of an embedding of E as a direct summand in a trivial bundle. Observe that smooth sections of a rank-k trivial bundle are just k-tuples of smooth functions on X. So if $s \in \Gamma^\infty(X, E)$, we can define its "derivative" Ds to be the result of viewing s as a k-tuple of functions $(f_1, \cdots, f_k)$, taking the exterior derivatives $df_1, \cdots, df_k$, and projecting back down (to the space of "E-valued 1-forms"). However, even though $d^2 f_j = 0$, the result of applying D twice is usually non-zero, since there is no reason why d should commute with the projection p onto E. The operator D^2 is basically what is called the **curvature** of E. We may view it as a $k \times k$ matrix of (ordinary scalar-valued) differential 2-forms on X.

If we think of p as an idempotent matrix in $M_k(R)$, then we may compute the curvature as follows. We take the exterior derivative of a matrix of forms by differentiating each of its entries (so that dp is a $k \times k$ matrix of differential 1-forms on X), and we denote matrix multiplication by $\cdot$. First of all, since $p \cdot p = p$, differentiating gives $dp \cdot p + p \cdot dp = dp$, or $dp \cdot (1-p) = p \cdot dp$, an identity we will need later. Similarly, for a section s of E, $s = p \cdot s$ and $ds = dp \cdot s + p \cdot ds$, or $(1-p) \cdot ds = dp \cdot s$. Thus

$$\begin{aligned} D^2 s &= (p \circ d \circ p)^2 s \\ &= (p \circ d)(p \cdot ds) \\ &= p \cdot (dp \cdot ds + p \cdot d^2 s) \\ &= (p \cdot dp) \cdot ds = (dp \cdot (1-p)) \cdot ds \\ &= dp \cdot ((1-p) \cdot ds) \\ &= dp \cdot (dp \cdot s) \\ &= (dp \cdot dp) \cdot s. \end{aligned}$$

In other words, the actions of D^2 on the image of p (the sections of E) is given by left multiplication by $dp \cdot dp$. Since we want to think of the curvature of E as being 0 on the image of $1-p$ (the sections of the complement), the curvature of E (computed for this particular choice of p) is really left multiplication by

$$dp \cdot dp \cdot p = dp \cdot (1-p) \cdot dp = p \cdot dp \cdot dp. \tag{6.2.1}$$

The component of the Chern character of E in degree $2n$ is now obtained (up to a certain scalar normalizing factor which we won't worry about at the moment) by taking the de Rham cohomology class of

$$\operatorname{Tr}(D^2)^n = \operatorname{Tr}\left((p \cdot dp \cdot dp)^n\right) = \operatorname{Tr}\left(p \cdot (dp)^{2n}\right). \tag{6.2.2}$$

(Here we've again used the trick of moving all the p's across the dp's.) Of course, for the de Rham class to make sense, we have to check that this form is closed, but

$$\begin{aligned} d\left(\operatorname{Tr}\left(p \cdot (dp)^{2n}\right)\right) &= \operatorname{Tr}\left((dp)^{2n+1}\right) \\ &= \operatorname{Tr}\left((dp \cdot p + p \cdot dp) \cdot (dp)^{2n}\right) \\ &= \operatorname{Tr}\left((1-p) \cdot (dp)^{2n+1}\right) + \operatorname{Tr}\left(p \cdot (dp)^{2n+1}\right) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Tr}\left(p \cdot (dp)^{2n+1}\right) &= \operatorname{Tr}\left(p^2 \cdot (dp)^{2n+1}\right) \\ &= \operatorname{Tr}\left(p \cdot (dp)^{2n+1} \cdot (1-p)\right) \\ &= \operatorname{Tr}\left((1-p) \cdot p \cdot (dp)^{2n+1}\right) = 0, \end{aligned}$$

and similarly with the other term. Here in the last step we've used the fundamental property of the trace, which is invariance under cyclic permutations.

Since the curvature depends on the choice of the projection p, it is also important to check that the result of this procedure gives an invariant of E and doesn't depend on the choices made. The classical argument for this is based on "homotopy invariance." Namely, suppose p_t, $0 \le t \le 1$, is a one-parameter family of idempotents, so that the vector bundles they define are all equivalent to one another (by Corollary 1.6.12). We show that the de Rham class of $\mathrm{Tr}\left(p_t \cdot (dp_t)^{2n}\right)$ doesn't change with t. It is obviously enough by calculus to show that the derivative of this form (with respect to t) is exact, hence represents 0 in de Rham theory. If $\dot{p}_t$ denotes the derivative of p_t with respect to t, then $d\dot{p}_t$ is the derivative of dp_t with respect to t, and thus (leaving the subscript t's off the p's for simplicity of notation)

$$\frac{d}{dt}\left(\mathrm{Tr}\left(p \cdot (dp)^{2n}\right)\right) = \mathrm{Tr}\left(\dot{p} \cdot (dp)^{2n}\right) + \sum_{i=1}^{2n} \mathrm{Tr}\left(p \cdot (dp)^{i-1} \cdot d\dot{p} \cdot (dp)^{2n-i}\right).$$

Since $p^2 = p$, we have $\dot{p} \cdot p + p \cdot \dot{p} = \dot{p}$ or $p \cdot \dot{p} = \dot{p} \cdot (1-p)$. Thus

$$\begin{aligned} \mathrm{Tr}\left(p \cdot \dot{p} \cdot (dp)^{2n}\right) &= \mathrm{Tr}\left(p \cdot p \cdot \dot{p} \cdot (dp)^{2n}\right) \\ &= \mathrm{Tr}\left(p \cdot \dot{p} \cdot (1-p) \cdot (dp)^{2n}\right) \\ &= \mathrm{Tr}\left(p \cdot \dot{p} \cdot (dp)^{2n} \cdot (1-p)\right) \\ &= \mathrm{Tr}\left((1-p) \cdot p \cdot \dot{p} \cdot (dp)^{2n}\right) = 0, \end{aligned}$$

and similarly

$$\mathrm{Tr}\left((1-p) \cdot \dot{p} \cdot (dp)^{2n}\right) = 0.$$

So the first term in the derivative vanishes. The remaining terms add up to something exact, since

$$\begin{aligned} \sum_{i=1}^{2n} \mathrm{Tr}\left(p \cdot (dp)^{i-1} \cdot d\dot{p} \cdot (dp)^{2n-i}\right) &= \sum_{i=1}^{2n} \mathrm{Tr}\left((dp)^{2n-i} \cdot p \cdot (dp)^{i-1} \cdot d\dot{p}\right) \\ &= \sum_{i=1}^{2n} \mathrm{Tr}\left((dp)^{2n-1} \cdot \begin{pmatrix} p, & i \text{ odd} \\ 1-p, & i \text{ even} \end{pmatrix} \cdot d\dot{p}\right) \\ &= n\,\mathrm{Tr}\left((dp)^{2n-1} \cdot d\dot{p}\right), \end{aligned}$$

which is exact. This completes the proof that the Chern classes are homotopy invariants of p. From this it is not too hard to see that they only depend on the isomorphism class of the vector bundle E.

6.2.3. Example. To show how this works, let's note that the set of rank-one self-adjoint idempotents in $M_2(\mathbb{C})$ may be identified with the set of one-dimensional subspaces of $\mathbb{C}^2$ (since there is a unique projection onto

each subspace annihilating the orthogonal complement), or in other words with $\mathbb{P}^1(\mathbb{C}) \cong S^2$. Thus if $X = \mathbb{P}^1(\mathbb{C}) \cong S^2$, there is a smooth idempotent map $p : X \to M_2(\mathbb{C})$ corresponding to this identification, and p will define a one-dimensional complex vector bundle E, which (modulo a trivial bundle) is the generator of $\tilde{K}^0(S^2) \cong \mathbb{Z}$. If we view S^2 as $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum x_i^2 = 1\}$, the map p may be given by

$$p(x) = \frac{1}{2}\begin{pmatrix} 1+x_1 & x_2+ix_3 \\ x_2-ix_3 & 1-x_1 \end{pmatrix}.$$

Then

$$dp = \frac{1}{2}\begin{pmatrix} dx_1 & dx_2 + i\,dx_3 \\ dx_2 - i\,dx_3 & -dx_1 \end{pmatrix}$$

and

$$dp \cdot dp = \frac{1}{4}\begin{pmatrix} -2i\,dx_2 \wedge dx_3 & 2(dx_1 \wedge dx_2 + i\,dx_1 \wedge dx_3) \\ 2(-dx_1 \wedge dx_2 + i\,dx_1 \wedge dx_3) & 2i\,dx_2 \wedge dx_3 \end{pmatrix}.$$

So

$$\begin{aligned} &p \cdot dp \cdot dp \\ &= \frac{1}{4}\begin{pmatrix} -i(1+x_1)\,dx_2 \wedge dx_3 + (x_2+ix_3)(-dx_1 \wedge dx_2 + i\,dx_1 \wedge dx_3) & * \\ * & * \end{pmatrix}, \end{aligned}$$

which has trace

$$\frac{-i}{2}\left(x_1\,dx_2 \wedge dx_3 + x_3\,dx_1 \wedge dx_2 + x_2\,dx_3 \wedge dx_1\right).$$

Since this integrates to something non-zero over S^2, $\mathrm{Ch}(E) \neq 0$. Note that we could not detect non-triviality of E merely from $\mathrm{Tr}(p)$, which is $\equiv 1$.

The Chern Character on K_0. Now we go back to the case of a general k-algebra R and study ways of detecting non-zero elements of $K_0(R)$ using cyclic homology. There are various Chern characters, depending on what one wants to use as the target of one's homomorphism from $K_0(R)$: $HH_0(R) = HC_0(R)$, $HP_0(R) \cong HC_0^-(R)$, or $H_0^{\mathrm{de\,R}}(R)$. These are related by the homomorphisms that relate these various cyclic homology groups.

We begin by discussing the simplest of these maps, the one into $HH_0(R) = HC_0(R)$. Unfortunately it is not always powerful enough to be very useful.

6.2.4. Definition. Let R be any ring (with identity). The **trace map** $\mathrm{Tr} : K_0(R) \to R/[R, R]$ (by Corollary 6.1.6 and Theorem 6.1.19, this is $HH_0(R) = HC_0(R)$ for any choice of a ground ring k) is defined to be the map induced by sending the class of an idempotent $p \in M_n(R)$ to the image of its trace (the sum of the diagonal entries) in the quotient $R/[R, R]$.

6.2.5. Proposition. *If R is any ring, the trace map $K_0(R) \to R/[R, R]$ as just defined is well defined and is a homomorphism, which is functorial in the ring R.*

Proof. Obviously $\mathrm{Tr}(p)$ doesn't change if we enlarge the matrix p by adding 0's in the lower right, and clearly Tr is functorial in R. By Lemma 1.2.1 and the universal property (Theorem 1.1.3) of the Grothendieck group, to show that we have a well-defined homomorphism on K_0, it is enough to show that Tr is conjugation-invariant and additive under block direct sums. The latter is obvious, so we just need to see that if $g \in GL(n, R)$ and p is an idempotent in $M_n(R)$, then $\mathrm{Tr}(gpg^{-1}) = \mathrm{Tr}(p)$. The proof is the same as in elementary linear algebra. We let $b = pg^{-1}$ and need to show the sum of the diagonal entries of gb differs from the sum of the diagonal entries of $bg = p$ by a sum of commutators. Let b_{ij} and g_{ij} be the respective matrix entries of b and g. Then the i-th diagonal entry of gb is $\sum_j g_{ij}b_{ji}$ and the j-th diagonal entry of bg is $\sum_i b_{ji}g_{ij}$. Thus the difference between the sum of the diagonal entries of gb and the corresponding sum for bg is

$$\sum_i \left(\sum_j g_{ij}b_{ji}\right) - \sum_j \left(\sum_i b_{ji}g_{ij}\right)$$
$$= \sum_{i,j} (g_{ij}b_{ji} - b_{ji}g_{ij}) = \sum_{i,j} [g_{ij}, b_{ji}] \in [R, R]. \quad \square$$

6.2.6. Examples.

(1) If R is a field, $K_0(R) \cong \mathbb{Z}$ and Tr is the usual trace. Since $\mathrm{Tr}(1_n) = n$, the map $\mathrm{Tr} : K_0(R) \to R$ has as its image the canonical image of $\mathbb{Z}$ in R. Thus Tr is injective if and only if R is of characteristic 0. It is surjective if and only if $R = \mathbb{F}_p$ for some prime number p.

(2) If $R = C^{\mathbb{C}}(X)$ for some compact Hausdorff space X, so that $K_0(R) \cong KU^0(X)$, then any idempotent $p \in M_n(R)$ is just a continuous function from X to the idempotent matrices over $\mathbb{C}$, for which the ordinary trace is just the rank of the idempotent. This must be locally constant, so that Tr just gives the rank function

$$\mathrm{rank} : KU^0(X) \to H^0(X, \mathbb{Z}) = C(X, \mathbb{Z}) \subseteq C(X, \mathbb{C}) = R.$$

Thus if X is connected, Tr takes its values in $\mathbb{Z}$ and vanishes on $\tilde{K}^0(X)$.

(3) If $R = M_n(k)$ is a matrix algebra over a field k, $K_0(R) \cong \mathbb{Z}$ with generator corresponding to a rank-one idempotent in R, and Tr is again the usual trace. So again the map $\mathrm{Tr} : K_0(R) \to R$ has as its image the canonical image of $\mathbb{Z}$ in R.

(4) Let k be a commutative ring, let G be a group, and let $R = kG$. By Exercise 6.1.45, $HH_0(R)$ can be identified with the free k-module on the conjugacy classes in G. If G is a finite group and $k = \mathbb{C}$, then R splits as a direct sum of matrix algebras and Tr is injective,

and in fact Tr induces an isomorphism $K_0(R) \otimes_{\mathbb{Z}} \mathbb{C} \to HH_0(R)$. On the other hand, if G is cyclic of prime order and $k = \mathbb{Z}$, then $\tilde{K}_0(R)$ is computed in Example 3.3.5(b) to be the finite class group of the associated cyclotomic field. Since $HH_0(R)$ is torsion-free, the trace map vanishes on $\tilde{K}_0(R)$ in this case.

(5) Here is maybe a more interesting example of the trace map. Let k be a field and let

$$R = \varinjlim(M_{2^n}(k),\, \phi_n),$$

where $\phi_n : M_{2^n}(k) \to M_{2^{n+1}}(k)$ is defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ as in Exercise 1.2.7. It was computed in that Exercise that $K_0(R) \cong \mathbb{Z}[\frac{1}{2}]$. Now if the characteristic of k is not 2, we can define a map $\psi_n : M_{2^n}(k) \to k$ to be 2^{-n} times the usual trace. Since ψ_n vanishes on commutators and $\psi_n = \psi_{n+1} \circ \phi_n$, the maps ψ_n give a well-defined map on the inductive limit:

$$\psi : R/[R, R] \to k.$$

When k is of characteristic 0, the composite $\psi \circ \mathrm{Tr}$ can be seen to be just the identity map $\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q} \subseteq k$, and thus Tr is injective.

To get a more powerful Chern character, we next show that the trace map factors through $HP_0(R)$ or $HC_0^-(R)$ via the canonical map $HP_0(R) \to HC_0(R)$. The canonical map $HC_0^-(R) \to HP_0(R)$ is always an isomorphism by Exercise 6.1.41. However, since S is an isomorphism on $HP_\bullet(R)$, the Chern character into $HP_0(R)$ can be viewed as mapping into $HP_{2n}(R)$ for any $n \geq 0$, and can then be mapped into $HC_{2n}(R)$, giving something that looks much more like the classical Chern character into even-dimensional de Rham cohomology.

6.2.7. Proposition. *Let k be a commutative ring. For any k-algebra R, there is a homomorphism* Ch : $K_0(R) \to HC_0^-(R) \cong HP_0(R)$ *called the Chern character, functorial in R, such that* Tr : $K_0(R) \to HC_0(R)$ *is just the composite of* Ch *followed by the canonical map* $HP_0(R) \to HC_0(R)$. *The map* Ch *is uniquely determined by these properties.*

Proof. Let $A = k[e]/(e^2 - e)$, the universal unital k-algebra generated by a single idempotent e. Since any element of A can be written uniquely in the form $a(1-e) + be$, A factors naturally as a direct product of rings $k \times k$, and (by the result of Exercise 6.1.44), the cyclic homology groups of A are just direct sums of two copies of the cyclic homology groups of k. By Example 6.1.16, the canonical map $HP_0(k) \to HC_0(k) \cong k$ is an isomorphism, and so is $HP_0(A) \to HC_0(A) \cong k \oplus k$. Thus the Proposition is true for A.

Now recall that $K_0(R)$ is generated as an abelian group by classes $[f]$, where f is an idempotent in $M_n(R)$ for some n. Note that f defines a unital homomorphism $\varphi : A \to M_n(R)$ sending e to f, and the class $[f] \in K_0(R)$ is

just the image under φ_* of the class $[e] \in K_0(A)$, followed by the canonical isomorphism $K_0(M_n(R)) \to K_0(R)$ of Theorem 1.2.4. Therefore we can define $\mathrm{Ch}([f])$ by first taking $\mathrm{Ch}([e]) \in HP_0(A)$ to be the inverse image of $\mathrm{Tr}([e]) \in HC_0(A)$ under the isomorphism $HP_0(A) \to HC_0(A)$, and then letting $\mathrm{Ch}([f]) = \varphi_*(\mathrm{Ch}([e])) \in HP_0(M_n(R))$, followed by the isomorphism $HP_0(M_n(R)) \to HP_0(R)$ provided by Theorem 6.1.37. Then from the commutative diagram

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\varphi_*} & K_0(M_n(R)) & \xrightarrow{\cong} & K_0(R) \\ {\scriptstyle \mathrm{Ch}}\downarrow & & & & \\ HP_0(A) & \xrightarrow{\varphi_*} & HP_0(M_n(R)) & \xrightarrow{\cong} & HP_0(R) \\ {\scriptstyle \cong}\downarrow & & \downarrow & & \downarrow \\ HC_0(A) & \xrightarrow{\varphi_*} & HC_0(M_n(R)) & \xrightarrow{\cong} & HC_0(R), \end{array}$$

we see that $\mathrm{Tr}([f])$ is the image of $\mathrm{Ch}([f])$ under the canonical map $HP_0(R) \to HC_0(R)$.

It is clear that if there is any functorial map $\mathrm{Ch} : K_0(R) \to HC_0^-(R) \cong HP_0(R)$ with the property stated in the Proposition, then it must agree with this one (since $K_0(R)$ is generated by elements coming from $K_0(A)$, and $HP_0(A) \to HC_0(A)$ is an isomorphism). To complete the proof we only need to see that the Chern character as we've defined it is a homomorphism, since functoriality is automatic from our definition. For this purpose it is enough to show that Ch is additive on the submonoid of $K_0(R)$ generated by classes of idempotents. But any two elements in this submonoid of $K_0(R)$ can be written in the form $[f_1]$, $[f_2]$ for some commuting idempotents f_1, f_2 in some $M_n(R)$. If we consider the universal k-algebra $B = k[e_1, e_2]/(e_1^2 - e_1, e_2^2 - e_2)$ on two commuting idempotents, then there is a unique homomorphism $B \to R$ with $e_j \mapsto f_j$, and so by functoriality it is enough to show Ch is additive for B. Since B also splits as a direct product of (4) copies of k, $HP_0(B) \to HC_0(B)$ is an isomorphism and additivity of Ch for B follows from additivity of the trace Tr. Thus the result follows. □

6.2.8. Remark. A similar technique (working with the universal example A) shows that one can construct functorial Chern characters $K_0(R) \to HC_{2n}(R)$ for any n. We have not listed this fact as a separate proposition since it's really contained in Proposition 6.2.7: we merely compose $\mathrm{Ch} : K_0(R) \to HP_0(R)$ with the canonical map $HP_0(R) \to HC_{2n}(R)$ coming from $(S^{-1})^n$. Again, the fact that $S^n : HC_{2n}(A) \to HC_0(A)$ is an isomorphism shows that the Chern character $K_0(R) \to HC_{2n}(R)$ is uniquely determined. Furthermore, since $B : HC_{2n}(A) \to HH_{2n+1}(A)$ is the 0-map for any n (the Hochschild homology of A is concentrated in degree 0), we see by functoriality that the image of the Chern character lies in the kernel of B. Thus, when $k \supseteq \mathbb{Q}$ (at least for augmented algebras—this restriction is really not necessary), we can by Theorem 6.1.40 think of the

Chern character, after forgetting the image of $K_0(k)$ in $K_0(R)$, as mapping into even-dimensional de Rham homology. This makes the Chern character as we've defined it look much more like the classical Chern character.

6.2.9. Example. Let $k = \mathbb{C}$, let X be a compact C^∞ manifold, and let $R = C^\infty(X)$ be the k-algebra of smooth functions on X. Then $K_0(R) \cong KU^0(X)$. Recall that there is a quotient map from $\Omega^\bullet_{\rm ab}R$ onto $\Omega^\bullet X$, the usual exterior differential forms on X, which induces a homomorphism $\Phi : H^{\rm de\ R}_\bullet(R) \to H^\bullet_{\rm de\ R}(X)$. We want to show that there are universal constants $c_n \neq 0$ (independent of X) such that for any smooth vector bundle E over X defined by an idempotent $p \in M_j(R)$,

$$\Phi \circ \mathrm{Ch}_{2n}([E]) = c_n \left[\mathrm{Tr}\left(p \cdot (dp)^{2n}\right)\right].$$

Here Ch_{2n} denotes the component of Ch in $H^{\rm de\ R}_{2n}(R)$ (viewed as in Remark 6.2.8 above) and the right-hand side is the de Rham class of the indicated differential form, as in the earlier section on the classical Chern character.

When $n = 0$, there is nothing to prove, since by Example 6.2.6(2), $\mathrm{Ch}_0([E]) = \mathrm{Tr}\, p = \mathrm{rank}\, E$. So assume $n > 0$. Again look at the universal example $A = \mathbb{C}[e]/(e^2 - e)$, which corresponds to taking $X = S^0$. Then $\mathrm{Ch}_{2n}([e])$ must be a multiple of the generator of $H^{\rm de\ R}_{2n}(A)$ (which is one-dimensional). One can take the generator to be the class of the "non-commutative differential form" $e \cdot (de)^{2n}$, so there is a universal constant c_n (computed in [KaroubiHomCyc, §II.20]) for which $\mathrm{Ch}_{2n}([e]) = c_n \left[e \cdot (de)^{2n}\right]$ in $H^{\rm de\ R}_{2n}(A)$. By functoriality, $\mathrm{Ch}_{2n}([p]) = c_n \left[p \cdot (dp)^{2n}\right]$ in $H^{\rm de\ R}_{2n}(M_j(R))$. Applying Tr and then Φ, we see that

$$\Phi \circ \mathrm{Ch}_{2n}([E]) = c_n \left[\mathrm{Tr}\left(p \cdot (dp)^{2n}\right)\right].$$

6.2.10. Examples. (a) Suppose k is a field of characteristic 0 and $R = k[t]$. By Example 6.1.20(a), $HC_{2n}(R) \cong k$ for $n > 0$, while $HC_0(R) = R$ is of infinite rank over k. Also, $S : HC_2(R) \to HC_0(R) = R$ is injective, with image $k \cdot 1$. Since $K_0(R) \cong \mathbb{Z}$, it's easy to see that Ch_{2n} is injective for any n, and is an isomorphism after tensoring over $\mathbb{Z}$ with k when $n > 0$ (though not for $n = 0$). In this case, $HP_0(R)$ is smaller that $HC_0(R)$, but gives a better approximation to $K_0(R)$.

(b) Let Δ be a square-free integer with $\Delta \not\equiv 1 \mod 4$, and let $k = \mathbb{Z}$, $R = \mathbb{Z}[\xi]/(\xi^2 - \Delta)$, which is the ring of integers in $\mathbb{Q}(\sqrt{\Delta})$ and thus a Dedekind domain. (See Exercise 1.4.24.) A calculation similar to that in Example 6.1.7(b) shows that if $S = R \otimes_{\mathbb{Z}} R^{\rm op}$, R has a resolution as an S-module that is periodic of period 2, by free S-modules of rank 1. From this, one sees that $HH_{2n}(R) = 0$ for $n > 0$ and that $HH_{2n+1}(R)$ is a finite group (isomorphic to $(\mathbb{Z}/2\Delta) \oplus (\mathbb{Z}/2)$) for $n \geq 0$. In fact, we can think of $HH_1(R)$ as the R-module generated by $d\xi$ subject to the relation $2\xi\, d\xi = d\Delta = 0$ (which imples $2\Delta\, d\xi = 0$ since $\Delta = \xi^2$), and $B : HC_0(R) \cong R \to HH_1(R)$ annihilates $\mathbb{Z}$ and sends ξ to $d\xi$. So from the Connes exact sequence, $HC_2(R)$ is torsion-free of rank 2 and $HC_1(R)$ is a finite abelian group. Plugging this back into the exact sequence and continuing by induction,

we find that $HC_{2n}(R)$ is torsion-free of rank 2 and $HC_{2n+1}(R)$ is a finite abelian group for all $n \geq 0$. On the other hand, $\tilde{K}_0(R)$ is the class group of $\mathbb{Q}(\sqrt{\Delta})$, which is finite, so Ch_{2n} is trivial on $\tilde{K}_0(R)$ for all n. In fact, since $HH_{2n}(R) = 0$ for $n > 0$, $S : HC_{2n+1}(R) \to HC_{2n-1}(R)$ is surjective for all $n > 0$, so the Mittag-Leffler condition is satisfied for odd-degree cyclic homology. Hence by Theorem 6.1.22, $HP_{\text{even}}(R) \cong \varprojlim_S HC_{\text{even}}(R)$, and $\mathrm{Ch} : K_0(R) \to HC_0^-(R)$ is trivial on $\tilde{K}_0(R)$.

(c) The calculation in Example 6.2.3 doesn't really depend on the use of C^∞ functions, since all the functions that appeared were polynomials. Thus if $R = \mathbb{R}[x_1, x_2, x_3]/(\sum x_i^2 - 1)$, the calculation we have already done shows there is an element of $K_0(R)$ detected by Ch_2 and not by Ch_0.

The Chern Character on Higher K-Theory. The Chern character on higher K-theory gives homomorphisms from the higher K-groups, as defined using the +-construction, to cyclic homology groups. We will mention along the way an earlier precursor of the Chern character, called the "Dennis trace map," which maps into Hochschild homology. As pointed out in [HoodJones], to define the Chern character on higher K-theory correctly, it is really best to work with $HC_\bullet^-$ rather than with $HC_\bullet$. First we need to define the "assembly maps" for group rings, which have other interesting applications, anyway. (See the last subsection of §6.3.)

6.2.11. Definition. Let G be a group and let k be a commutative ring. The **assembly map** $H_\bullet(G, k) \to HH_\bullet(kG)$ is defined by identifying $HH_\bullet(kG)$ as in Exercise 6.1.45 with $H_\bullet(G, kG)$ (with G acting by conjugation) and taking the map on homology induced by the inclusion $k \cdot 1 \hookrightarrow kG$. As shown in Exercise 6.1.45, this map is a split injection. To be more explicit, it corresponds to the inclusion into the Hochschild complex of kG of the subcomplex $C_\bullet(G)$ spanned (over k) by elementary tensors of the form $g_0 \otimes g_1 \otimes \cdots \otimes g_n$ with $g_i \in G$, $g_0 g_1 \cdots g_n = 1$. This does not give exactly the same complex for computing $H_\bullet(G, k)$ that we gave in Definition 4.1.7, but another equivalent one coming from tensoring over G with k on the right the resolution of k as a trivial (right) G-module given by

$$\cdots \xrightarrow{b'} \sum_{\substack{g_i \in G \\ g_0 g_1 \cdots g_n = 1}} k\,(g_0 \otimes g_1 \otimes \cdots \otimes g_n) \xrightarrow{b'} \cdots \xrightarrow{b'} \sum_{g_0 = 1} k g_0 = k, \tag{6.2.12}$$

with the G-module structure given by

$$(g_0 \otimes g_1 \otimes \cdots \otimes g_n) \cdot g = g^{-1} g_0 \otimes g_1 \otimes \cdots \otimes g_n g.$$

(The complex (6.2.12) is acyclic since (6.1.5) again gives a contracting homotopy.) The assembly map $H_\bullet(G, \mathbb{Z}) \to HH_\bullet(kG)$ is defined to be the map $H_\bullet(G, \mathbb{Z}) \to H_\bullet(G, k)$ induced by the canonical map of rings $\mathbb{Z} \to k$, followed by the above assembly map. The assembly map $H_\bullet(G, k) \to HC_\bullet(kG)$ is the assembly map in Hochschild homology followed by the canonical map of Connes, $I : HH_\bullet(kG) \to HC_\bullet(kG)$.

There is also a "shifted assembly map" $H_\bullet(G, k) \to HC_{\bullet+2j}(kG)$ which when followed by the periodicity operator S^j gives the above map, but since this is turn comes from the assembly map on $HP_\bullet$ and on $HC_\bullet^-$ we mention these first. The "shifted assembly map" on $HC_\bullet$ will then be the assembly map $H_\bullet(G, k) \to HC_\bullet^-(kG)$ followed by the maps

$$HC_\bullet^-(kG) \to HP_\bullet(kG) \xrightarrow[\cong]{S^{-j}} HP_{\bullet+2j}(kG) \to HC_{\bullet+2j}(kG).$$

To define the assembly map $H_\bullet(G, k) \to HC_\bullet^-(kG)$, we note that just as the Hochschild complex of a ring R can be included as the 0-th column of the cyclic double complex of R, the complex $C_\bullet(G)$ can be included as the 0-th column of a double complex $CC_\bullet^-(G)$ with columns indexed by the integers ≤ 1, with even columns given by $C_\bullet(G)$, with odd columns the same as (6.2.12) except that the sign of the differential is changed, and with horizontal maps defined using N and $1-t$. (The key point here is that $C_\bullet(G)$ is invariant under the cyclic operators t_n, since, in a group, $g_0g_1\cdots g_n = 1$ implies $g_ng_0g_1\cdots g_{n-1} = 1$.) The double complex $CC_\bullet^-(G)$ is a subcomplex of the double complex used to compute $HC_\bullet^-(kG)$, and making this double complex into a single complex as in (6.1.15) gives us homology groups $HC_\bullet^-(G)$ which naturally map to $HC_\bullet^-(kG)$. (Similarly we get groups $HC_\bullet(G)$ mapping to $HC_\bullet(kG)$ and $HP_\bullet(G)$ mapping to $HP_\bullet(kG)$.) On the other hand, we can show, following [KaroubiHomCyc, Proposition 2.22], that

$$HC_n^-(G) \cong \prod_{j=0}^{\infty} H_{n+2j}(G, k),$$

and similarly

$$HC_n(G) \cong \sum_{j=0}^{[\frac{n}{2}]} H_{n-2j}(G, k), \qquad HP_n(G) \cong \prod_{j=[\frac{n}{2}]}^{\infty} H_{n+2j}(G, k).$$

To prove this, introduce the double complex of free right G-modules (6.2.13)

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \Big\downarrow b_3' & & \Big\downarrow -b_3'' \\
\cdots \xleftarrow{1-t_1} & & \sum_{g_0g_1g_2=1} k\,(g_0 \otimes g_1 \otimes g_2) & \xleftarrow{N_1} & \sum_{g_0g_1g_2=1} k\,(g_0 \otimes g_1 \otimes g_2) \\
 & & \Big\downarrow b_2' & & \Big\downarrow -b_2'' \\
\cdots \xleftarrow{N_0} & & \sum_{g_0g_1=1} k\,(g_0 \otimes g_1) & \xleftarrow{1-t_0} & \sum_{g_0g_1=1} k\,(g_0 \otimes g_1) \\
 & & & & \\
 & & i=0 & & i=1\,,
\end{array}$$

where the even-numbered columns are copies of (6.2.12) with the last k left off, and the odd-numbered columns are similar except that we leave off the last term $\pm d^n_{n-1}$ in the formula for b'. Note that (6.2.13) $\otimes_{kG} k$ is precisely $CC^-_\bullet(G)$. On the other hand, it is easy to see that all the columns of (6.2.13) are acyclic, except for the $j = 0$ row in the even columns. So the same argument as in the proof of Example 6.1.16 shows that if we make (6.2.13) into a single complex as in (6.1.15), we get a product of free G-module resolutions of k, shifted in degrees by 2. This gives the desired result $HC^-_n(G) \cong \prod_{j=0}^{\infty} H_{n+2j}(G, k)$. Now it is easy to see how to define the assembly map; it is the obvious injection of $H_\bullet(G, k)$ into this product. In particular the assembly map is split injective, and the assembly map in Hochschild homology is just the assembly map in $HC^-_\bullet$ followed by the canonical map $HC^-_\bullet \to HH_\bullet$.

6.2.14. Definition. Let k be a commutative ring and let R be a k-algebra. For convenience let $G = GL(R)$. The **Dennis trace map** is the collection of homomorphisms $K_n(R) \to HH_n(R)$, $n \geq 1$, defined as the composites

$$K_n(R) = \pi_n(BG^+) \xrightarrow{\text{Hurewicz}} H_n(BG^+; \mathbb{Z}) \xrightarrow{\cong} H_n(G, \mathbb{Z}) \xrightarrow{\text{assembly}} HH_n(kG) \xrightarrow{\Phi} HH_n(R).$$

The **Chern character** (on higher K-theory) is the collection of homomorphisms $K_n(R) \to HC^-_n(R)$, $n \geq 1$, defined as the composites

$$K_n(R) = \pi_n(BG^+) \xrightarrow{\text{Hurewicz}} H_n(BG^+; \mathbb{Z}) \xrightarrow{\cong} H_n(G, \mathbb{Z}) \xrightarrow{\text{assembly}} HC^-_n(kG) \xrightarrow{\Phi} HC^-_n(R).$$

In both cases the first map is the Hurewicz map of Exercise 5.2.15. The second map is defined in Definition 6.2.11. The map Φ is obtained by passing to the limit as $r \to \infty$ in the following simple construction: since $GL(r, R) \subseteq M_r(R)$, there is an obvious algebra homomorphism

$$\varphi_r : kGL(r, R) \to M_r(R)$$

obtained by sending k to $k \cdot 1$ and any group element to itself (viewed as an element of $M_r(R)$). Thus by Theorems 6.1.36 and 6.1.37 we obtain composites

$$HH_n(kGL(r, R)) \xrightarrow{\varphi_{r*}} HH_n(M_r(R)) \xrightarrow[\cong]{\text{Tr}} HH_n(R)$$

or

$$HC^-_n(kGL(r, R)) \xrightarrow{\varphi_{r*}} HC^-_n(M_r(R)) \xrightarrow[\cong]{\text{Tr}} HC^-_n(R).$$

Then we can check that the diagram

$$\begin{array}{ccc} HH_n(kGL(r, R)) & \xrightarrow{\text{Tr}\circ\varphi_{r*}} & HH_n(R) \\ \downarrow & & \| \\ HH_n(kGL(r+1, R)) & \xrightarrow{\text{Tr}\circ\varphi_{r+1*}} & HH_n(R) \end{array}$$

commutes, and similarly with $HC_\bullet^-$. As pointed out in [LodayCH, p. 267], there is a slight subtlety here, since the map $GL(r, R) \hookrightarrow GL(r+1, R)$ is obtained by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, and we need to check that the 1 added in the lower right doesn't affect the class we get in $HH_n(R)$. The point however is that

$$(6.2.15)\quad \operatorname{Tr}\left(\begin{pmatrix} a_0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} a_n & 0 \\ 0 & 1 \end{pmatrix}\right) = \operatorname{Tr}(a_0 \otimes \cdots \otimes a_n) + 1 \otimes \cdots \otimes 1,$$

and the term $1 \otimes \cdots \otimes 1$ comes from the image of the map $HH_n(k) \to HH_n(R)$, which is 0 for $n \geq 1$. The Dennis trace map and the Chern character Ch are functorial in R since each individual map in the composites is functorial.

Definitions 6.2.11 and 6.2.14 are so abstract and complicated that it would be nice to have a much more concrete idea of what the Chern character does, especially on K_1. It turns out that there is a quite simple description for the case of K_1, similar to the one we found in Example 6.2.9 in the case of K_0. To find it, we again introduce a "universal example," the Laurent polynomial ring, or alternatively, the group ring of an infinite cyclic group.

6.2.16. Theorem. *Let k be a commutative ring and let $R = k[v, v^{-1}]$ be the Laurent polynomial ring on one variable v. By Exercise 6.1.43(1) (this part doesn't require any assumption on k), $HH_n(R) = 0$ for $n > 1$ and $HH_1(R)$ is the free R-module on a generator dv. The Dennis trace map sends the class of $v \in R^\times \subseteq K_1(R)$ to $v^{-1}dv \in HH_1(R) \cong \Omega^1_{\mathrm{ab}}(R)$, the "logarithmic derivative" of v.*

Now assume k is a field of characteristic 0. Then by Exercise 6.1.43, $HC_\bullet^-(R)$ is as a $k[u]$-module isomorphic to a direct sum of two copies of $k[u]$, one of which has been shifted to have its generator in degree 1. This generator may be chosen to map to $v^{-1}dv$ in $HH_1(R)$, and this determines it uniquely. The Chern character sends $[v] \in K_1(R)$ to this generator of $HC_1^-(R)$, and the shifted Chern character $K_1(R) \to HC_{2j+1}(R)$ may be viewed as (up to a non-zero constant) mapping to the class in $H^{\mathrm{de\,R}}_{2j+1}(R)$ of the non-commutative differential form $\left(v^{-1}dv\right)^{2j+1}$.

Proof. Let G be the infinite cyclic group with generator v. Then $G \subseteq R^\times$ and $kG = R$. So to compute the Dennis trace map or the Chern character on $[v]$, we only need to look at the assembly map for G. (The map Φ for G in Definition 6.2.14 is just the identity map.) In Hochschild homology, it sends $[v]$ to the class of the cycle corresponding to v in $C_1(G)$, which is just $v^{-1} \otimes v$. When we identify $HH_1(R)$ with $\Omega^1_{\mathrm{ab}}(R)$, this element goes to $v^{-1} \otimes v$. In cyclic homology, it sends $[v]$ to a class in $HC_1^-(R)$ mapping to $v^{-1}dv$ in $HH_1(R)$, and this class is unique by Exercise 6.1.43 if k is a field of characteristic 0.

Now consider the shifted Chern character $K_1(R) \to HC_{2j+1}(R)$. Since $HC_{2j+1}(k) = 0$, we may view this as mapping into $HC_{2j+1}(R, I)$, where I

is the augmentation ideal generated by $v-1$, and it also maps into the kernel of $B : HC_{2j+1}(R) \to HH_{2j+2}(R)$ since $HH_n(R) = 0$ for $n > 1$. So the shifted Chern character may be viewed as a map $K_1(R) \to H^{\text{de R}}_{2j+1}(R)$. It must be non-zero, since S^j takes us back to the (unshifted) Chern character. But if k is a field of characteristic 0, $H^{\text{de R}}_{2j+1}(R)$ is one-dimensional (by Exercise 6.1.43(5)). So we only need to show that the non-commutative differential form $\left(v^{-1}dv\right)^{2j+1}$ defines a non-zero class in this group (cf. [KaroubiHomCyc, Proposition 2.34]). First let's show that its image (in $\Omega^\bullet_{\text{ab}}(R)$) is in the kernel of d. Since $v^{-1}v = 1$,

$$d(v^{-1}) \cdot v + v^{-1}dv = 0,$$

i.e., $d(v^{-1}) = -v^{-1}dv \cdot v^{-1}$. Thus in ΩR,

$$d\left(v^{-1}dv\right) = d\left(v^{-1}\right)dv = -v^{-1}dv \cdot v^{-1}dv = -\left(v^{-1}dv\right)^2,$$

and thus (since d is a graded derivation)

$$d\left(v^{-1}dv\right)^2 = d\left(v^{-1}dv\right) \cdot \left(v^{-1}dv\right) - \left(v^{-1}dv\right) \cdot d\left(v^{-1}dv\right) = 0.$$

Then

$$\begin{aligned} d\left(v^{-1}dv\right)^3 &= d\left(v^{-1}dv\right)^2 \cdot \left(v^{-1}dv\right) + \left(v^{-1}dv\right)^2 \cdot d\left(v^{-1}dv\right) \\ &= -\left(v^{-1}dv\right)^4, \end{aligned}$$

and so by induction

$$d\left(v^{-1}dv\right)^{2j+1} = -\left(v^{-1}dv\right)^{2j+2}$$

in ΩR. But in $\Omega^\bullet_{\text{ab}}(R)$, even powers of a form of degree 1 vanish since they are graded commutators, and thus $\left(v^{-1}dv\right)^{2j+1}$ defines a class in $H^{\text{de R}}_{2j+1}(R)$.

There are a few ways to show this class is non-zero. One, used in [KaroubiHomCyc, Proposition 2.34], is to show that it can map to a non-zero class in the de Rham cohomology of S^{2j+1}. Another is to rewrite

$$\begin{aligned} \left(v^{-1}dv\right)^{2j+1} &= v^{-1}dv \cdot v^{-1}dv \cdot \cdots \cdot v^{-1}dv \\ &= v^{-1}dv\left(\left(v^{-1}dv \cdot v^{-1}\right)dv\right)^j \\ &= (-1)^j v^{-1}dv\left(d(v^{-1})dv\right)^j, \end{aligned}$$

which corresponds to the class in $H^\lambda_{2j+1}(R)$ of

$$(-1)^j v^{-1} \otimes v \otimes v^{-1} \otimes v \otimes \cdots \otimes v^{-1} \otimes v.$$

Tracing the definition of S through the isomorphism of $HC_\bullet$ with $H^\lambda_\bullet$, one can check that under S^j this goes to a non-zero multiple of the class

of $v^{-1} \otimes v$, which corresponds to a generator of $HC_1(R)$. We adopt still another approach used in [LodayCH, Lemma 8.4.8], which is to write down a specific class $[a]$ in $HC_1^-(R)$ which maps to a non-zero multiple of the class of $(v^{-1}dv)^{2j+1}$ for each j. Since $HC_1^-(R)$ is one-dimensional and $[a]$ maps to the generator $v^{-1} \otimes v$ of $HC_1(R)$, $[a]$ generates $HC_1(R)$ and thus $(v^{-1}dv)^{2j+1}$ generates $H^{\text{de R}}_{2j+1}(R)$ for each j. To write down the class $[a]$, recall $HC_\bullet^-(R)$ is the homology of $C_\bullet(R)[[u]]$ with respect to the differential $1{\otimes}b{+}u{\otimes}B$. It is actually more convenient to reduce and use the normalized Hochschild complex (isomorphic to ΩR) introduced in Definition 6.1.38 in place of $C_\bullet(R)$; this doesn't change the result. Then the class $[a]$ is represented by the formal power series

$$(6.2.17)\quad a = (v^{-1} \otimes v) - u(v^{-1} \otimes v)^{\otimes 2} + 2u^2(v^{-1} \otimes v)^{\otimes 3} - 6u^3(v^{-1} \otimes v)^{\otimes 4} + \cdots .$$

To check that this works, note that

$$\begin{aligned} b\left(v^{-1} \otimes v \otimes \cdots \otimes v^{-1} \otimes v\right) &= \left(1 \otimes v^{-1} \otimes v \otimes \cdots \otimes v^{-1} \otimes v\right) \\ &\quad - \left(v \otimes 1 \otimes v^{-1} \otimes v \otimes \cdots \otimes v^{-1} \otimes v\right) \\ &\quad + \cdots - \left(1 \otimes v \otimes v^{-1} \otimes \cdots \otimes v^{\otimes} v{-}1\right). \end{aligned}$$

Here the terms represented by $\cdots$ all contain a 1 past the first slot, and so represent 0 in the normalized complex. On the other hand,

$$N\left(v^{-1} \otimes v\right)^{\otimes j} = j\left(v^{-1} \otimes v\right) - j\left(v \otimes v^{-1}\right),$$

so

$$\begin{aligned} B\left(v^{-1} \otimes v\right)^{\otimes j} &= (1-t)sN\left(v^{-1} \otimes v\right)^{\otimes j} \\ &= j(1-t)\left(1 \otimes \left(v^{-1} \otimes v\right)^{\otimes j} - 1 \otimes \left(v \otimes v^{-1}\right)^{\otimes j}\right) \\ &= j\left(1 \otimes \left(v^{-1} \otimes v\right)^{\otimes j} - 1 \otimes \left(v \otimes v^{-1}\right)^{\otimes j}\right) \\ &\qquad + \text{terms going to 0 in the reduced complex} \\ &= jb\left(v^{-1} \otimes v\right)^{\otimes j+1}, \end{aligned}$$

and a is a cycle in the reduced mixed complex. The image of $[a]$ in $H^{\text{de R}}_{2j+1}(R)$ is evidently

$$(-1)^j j!\, v^{-1}dv\left(d(v^{-1})dv\right)^j,$$

which differs from $(v^{-1}dv)^{2j+1}$ by a factor of $j!$. This is non-zero since we are in characteristic 0. □

6.2.18. Corollary. *Let k be a commutative ring and let R any commutative k-algebra. Then the Dennis trace map vanishes on $SK_1(R)$ and sends any $g \in R^\times \subseteq K_1(R)$ to the "logarithmic derivative" of g, $g^{-1}dg \in HH_1(R) \cong \Omega^1_{\text{ab}}(R)$.*

Now assume k is a field of characteristic 0 and let R be any k-algebra. For $g \in GL(n, R)$, the shifted Chern character $K_1(R) \to HC_{2j+1}(R)$ sends $[g] \in K_1(R)$ to a non-zero multiple (not depending on g or R) of the class in $H^{\text{de R}}_{2j+1}(R)$ of the non-commutative differential form $\operatorname{Tr}\left((g^{-1}dg)^{2j+1}\right)$.

Proof. If $g \in GL(n, R)$, then $[g] \in K_1(R)$ is the image in $K_1(M_n(R))$ of $[v] \in K_1(k[v, v^{-1}])$ under the unique k-algebra homomorphism $\varphi_g : k[v, v^{-1}] \to M_n(R)$ with $v \mapsto g$, followed by the Morita isomorphism $K_1(M_n(R)) \xrightarrow{\cong} K_1(R)$ of Exercise 2.1.8. By naturality of the Dennis trace map and the Chern character, we therefore have commutative diagrams

$$\begin{array}{ccccc}
K_1(k[v, v^{-1}]) & \xrightarrow{\varphi_{g*}} & K_1(M_n(R)) & \xrightarrow[\text{Morita}]{\cong} & K_1(R) \\
\Big\downarrow{\scriptstyle \text{Dennis}} & & \Big\downarrow{\scriptstyle \text{Dennis}} & & \\
HH_1(k[v, v^{-1}]) & \xrightarrow{\varphi_{g*}} & HH_1(M_n(R)) & \xrightarrow[\text{Tr}]{\cong} & HH_1(R)
\end{array}$$

and

$$\begin{array}{ccccc}
K_1(k[v, v^{-1}]) & \xrightarrow{\varphi_{g*}} & K_1(M_n(R)) & \xrightarrow[\text{Morita}]{\cong} & K_1(R) \\
\Big\downarrow{\scriptstyle \text{Ch}} & & \Big\downarrow{\scriptstyle \text{Ch}} & & \\
HC_1^-(k[v, v^{-1}]) & \xrightarrow{\varphi_{g*}} & HC_1^-(M_n(R)) & \xrightarrow[\text{Tr}]{\cong} & HC_1^-(R),
\end{array}$$

and we can apply Theorem 6.2.16 to the vertical arrows on the left. We conclude for instance that the Dennis trace map sends any $g \in R^\times \subseteq K_1(R)$ to $g^{-1}dg \in HH_1(R) \cong \Omega^1_{\text{ab}}(R)$.

The next step is to see that the diagrams

$$\begin{array}{ccc}
K_1(M_n(R)) & \xrightarrow[\text{Morita}]{\cong} & K_1(R) \\
\Big\downarrow{\scriptstyle \text{Dennis}} & & \Big\downarrow{\scriptstyle \text{Dennis}} \\
HH_1(M_n(R)) & \xrightarrow[\text{Tr}]{\cong} & HH_1(R)
\end{array}
\qquad
\begin{array}{ccc}
K_1(M_n(R)) & \xrightarrow[\text{Morita}]{\cong} & K_1(R) \\
\Big\downarrow{\scriptstyle \text{Ch}} & & \Big\downarrow{\scriptstyle \text{Ch}} \\
HC_1^-(M_n(R)) & \xrightarrow[\text{Tr}]{\cong} & HC_1^-(R)
\end{array}$$

commute. This will then imply the statement about the shifted Chern character. For this we have to look back at the isomorphisms Tr of Theorems 6.1.36 and 6.1.37. They have natural inverses defined by the non-unital inclusion of R into the upper left-hand corner of $M_n(R)$, whereas the isomorphism $K_1(M_n(R)) \xrightarrow{\cong} K_1(R)$ has an inverse given by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. While these don't appear to be the same because of the 1 in the lower right, the calculation in (6.2.15) again shows that the diagram commutes.

To conclude the proof, we have to show that if R is commutative, the Dennis trace map vanishes on $SK_1(R)$. Note that if $g \in SL(n, R)$, then the corresponding class in $HH_1(M_n(R))$ is the image under φ_{g*} of the Hochschild class of $v^{-1} \otimes v$, in other words the class of $g^{-1} \otimes g$. So we need to show that $\mathrm{Tr}(g^{-1} \otimes g)$ is trivial in $HH_1(R) \cong \Omega^1_{\mathrm{ab}}(R)$. This is a consequence of the "cofactor expansion" of determinants from elementary linear algebra. Let $h = g^{-1}$ and let h_{ij} be its (i, j)-entry. Since $\det g = 1$, h_{ji} is $(-1)^{i+j}$ times the determinant of the submatrix of g obtained by deleting the i-th row and j-th column, which is exactly the coefficient of g_{ij} when we write $\det g$ as a polynomial in the matrix entries. Since $\det g = 1$, differentiating gives

$$\begin{aligned} 0 = d\,(\det g) &= \sum_{ij} h_{ji}\, dg_{ij} \\ &=_{\mathrm{def}} \mathrm{Tr}\,(h \otimes g)\,. \quad \square \end{aligned}$$

Remark. Even when one assumes nothing about k and R, the proofs of Theorem 6.2.16 and Corollary 6.2.17 still yield formulas for the Dennis trace map and the Chern character on $K_1(R)$; they are only slightly harder to use. Given $g \in GL(n, R)$, the Dennis trace map applied to $[g]$ yields the Hochschild class in $HH_1(R)$ of $\mathrm{Tr}(g^{-1} \otimes g)$, and the Chern character applied to the same class yields $\mathrm{Tr}\left(\varphi_{g*}[a]\right)$, where $\varphi_g : k[v, v^{-1}] \to M_n(R)$ sends v to g and $[a] \in HC_1^-(k[v, v^{-1}])$ is given by (6.2.18).

6.2.19. Examples. (a) If $R = k$, then $HC_1^-(R)$ vanishes, so the Chern character vanishes identically on $K_1(k)$. Furthermore, $HH_n(k)$ vanishes for $n > 0$, so the Dennis trace map cannot detect any of the higher K-theory of k. While these facts may seem unfortunate, they have one advantage, which is that for general k-algebras R, the Chern character on $K_1(R)$ ignores the image of $K_1(k)$ in $K_1(R)$, and thus only detects the "interesting" part of $K_1(R)$. Similarly the Dennis trace map on $K_n(R)$, $n > 0$, ignores the image of $K_1(k)$ in $K_1(R)$.

(b) If $k = \mathbb{Z}$ and R is the ring of integers in a real quadratic number field, for example, $R = k[\xi]$ with $\xi^2 = \Delta$ a square-free positive number ($\not\equiv 1 \mod 4$), then as we saw in Example 6.2.10(b), $HH_1(R)$ is a non-trivial finite group. In this case, the Dennis trace map on $K_1(R) \cong R^\times$ has a large kernel but is not identically 0. In this case $R^\times$ is the product of $\{\pm 1\}$ by an infinite cyclic group, by a special case of the Dirichlet Unit Theorem (Theorem 2.3.8). For example, if $\Delta = 2$ and we consider the fundamental unit $u = \xi + 1$ (with inverse $\xi - 1$), then $u^{-1}du = (\xi - 1)d\xi$, which is an element of order 4 in $HH_1(R)$.

(c) Let k be an algebraically closed field of characteristic 0, $\mathbb{C}$ for example, and let $R = k(t)$, the field of rational functions over R in one variable t. As in Example 6.1.7(a), one finds that $HH_n(R)$ vanishes for $n > 1$, and since any non-zero element of R can be written uniquely as a product of an element of $k^\times$ and a finite product of (positive or negative) powers of polynomials of the form $t - a$, $a \in k$, we see $HH_1(R) \cong \Omega^1_{\mathrm{ab}}(R)$ can be

identified with a free R-module on one generator dt. By partial fractions, any element of $HH_1(R)$ is a k-linear combination of terms of the form $(t-a)^m dt$, where $m \in \mathbb{Z}$. Such a differential can be "integrated" exactly when $m \neq -1$, so $HC_1(R)$, which is the cokernel of $d : R \to \Omega^1_{\text{ab}}(R)$, is the free k-vector space on the basis $\{(t-a)^{-1}dt : a \in k\}$. The Dennis trace map $K_1(R) \cong R^\times \to HH_1(R)$ kills $k^\times$ and sends $(t-a)^m$ to its logarithmic derivative $m(t-a)^{-1}dt$. While this is far from an isomorphism, the map $\text{Ch}_1 : K_1(R) \cong R^\times \to HC_1(R)$ has each basis element of $HC_1(R)$ in its image, and induces an isomorphism from $(R^\times/k^\times) \otimes_{\mathbb{Z}} k$ onto $HC_1(R)$. By the remarks in (a), this is the best for which one could possibly hope.

(d) Let $k = \mathbb{C}$ and

$$R = \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1),$$

the ring of polynomial functions on S^3. There is an element of $SK_1(R)$ defined by the element

$$g = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

in $SL(2, R)$. Under the obvious inclusion of R into $C^{\mathbb{C}}(S^3)$, this element maps to a generator of

$$K_1^{\text{top}}(C^{\mathbb{C}}(S^3)) \cong KU^{-1}(S^3) \cong \mathbb{Z},$$

so it must have infinite order. An alternative way of checking this is to use the Chern character $K_1(R) \to H_3^{\text{de R}}(R)$, followed by the obvious map into the de Rham cohomology $H^3_{\text{de R}}(S^3)$ of S^3. Note that it is essential here to use HC_3 and not just the Dennis trace map into $HH_1(R)$, since we know from Corollary 6.2.18 that the latter vanishes identically on $SK_1(R)$.

To do the calculation, we need to show that

$$\int_{S^3} \text{Tr}\left(\left(g^{-1}dg\right)^3\right) \neq 0.$$

But

$$g^{-1}dg = \begin{pmatrix} x_0 - ix_1 & -x_2 - ix_3 \\ x_2 - ix_3 & x_0 + ix_1 \end{pmatrix} \begin{pmatrix} dx_0 + idx_1 & dx_2 + idx_3 \\ -dx_2 + idx_3 & dx_0 - idx_1 \end{pmatrix},$$

which with the complex notation $z_0 = x_0 + ix_1$, $z_1 = x_2 + ix_3$, $\bar{z}_0 = x_0 - ix_1$, $\bar{z}_1 = x_2 - ix_3$ becomes simply

$$\begin{aligned} g^{-1}dg &= \begin{pmatrix} \bar{z}_0 & -z_1 \\ \bar{z}_1 & z_0 \end{pmatrix} \begin{pmatrix} dz_0 & dz_1 \\ -d\bar{z}_1 & d\bar{z}_0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{z}_0 dz_0 + z_1 d\bar{z}_1 & \bar{z}_0 dz_1 - z_1 d\bar{z}_0 \\ \bar{z}_1 dz_0 - z_0 d\bar{z}_1 & z_0 d\bar{z}_0 + \bar{z}_1 dz_1 \end{pmatrix} \\ &= \begin{pmatrix} \omega_0 & -\bar{\omega}_1 \\ \omega_1 & \bar{\omega}_0 \end{pmatrix}, \end{aligned}$$

where $\omega_0 = \bar{z}_0 dz_0 + z_1 d\bar{z}_1$ and $\omega_1 = \bar{z}_1 dz_0 - z_0 d\bar{z}_1$. Then

$$(g^{-1}dg)^2 = \begin{pmatrix} -\bar{\omega}_1 \wedge \omega_1 & \bar{\omega}_1 \wedge (\omega_0 - \bar{\omega}_0) \\ \omega_1 \wedge (\omega_0 - \bar{\omega}_0) & \bar{\omega}_1 \wedge \omega_1 \end{pmatrix},$$

$$(g^{-1}dg)^3 = \begin{pmatrix} (\bar{\omega}_0 - 2\omega_0) \wedge \bar{\omega}_1 \wedge \omega_1 & * \\ * & (2\bar{\omega}_0 - \omega_0) \wedge \bar{\omega}_1 \wedge \omega_1 \end{pmatrix},$$

and

$$\begin{aligned} \operatorname{Tr}\left((g^{-1}dg)^3\right) &= 3(\bar{\omega}_0 - \omega_0) \wedge \bar{\omega}_1 \wedge \omega_1 \\ &= 12 \operatorname{Im}\omega_0 \wedge \operatorname{Re}\omega_1 \wedge \operatorname{Im}\omega_1 \\ &= 12(x_0 dx_1 - x_1 dx_0 + x_3 dx_2 - x_2 dx_3) \\ &\quad \wedge (x_2 dx_0 + x_3 dx_1 - x_0 dx_2 - x_1 dx_3) \\ &\quad \wedge (x_2 dx_1 - x_3 dx_0 - x_1 dx_2 + x_0 dx_3) \\ &= 12\left((x_1^2 x_3 + x_0^2 x_3 + x_3 x_2^2 + x_0 x_1 x_2 - x_1 x_0 x_2 + x_3^3)\right. \\ &\quad dx_0 \wedge dx_1 \wedge dx_2 \\ &\quad + (\cdots)\, dx_0 \wedge dx_1 \wedge dx_3 + (\cdots)\, dx_0 \wedge dx_2 \wedge dx_3 \\ &\quad \left. + (\cdots)\, dx_1 \wedge dx_2 \wedge dx_3\right) \\ &= 12\,(x_3 dx_0 \wedge dx_1 \wedge dx_2 - x_2 dx_0 \wedge dx_1 \wedge dx_3 + \cdots). \end{aligned}$$

This is a multiple of the usual volume form on S^3 so its integral is non-zero. Thus our element of SK_1 has infinite order.

6.2.20. Exercise. Show that $\operatorname{Tr} : K_0(R) \to R/[R, R]$ is injective if k is a field of characteristic 0 and R is a finite-dimensional semisimple algebra over k. (By Wedderburn theory, R splits as a direct sum of matrix algebras over division algebras. You can reduce to the case where all the division algebras are fields by going up to a splitting field.) Can you describe the kernel of Tr when k is a field of characteristic p?

6.2.21. Exercise. Show that Tr and Ch are additive on direct products of k-algebras. In other words, if $R = R_1 \times R_2$ as in Exercise 6.1.44, show that the trace and Chern character maps for R split up as direct sums of the corresponding maps for R_1 and R_2.

6.2.22. Exercise. Show how to define a relative Chern character $K_\bullet(R, I) \to HC_\bullet^-(R, I)$, where relative cyclic homology is defined in Remark 6.1.17.

6.2.23. Exercise. Let k be a field of characteristic 0 and let $R = k[u, u^{-1}, v, v^{-1}]$. Compute $HH_\bullet(R)$ and show that it vanishes past degree 2. (R is the group ring of a free abelian group on two generators u and v, so you can use Exercise 6.1.45.) Show that the Steinberg symbol $\{u, v\} \in K_2(R)$ has infinite order by showing that it maps under the Dennis trace map to a non-zero element of $HH_2(R)$, which is a vector space over k and thus a torsion-free abelian group.

In this problem you need to trace through Definition 6.2.14. In this regard, note first that the Hurewicz map is just the corestriction map $K_2(R) = H_2(E(R), \mathbb{Z}) \to H_2(GL(R), \mathbb{Z})$.

6.2.24. Exercise. Let k be a field and let R be a k-algebra. Show how to define a Chern character $K_{-1}(R) \to HC^{-}_{-1}(R) \cong HP_{\text{odd}}(R)$, using the result of Exercise 6.1.49. Can you find an example where this Chern character is non-zero? (Recall $K_0(R) \oplus K_{-1}(R)$ is a direct summand in $K_0(R[u, u^{-1}])$, and note (as a consequence of the result of Exercise 6.1.49) that $HP_{\text{even}}(R[u, u^{-1}]) \cong HP_{\text{even}}(R) \oplus HP_{\text{odd}}(R)$. You just need to show the pieces match up under the Chern character for $K_0(R[u, u^{-1}])$.)

One can similarly define a Chern character $K_{-n}(R) \to HC^{-}_{-n}(R) \cong HP_{-n}(R)$ for all $n \geq 1$. Note that it is essential here to use HC^{-} or HP, since HC vanishes in negative degrees.

6.2.25. Exercise. Use Theorem 5.3.32 and Remark 5.3.33, which identify $K_n(R)$ with equivalence classes of virtual flat R-bundles over S^n, to give another, perhaps more concrete, approach to the Chern character. (If X is a homology n-sphere, a flat R-bundle over X corresponds to a map $X \to BGL(R)$. The "fundamental class" of X then maps to a class in $H_n(GL(R), \mathbb{Z})$, which we can map to $HC^{-}_n(R)$ by the assembly map.)

3. Some applications

Our aim in this last section is to give a few examples that show how the Chern character can be used to study problems in K-theory. We have deliberately chosen fairly simple examples to illustrate some of the possible ideas, and the interested reader is invited to look for more complicated and more interesting examples either in the current mathematical literature (in which cyclic homology and the Chern character are "hot" subjects) and on his or her own.

Non-vanishing of Class Groups and Whitehead Groups. Since some of the oldest applications of algebraic K-theory come from the study of class groups of number fields or of finiteness obstructions and Whitehead torsion in topology, it seems natural to look for applications of cyclic homology to these classical areas. Unfortunately, first tries don't seem very promising, since in many cases of interest (such as Examples 6.2.10(b) and 6.2.19(b)), the Dennis trace map and Chern character turn out to be trivial or almost trivial. For example, if we take $k = \mathbb{Z}$ and R to be the ring of integers in a number field (a finite extension of $\mathbb{Q}$), then R is a Dedekind domain and a finitely generated free k-module. So $HC_0(R) \cong HH_0(R) \cong R$ is torsion-free and $\text{Tr} : K_0(R) \to R$ cannot detect any of $\tilde{K}_0(R)$, which is a torsion group. In fact, Example 6.2.10(b) shows that for many quadratic fields, $HC^{-}_0(R)$ is also torsion-free and thus even the more sophisticated Chern character $\text{Ch} : K_0(R) \to HC^{-}_0(R)$ cannot detect any of $\tilde{K}_0(R)$.

The following idea for getting around these difficulties was suggested by Karoubi [KaroubiNumThy], and involves K-theory with finite coefficients as introduced in Theorem 5.3.7. However, as we noted there, the definition of K-theory with finite coefficients in [Browder] is not the correct one in low degrees for general rings R. Since we will need a group $K_1(R; \mathbb{Z}/n)$

fitting into an exact sequence

$$(6.3.1) \qquad K_1(R) \xrightarrow{x \mapsto x^n} K_1(R) \to K_1(R;\, \mathbb{Z}/n) \to K_0(R) \xrightarrow{n} K_0(R),$$

a more suitable definition in general is to take

$$K_1(R;\, \mathbb{Z}/n) =_{\text{def}} \pi_2(\mathcal{Q}(R);\, \mathbb{Z}/n) =_{\text{def}} \left[M_n^2,\, \mathcal{Q}(R)\right],$$

where M_n^2 is the 2-dimensional Moore space with fundamental group $\mathbb{Z}/n$. (For example, $M_2^2 \cong \mathbb{P}^2(\mathbb{R})$.) However, since this definition is a bit hard to deal with and our purpose here is just to work out a few examples in which R is a Dedekind domain, we will use a somewhat *ad hoc* substitute with an easier definition, without proving that it coincides with $K_1(R;\, \mathbb{Z}/n)/\operatorname{im} SK_1(R)$.

6.3.2. Definition. Let R be a Dedekind domain with field of fractions F, and let $n > 1$ be a positive integer. Define a group $U(R;\, \mathbb{Z}/n)$ as follows:

$$U(R;\, \mathbb{Z}/n) = \{a \in F^\times \mid \text{there is a fractional ideal } I \text{ of } R \text{ with } I^n = (a)\} \,/\, \left(F^\times\right)^n.$$

This is a group under multiplication since if $I^n = (a)$ and $J^n = (b)$, then $(I^{-1})^n = (a^{-1})$ and $(IJ)^n = (ab)$. Note there is a natural surjective homomorphism of $U(R;\, \mathbb{Z}/n)$ onto the set of n-torsion elements of $C(R)$, since any n-torsion element of $C(R)$ is represented by a fractional ideal I with I^n a principal fractional ideal. In the other direction, if $a \in F^\times$ and I is a fractional ideal with $I^n = (a)$, then $[I]$ is an n-torsion element in $C(R)$, while, in addition, a modulo $\left(F^\times\right)^n$ determines I up to principal fractional ideals since if $I^n = (a)$ and $J^n = (ab^n)$, then $(JI^{-1})^n = (b^n) = (b)^n$, hence $JI^{-1} = (b)$ by unique factorization of ideals (Theorem 1.4.7). Furthermore, the kernel of the map $U(R;\, \mathbb{Z}/n) \to C(R)$ is

$$\begin{aligned}
&\{a \in F^\times \mid \\
&\qquad \text{there is a principal fractional ideal } I \text{ of } R \text{ with } I^n = (a)\} \,/\, \left(F^\times\right)^n \\
&= \{a \in F^\times \mid \exists b \in F^\times,\, (b)^n = (a)\} \,/\, \left(F^\times\right)^n \\
&= R^\times /\left(R^\times\right)^n.
\end{aligned}$$

So we have a natural exact sequence

$$R^\times \xrightarrow{x \mapsto x^n} R^\times \to U(R;\, \mathbb{Z}/n) \to C(R) \xrightarrow{n} C(R),$$

reminiscent of (6.3.1). The map $U(R;\, \mathbb{Z}/n) \to C(R)$, though sometimes a split surjection, cannot have a natural splitting since there is no natural way to choose generators for principal ideals.

For future use, note that we may give an alternative description of $U(R;\, \mathbb{Z}/n)$ using only R and not F as

$$U(R;\, \mathbb{Z}/n) = \{a \in R \setminus \{0\} : \exists I \trianglelefteq R,\, I^n = (a)\} \,/\, (R \setminus \{0\})^n.$$

Here $R \setminus \{0\}$ is only a monoid (under multiplication) and not a group, but the "quotient" makes sense as a group anyway, since for $a \in R \setminus \{0\}$, a^{n-1} is a multiplicative inverse for a modulo n-th powers.

Similarly, we define

$$U_1(R;\, \mathbb{Z}/n) = \frac{\{a \in R \setminus \{0\} : a \text{ invertible mod } n,\ \exists I \trianglelefteq R,\ I^n = (a)\}}{\{a \in R \setminus \{0\} : a \text{ invertible mod } n\}^n},$$

and this is also a group under multiplication with an obvious (injective) map to $U(R;\, \mathbb{Z}/n)$. The image is the inverse image in $U(R;\, \mathbb{Z}/n)$ of the n-torsion classes in $C(R)$ represented by $[I]$ with $I \trianglelefteq R$, $I + (n) = R$.

Now we want to introduce Hochschild homology and cyclic homology with finite coefficients.

6.3.3. Definition. Let $k = \mathbb{Z}$ and let R be a k-algebra. The Hochschild complex of R with coefficients mod n is defined to be

$$C_\bullet(R;\, \mathbb{Z}/n) =_{\text{def}} C_\bullet(R) \otimes_{\mathbb{Z}} \mathbb{Z}/(n),$$

with boundary map induced by the Hochschild boundary b. The cyclic double complex of R with coefficients mod n is defined similarly to be

$$CC_{\bullet\bullet}(R;\, \mathbb{Z}/n) =_{\text{def}} CC_{\bullet\bullet}(R) \otimes_{\mathbb{Z}} \mathbb{Z}/(n).$$

It is clear that one has a short exact sequence of chain complexes

$$0 \to C_\bullet(R) \xrightarrow{n} C_\bullet(R) \to C_\bullet(R;\, \mathbb{Z}/n) \to 0,$$

and similarly a short exact sequence of cyclic double complexes

$$0 \to CC_{\bullet\bullet}(R) \xrightarrow{n} CC_{\bullet\bullet}(R) \to CC_{\bullet\bullet}(R;\, \mathbb{Z}/n) \to 0,$$

so that taking homology in the usual way, we obtain $\mathbb{Z}/(n)$-modules

$$HH_\bullet(R;\, \mathbb{Z}/n),\ HC_\bullet(R;\, \mathbb{Z}/n), \text{ and so on,}$$

as well as natural exact sequences (coming from Theorem 1.7.6)

$$\cdots \xrightarrow{\partial} HH_j(R) \xrightarrow{n} HH_j(R) \to HH_j(R;\, \mathbb{Z}/n)$$
$$\xrightarrow{\partial} HH_{j-1}(R) \xrightarrow{n} HH_j(R) \to \cdots,$$

and similarly for cyclic homology.

6.3.4. Proposition. *Let R be a Dedekind domain, and suppose $n > 1$ is invertible in the field of fractions F of R (i.e., n is not divisible by the*

characteristic of F). We view R as a k-algebra with $k = \mathbb{Z}$. Let $U_1(R; \mathbb{Z}/n)$ be the subgroup of $U(R; \mathbb{Z}/n)$ defined in Definition 6.3.2. Then there is a natural homomorphism, which we can think of as the "mod-n trace map," $\varphi : U_1(R; \mathbb{Z}/n) \to HH_1(R; \mathbb{Z}/n)$ making the diagram

$$\begin{array}{ccccc} R^\times & \xrightarrow{x \mapsto x^n} & R^\times & \longrightarrow & U_1(R; \mathbb{Z}/n) \\ \text{Tr}\downarrow & & \text{Tr}\downarrow & & \varphi\downarrow \\ HH_1(R) & \xrightarrow{n} & HH_1(R) & \longrightarrow & HH_1(R; \mathbb{Z}/n) \end{array}$$

commute.

Proof. Since multiplication by n is injective on R, the boundary map $HH_1(R; \mathbb{Z}/n) \xrightarrow{\partial} HH_0(R) = R$ is trivial and thus

$$HH_1(R; \mathbb{Z}/n) = HH_1(R)/n.$$

From our description of $HH_1(R)$ as $\Omega^1_{\text{ab}}(R)$, $HH_1(R; \mathbb{Z}/n)$ is therefore the universal $R/(n)$-module on generators da, $a \in R$, for which $a \mapsto da$ is a linear derivation. Let $a \in R \smallsetminus \{0\}$ with $(a) = I^n$ and a invertible mod n be as in the definition of $U_1(R; \mathbb{Z}/n)$. By assumption there is an $a' \in R \smallsetminus \{0\}$ with $aa' - 1 \in (n)$. Define φ to be the "logarithmic derivative mod n," in other words, $\varphi(a) = a'\,da$ in $HH_1(R; \mathbb{Z}/n)$. We have to see that this is well defined and multiplicative. First of all, a' is determined up to an element of (n) so $a'\,da$ in $HH_1(R; \mathbb{Z}/n)$ only depends on a. Secondly, if we change a to ac^n with $c \in R$ and c invertible mod n, say $cc' - 1 \in (n)$, then $a'c'^n$ is an inverse for ac^n mod n and

$$a'c'^n d(ac^n) = a'c'^n \left(c^n da + nac^{n-1}cd\right) = a'da$$

in $HH_1(R; \mathbb{Z}/n)$, so that the "logarithmic derivative mod n" for ac^n is the same as for a. Thus φ is well defined. It is multiplicative since if $aa' - 1 \in (n)$ and $bb' - 1 \in (n)$, then $a'b'$ is an inverse for ab mod n and

$$a'b'd(ab) = a'b'\,(a\,db + b\,da) = (aa')b'db + (bb')a'da = b'db + a'da$$

in $HH_1(R; \mathbb{Z}/n)$. Thus we have a well-defined homomorphism.

The commutativity of the diagram is obvious, since if a is actually invertible and not just invertible mod n, then we can take $\varphi(a) = a^{-1}da$. □

6.3.5. Corollary. *Let R be the ring of algebraic integers in some number field F. Then φ as defined in Proposition 6.3.4 gives a homomorphism from a subgroup of the n-torsion subgroup of $C(R)$ to*

$$HH_1(R; \mathbb{Z}/n)/\{a^{-1}da \mid a \in R^\times\}.$$

Proof. First of all, R is of characteristic 0 so we have no trouble on this account. By Proposition 6.3.4, φ gives a homomorphism $U_1(R; \mathbb{Z}/n) \to$

$HH_1(R;\, \mathbb{Z}/n)$ extending the logarithmic derivative map $\mathrm{Tr} : R^\times/(R^\times)^n \to HH_1(R;\, \mathbb{Z}/n)$, while we observed before that $U_1(R;\, \mathbb{Z}/n)/\operatorname{im} R^\times$ is just the group of n-torsion classes $[I]$ in $C(R)$ represented by ideals $I \trianglelefteq R$ with $I + (n) = R$. □

While what we have done is quite elementary, it may perhaps be surprising that it sometimes yields some non-trivial information about class groups of number fields.

6.3.6. Example. As in Example 6.2.10(b), let Δ be a square-free integer with $\Delta \not\equiv 1 \mod 4$, and let $R = \mathbb{Z}[\xi]/(\xi^2 - \Delta)$, which is the ring of integers in $F = \mathbb{Q}(\sqrt{\Delta})$. We computed in Example 6.2.10(b) that $HH_1(R)$ is the R-module generated by $d\xi$ subject to the relation $2\xi\, d\xi = 0$, which as an abelian group is $(\mathbb{Z}/2)\xi\, d\xi \oplus (\mathbb{Z}/2\Delta) d\xi$. Thus $HH_1(R;\, \mathbb{Z}/n)$ vanishes identically if n is relatively prime to 2Δ, and Corollary 6.3.5 is useless for studying torsion in $C(R)$ of order prime to 2Δ. However, we can sometimes detect torsion of order dividing 2Δ.

For simplicity assume further that we are in the imaginary quadratic case, *i.e.*, $\Delta = -D < 0$. Then by the Dirichlet Unit Theorem (Theorem 2.3.8), $R^\times$ is just the group of roots of unity in R, which (excluding the case of the Gaussian integers) is just $\{\pm 1\}$. Since $d(-1) = 0$, $\{a^{-1}da \mid a \in R^\times\}$ vanishes and we get a map from a certain subgroup of $C(R)$ into $HH_1(R;\, \mathbb{Z}/n)$. For example, let $D = 129 \equiv 1 \mod 4$, so $\Delta = -D \equiv 3 \mod 4$, and let $n = 6$. Modulo 5, $\xi^2 + 129$ becomes $\xi^2 - 1 = (\xi - 1)(\xi + 1)$, so (5) splits in R into a product of two prime ideals each of norm 5, $\mathfrak{p}_\pm = (5,\, 1 \pm \xi) = (5,\, 11 \pm \xi)$. By trial and error, we can find that $x^2 + 129y^2 = 5^6$ has the solution $x = 4$, $y = 11$, so that $4 \pm 11\xi$ has norm 5^6 and $\mathfrak{p}_\pm$ are the only possible prime divisors of $(4 + 11\xi)$. Thus $(4 + 11\xi) = \mathfrak{p}_-^a \mathfrak{p}_+^{6-a}$ for some a. Since $\mathfrak{p}_-\mathfrak{p}_+ = (5)$ and 5 does not divide $4 + 11\xi$, either $a = 0$ or $a = 6$. Since $(11 - \xi)^2 = -129 + 121 - 22\xi = -2(4 + 11\xi)$, it follows that in fact $\mathfrak{p}_-^6 = (4 + 11\xi)$. Since $\mathrm{N}_{F/\mathbb{Q}}(4 + 11\xi) = 5^6 \equiv 1 \mod 6$, $4 + 11\xi$ is invertible mod 6 (with inverse $-2 + \xi$) and defines an element of $U_1(R;\, \mathbb{Z}/6)$ which under φ maps to $(-2 + \xi)d(-2 - \xi) = (2 - \xi)d\xi$, which is an element of order 6. Thus $C(R)$ contains an element of order a multiple of 6.

6.3.7. Example. Here is an example with a cubic field. Suppose $D > 1$ is a positive square-free integer, and $F = \mathbb{Q}(\sqrt[3]{D})$. Then it is not too hard to show that if $D \not\equiv \pm 1 \mod 9$, then the ring of algebraic integers in F is exactly $R = \mathbb{Z}[\xi]$, where $\xi^3 = D$. Also, direct calculation shows that

$$\mathrm{N}_{F/\mathbb{Q}}(a + b\xi + c\xi^2) = a^3 + Db^3 + D^2c^3 - 3Dabc.$$

Suppose for instance that $D = 182 = 2 \cdot 7 \cdot 13$. A little trial and error with the norm formula shows that $\mathrm{N}_{F/\mathbb{Q}}(17 - 3\xi) = -1$ and that $\mathrm{N}_{F/\mathbb{Q}}(5 - 2\xi) = -11^3$. The first of these formulas shows that $17 - 3\xi \in R^\times$. By the Dirichlet Unit Theorem (Theorem 2.3.8), $R^\times$ is the product of a free abelian group by the roots of unity in R, and the latter just consist of ± 1. If we take $n = 3$, then $17 - 3\xi$ maps to 1 in $R/(3)$, but $17 - 3\xi$ cannot be the cube of another unit (since expanding out $(a + b\xi + c\xi^2)^3$ gives a constant term of

$a^3+b^3D+c^3D^2+6abcD$, which is congruent to a^3 mod D and thus mod 7, but 17 is not congruent to a cube mod 7). Thus $17-3\xi$ generates the free abelian part of $R^\times$ modulo cubes, and the image of $R^\times$ in $(R/(3))^\times$ is just ± 1. The polynomial x^3-182 factors over $\mathbb{F}_{11}$ as $(x+3)(x^2-3x+9)$, and the discriminant of the quadratic factor is not a quadratic residue mod 11, so (11) splits in R as $\mathfrak{p}_1\mathfrak{p}_2$, where $\mathfrak{p}_1=(11,\ \xi+3)$ has norm 11 and where $\mathfrak{p}_2=(11,\ \xi^2-3\xi+9)$ has norm 11^2. Since 11 and 11^2 are not cubic residues mod 7, they cannot be norms of elements of R, and so $\mathfrak{p}_1$ and $\mathfrak{p}_2$ cannot be principal ideals. The fact that $\mathrm{N}_{F/\mathbb{Q}}(5-2\xi)=-11^3$, together with the fact that $5-2\xi$ is not divisible by 11, then implies that $(5-2\xi)=\mathfrak{p}_1^3$, and thus $\mathfrak{p}_1$ gives an element of $C(R)$ of order 3, for which $\mathfrak{p}_2$ gives the inverse (since their product is a principal ideal).

We find that $HH_1(R)$ is the R-module on a generator $d\xi$ satisfying $3\xi^2 d\xi=0$, so $HH_1(R;\ \mathbb{Z}/3)$ is elementary abelian of rank 3. (Its generators as an abelian group are $d\xi$, $\xi\, d\xi$, and $\xi^2 d\xi$.) From what we said earlier, the image of $R^\times$ in $HH_1(R;\ \mathbb{Z}/3)$ is trivial. Under the map of Corollary 6.3.5, $[\mathfrak{p}_1]$ maps to the logarithmic derivative of $5-2\xi$, which mod 3 is $-1+\xi$ and has inverse $1+\xi+\xi^2$. So $[\mathfrak{p}_1]\mapsto(1+\xi+\xi^2)d\xi$, an element of $HH_1(R;\ \mathbb{Z}/3)$ of order 3. In fact it is known that $C(R)$ is an elementary abelian group of order 27 in this case, but it seems impossible to detect the whole group with such a simple-minded technique.

Idempotents in C*-Algebras. A very pretty application of the Chern character to the problem of studying idempotents in C*-algebras was given in [Connes1], and this has sparked a whole flurry of activity in using the Chern character in problems of functional analysis. In this subsection we will present some necessary background material and then Connes' argument.

6.3.8. Definition. A (concrete) **C*-algebra** is an algebra A of bounded operators on a (complex) Hilbert space $\mathcal{H}$, such that A is closed in the operator norm (for the action of operators on $\mathcal{H}$), and such that A contains the Hilbert space adjoint operator a^* for each of its elements a. Note that $a\mapsto a^*$ is a conjugate-linear anti-automorphism of A, and that A is a Banach algebra under the operator norm. An element a of A is called **self-adjoint** if $a=a^*$ and is called **positive** if it is of the form b^*b for some other operator b. (If such an operator b exists at all on $\mathcal{H}$, one can choose one lying in A, as a consequence of the Spectral Theorem to be mentioned below.) There is a partial order defined on the self-adjoint elements of A by saying that $a\geq b$ if and only if $a-b$ is a positive element.

For simplicity we shall suppose A contains the identity operator 1 on $\mathcal{H}$, which is then a unit element for A. By the Spectral Theorem for self-adjoint operators on a Hilbert space, if $a\in A$ is self-adjoint, then its spectrum (the set of complex numbers λ for which $a-\lambda 1$ is not invertible) is a compact subset X of $\mathbb{R}$, contained in $[0,\infty)$ if a is positive, and if f is a continuous function on X, $f(a)$ makes sense as an operator and also lies in A (in fact in the C*-subalgebra generated by a and 1). Also, the element $f(a)$ will be self-adjoint if f is real-valued, positive if $f\geq 0$. In this way

one gets an isomorphism $f \mapsto f(a)$ from the algebra $C(X)$ of continuous complex-valued functions on X to the C*-subalgebra of A generated by a and 1. It also follows that every self-adjoint element of a C*-algebra is a difference of two positive elements. (Write $a = f(a) - [f(a) - a]$, where $f(x) = \max(x, 0)$.) An idempotent $e \in A$ is called a **projection** if it is also self-adjoint; this means exactly that e is the orthogonal projection onto some subspace of $\mathcal{H}$.

6.3.9. Example. The simplest examples of (non-commutative) C*-algebras are the matrix algebras $M_n(\mathbb{C})$, which may be identified with the algebras of all linear operators on a finite-dimensional Hilbert space. In $M_n(\mathbb{C})$, the self-adjoint elements are the hermitian matrices, and the positive elements are the positive semidefinite matrices.

Similarly, if A is a C*-algebra on a Hilbert space $\mathcal{H}$, then $M_n(A)$ is a C*-algebra acting on the direct sum Hilbert space $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n summands).

The example of immediate interest to us comes from the case where G is a group (usually countable) and $\mathcal{H} = \ell^2(G)$, the Hilbert space completion of the group ring $\mathbb{C}G$, with respect to the inner product making G into an orthonormal basis. Elements of G act on $\mathcal{H} = \ell^2(G)$ by left multiplication, so we can identify them with operators on $\mathcal{H}$ satisfying $g^* = g^{-1}$. Thus the closed linear span of the elements of G is a C*-algebra on $\mathcal{H}$, called the **reduced (or regular) C*-algebra of the group**, denoted $C_r^*(G)$. The action of the complex group ring $\mathbb{C}G$ on $\mathcal{H}$ is clearly faithful (since we can recover a group ring element from its action on 1_G, the identity element of G viewed as a basis vector of $\mathcal{H}$), and thus $C_r^*(G)$ contains a natural copy of $\mathbb{C}G$.

6.3.10. Definition. Recall from Definition 6.1.33 that a **trace** on an algebra A (over $\mathbb{C}$, say) is a linear map $\varphi : A \to \mathbb{C}$ such that $\varphi(a_0 a_1) = \varphi(a_1 a_0)$ for all $a_0, a_1 \in A$. If A is a C*-algebra, we call the trace **self-adjoint** if it maps self-adjoint elements into $\mathbb{R}$, **positive** if it is self-adjoint, and also maps positive elements into $[0, \infty)$, **normalized** if $\varphi(1) = 1$. Note for example that the usual trace Tr on $M_n(\mathbb{C})$ is positive, since $\operatorname{Tr} a$ is the sum of the eigenvalues of a, and if a is positive semidefinite, its eigenvalues are all non-negative. But Tr is not normalized unless $n = 1$; it sends 1 to n.

A positive trace on a C*-algebra A is called **faithful** if, whenever $a \in A$ satisfies $a \geq 0$ but $a \neq 0$, then $\varphi(a) > 0$. For example, the usual trace on $M_n(\mathbb{C})$ is faithful since a matrix which is positive semidefinite and not identically zero must have at least one positive eigenvalue.

6.3.11. Example. Let $\mathcal{H} = \ell^2(G)$, $C_r^*(G)$ be as in Example 6.3.9, and define a map $\varphi : C_r^*(G) \to \mathbb{C}$ by $\varphi(a) = \langle a \cdot 1_G, 1_G \rangle$, where $\langle\ ,\ \rangle$ denotes the inner product on $\mathcal{H}$. If $a = b^* b$, then

$$\varphi(a) = \langle b^* b \cdot 1_G, 1_G \rangle = \langle b \cdot 1_G, b \cdot 1_G \rangle \geq 0,$$

so φ is positive. Also, for any group elements $g, h \in G$,

$$\varphi(gh) = \langle gh \cdot 1_G, 1_G \rangle = \langle gh, 1_G \rangle = \begin{cases} 1 & \text{if } gh = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

Since $gh = 1_G$ if and only if g and h are inverses, in which case also $hg = 1_G$, we see that $\varphi(gh) = \varphi(hg)$. Since elements of G generate $C_r^*(G)$ as a C*-algebra, it follows that φ is a positive trace. We claim it is also faithful. If $a \in A$, then $a \cdot 1_G \in \mathcal{H}$, and is thus an ℓ^2 formal linear combination $\sum_g c_g g$ of elements of G, and we may identify a with this element of $\ell^2(G)$ since the action of a on any other element h of G is given by

$$a \cdot h = \sum_g c_g(gh).$$

In this way we naturally identify $C_r^*(G)$ with a linear subspace of $\mathcal{H}$. (Caution: this subspace is usually not closed in the topology of $\mathcal{H}$.) Then we see that if $a \neq 0$,

$$\varphi(a^*a) = \langle a^*a \cdot 1_G,\, 1_G\rangle = \langle a \cdot 1_G,\, a \cdot 1_G\rangle = \|a \cdot 1_G\|^2 > 0,$$

and thus φ is faithful.

The notion of positivity, together with the pairing between HC^0 and K_0, can be used to get a little more information about K_0 of many C*-algebras.

6.3.12. Proposition (Kaplansky). *Let A be a C*-algebra, and let $e \in A$ be an idempotent. Then there is a projection $p \in A$ such that Ae and Ap are isomorphic projective modules over A. In particular, e and p represent the same class in $K_0(A)$.*

Proof. One can do the proof completely algebraically, but to get a better impression of what is going on, suppose A is acting on a Hilbert space $\mathcal{H}$. Then the image of e must be a closed subspace V of $\mathcal{H}$, and with respect to the decomposition $\mathcal{H} = V \oplus V^\perp$ of $\mathcal{H}$, e must have the matrix $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$, where $a : V^\perp \to V$ is a bounded operator. Then

$$e^* = \begin{pmatrix} 1 & 0 \\ a^* & 0 \end{pmatrix}, \qquad ee^* = \begin{pmatrix} 1 + aa^* & 0 \\ 0 & 0 \end{pmatrix},$$

and in particular, the spectrum of ee^* is contained in $\{0\} \cup [1, \infty)$. Thus if $f(0) = 0$ and $f(t) = 1$ for $t \geq 1$, f is continuous on the spectrum of ee^* and thus $p = f(ee^*)$ lies in A and is a self-adjoint projection with the same range as e. Now $ep = p$ and $pe = e$, so right multiplication by p gives an isomorphism $Ae \to Ap$, with inverse given by right multiplication by e. □

6.3.13. Corollary. *Let A be a C*-algebra with a unit 1, and let φ be a positive trace on A. Then the map it induces $K_0(A) \to \mathbb{C}$ has image contained in $\mathbb{R}$. Furthermore, the image under φ of the classes of idempotents in A is contained in $[0, \varphi(1)]$. Thus, if in addition φ is normalized and faithful, and if φ takes integer values on self-adjoint projections, then A contains no idempotents other than 0 and 1.*

Proof. By Proposition 6.2.5, any trace φ on A (positive or not) induces a homomorphism $K_0(A) \to \mathbb{C}$. Now $K_0(A)$ is generated by classes of

idempotents in matrix algebras $M_n(A)$ over A, and each $M_n(A)$ is itself a C*-algebra to which we can apply Proposition 6.3.12. So each idempotent is equivalent to a self-adjoint projection on which φ by assumption takes a non-negative value. Furthermore, if e is an idempotent in A equivalent to the self-adjoint projection p in A, then $1-p \geq 0$, so $\varphi(1) - \varphi(p) \geq 0$. Thus if φ is positive and normalized, $0 \leq \varphi(p) = \varphi(e) \leq 1$. Finally, if φ is faithful and $e \neq 0, 1$, then $p \neq 0$, $1-p \neq 0$, and so $\varphi(p) \neq 0, 1$. Hence if $\varphi(p)$ is an integer, we have a contradiction. □

6.3.14. Lemma. *Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathcal{F}(\mathcal{H}) \subset \mathcal{L}^1(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ be the algebras of finite-rank, trace-class, and compact operators on $\mathcal{H}$, respectively. (These are all ideals in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$; the definitions of $\mathcal{L}^1$ and of $\mathcal{K}$ may be found in Exercise 2.2.10.) Then the inclusion maps*

$$\mathcal{F}(\mathcal{H}) \hookrightarrow \mathcal{L}^1(\mathcal{H}) \hookrightarrow \mathcal{K}(\mathcal{H})$$

induce isomorphisms on K_0 for rings without unit, and the operator trace $\mathrm{Tr} : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$ induces an isomorphism $K_0(\mathcal{L}^1(\mathcal{H})) \to \mathbb{Z}$.

Proof. Since $M_n(\mathcal{K}(\mathcal{H})) \cong \mathcal{K}(\mathcal{H} \otimes \mathbb{C}^n) \cong \mathcal{K}(\mathcal{H})$ (recall $\mathcal{H}$ is infinite-dimensional, so $\mathcal{H} \otimes \mathbb{C}^n \cong \mathcal{H}$), and similarly with the others, it is not necessary to pass to matrices in the definition of K_0. Let $\tilde{\mathcal{F}} \subset \tilde{\mathcal{L}}^1 \subset \tilde{\mathcal{K}}$ be the algebras with unit adjoined (in other words, obtained by adding on multiples of the identity operator on $\mathcal{H}$). Then each idempotent in any of these algebras $\tilde{\mathcal{F}}$, $\tilde{\mathcal{L}}^1$, $\tilde{\mathcal{K}}$ is either of finite rank or of finite corank, and any two idempotents of the same finite rank or corank are conjugate (by an invertible operator in $\tilde{\mathcal{F}}$). Furthermore, if two idempotents are conjugate by an invertible operator in $\tilde{\mathcal{F}}$, they must have the same rank and corank. So the result follows when we compute from the split exact sequences

$$0 \to \mathcal{F},\, \mathcal{L}^1,\, \mathcal{K} \to \tilde{\mathcal{F}},\, \tilde{\mathcal{L}}^1,\, \tilde{\mathcal{K}} \leftrightarrows \mathbb{C} \to 0$$

using excision. □

Now we are ready for the argument of Connes, which shows that the hypotheses of Corollary 6.3.13 are satisfied for certain group C*-algebras of interest. We shall only give the argument for free groups, but similar arguments can be given for many other infinite torsion-free groups.

6.3.15. Theorem [Connes1, I, §1, Lemma 6 and Corollary 7]. *Let G be a free group on n generators $g_1, \ldots, g_n$, where $n \geq 1$. (We allow $n = \infty$.) Then $A = C^*_r(G)$ contains no idempotents other than* 0 *and* 1.

Proof. Let E be the set of two-element subsets $\{g, gg_j\}$ of G, where the two elements differ by multiplication on the right by one of the generators g_j. When we write the two elements of G in such a subset as reduced words in the generators and their inverses, then one will have length one less than the other. Note that we may identify G with the vertices of a "tree"

(a connected and simply connected one-dimensional simplicial complex), the so-called "Cayley graph," for which E is the set of edges, and that G acts on E by left translation. Define a map $\psi : G \setminus \{1_G\} \to E$ by $\psi(gg_j^{\pm}) = \{g, gg_j^{\pm}\}$, where g is obtained from the given reduced word $w = gg_j^{\pm}$ by deleting the last letter $g_j^{\pm}$, whatever it happens to be. By what we said above, ψ is a bijection. Note that if $g, h \in G$, then $\psi(gh) = g\psi(h)$ except when certain cancellation occurs, and in particular, for fixed g, this holds for all but finitely many $h \in G$.

Let $\mathcal{H}_0 = \ell^2(G)$ and let $\mathcal{H}_1 = \ell^2(E) \oplus \mathbb{C}$. We let G act on $\mathcal{H}_0$ and on $\ell^2(E)$ by left translation. Since E is in natural bijection with $G \times \{g_1, \ldots, g_n\}$, $\ell^2(E)$ is as a representation space for G just a multiple of $\ell^2(G)$, and the action of G extends to an action of the C*-algebra completion A. We also let A act on the $\mathbb{C}$ summand in $\mathcal{H}_1$ by the degenerate representation $A \to 0$ (**not** by the trivial representation $G \to 1$, which does not extend continuously to an action of A if $n > 1$). We will use these actions to show that the normalized trace φ on A of Example 6.3.11 (which we already know is positive and faithful) takes only integer values on $K_0(A)$.

Let $u : \mathcal{H}_0 \to \mathcal{H}_1 = \ell^2(E) \oplus \mathbb{C}$ be the unitary operator defined by

$$g \mapsto \begin{cases} (0, 1), & g = 1_G, \\ (\psi(g), 0), & g \neq 1_G. \end{cases}$$

(Since this is a bijection of orthonormal bases, it extends uniquely to an isometry of $\mathcal{H}_0$ onto $\mathcal{H}_1$.) Let Tr be the usual trace for trace-class operators on $\mathcal{H}_0$. We claim that for $a \in \mathbb{C}G \subset A$, $a - u^{-1}au$ is a finite-rank operator acting on $\mathcal{H}_0$, and

$$\text{(6.3.16)} \qquad \operatorname{Tr}(a - u^{-1}au) = \varphi(a).$$

But if $g \in G$, $g - u^{-1}gu$ has finite rank since $\psi(gh) = g\psi(h)$ for all but finitely many $h \in G$. If $g = 1_G$, then since 1_G acts as the identity on $\ell^2(E)$ but as 0 on $\mathbb{C}$, $1_G - u^{-1}1_Gu$ is a rank-one projection, and (6.3.16) holds. If $g \neq 1_G$, then $(u^{-1}gu)1_G = u^{-1}g(0, 1) = 0$, while if $h \neq 1_G$ has the reduced word expression $h = h_0g_j^{\pm}$, then $\psi(h) = e_h = \{h_0, h_0g_j^{\pm}\} \in E$ and $(u^{-1}gu)h = u^{-1}ge_h$ coincides with h if $\psi(gh) = g\psi(h)$ and otherwise is a different element of G. Thus $(g - u^{-1}gu)h$ is in any event either 0 or else an element of G other than h, and $\operatorname{Tr}(g - u^{-1}gu) = 0 = \varphi(g)$ so again (6.3.16) holds.

Since $a \mapsto a - u^{-1}au$ takes finite-rank values on $\mathbb{C}G \subset A$ and is continuous in the operator norm, it maps A into the compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H}_0)$ on $\mathcal{H}_0$. It also preserves self-adjointness. If B denotes the C*-algebra on $\mathcal{H}_0$ generated by A and by $\mathcal{K}$, then B contains $\mathcal{K}$ as a closed 2-sided ideal, and the quotient $B/\mathcal{K}$ is isometric to A. (To prove this, one needs to know that $A \cap \mathcal{K} = 0$. This can be proved easily by contradiction: each operator in A commutes with **right** translations by elements of G, but since each compact self-adjoint operator is diagonalizable, if $A \cap \mathcal{K}$ were non-zero,

then it would contain a self-adjoint operator with a non-zero eigenfunction ξ, which would then have to generate a one-dimensional representation of G under right translation. Thus ξ would have to transform on the right according to a character of G, so $|\xi(g)|$ would be constant, which is impossible since G is infinite and we need have to have $\xi \in \ell^2(G)$.) Now let $D = D(B, \mathcal{K})$, the double of B along $\mathcal{K}$ as defined in Definition 1.5.1. We see that

$$a \mapsto (a,\, u^{-1}au)$$

is a splitting of the exact sequence

$$0 \to \mathcal{K} \to D \to B/\mathcal{K} \cong A \to 0$$

different from the standard splitting given by the diagonal map. Because of Lemma 6.3.14, it thus induces a map

$$K_0(A) \to K_0(D) \cong K_0(A) \oplus K_0(\mathcal{K}) \cong K_0(A) \oplus \mathbb{Z}$$

whose first component is the identity and whose second component is a map $K_0(A) \to \mathbb{Z}$. If we can show that this map coincides with the map induced by φ, then we'll be done by Corollary 6.3.13.

Let

$$\mathcal{A} = \{a \in A : a - u^{-1}au \text{ is of trace class on } \mathcal{H}_0\},$$

equipped with the norm

$$\|a\|_{\mathcal{A}} = \|a\| + \|a - u^{-1}au\|_1.$$

(See Exercise 2.2.10 for the definition of the Schatten 1-norm $\|\ \|_1$ on the trace-class operators $\mathcal{L}^1(\mathcal{H}_0)$.) Then $\mathbb{C}G \subseteq \mathcal{A}$ and $\mathcal{A}$ is clearly a vector space closed under the $*$-operation. But in fact $\mathcal{A}$ is an algebra since if $a,\, b \in \mathcal{A}$,

$$ab - u^{-1}abu = (a - u^{-1}au)b + u^{-1}au(b - u^{-1}bu)$$

and $\mathcal{L}^1(\mathcal{H}_0)$ is a two-sided ideal. In fact the same calculation shows

$$\|ab\|_1 \le \|a - u^{-1}au\|_1\, \|b\| + \|u\|\, \|b - u^{-1}bu\|_1,$$

so adding the inequality

$$\|ab\| \le \|a\|\, \|b\|,$$

we see

$$\|ab\|_{\mathcal{A}} \le \|a\|_{\mathcal{A}}\, \|b\|_{\mathcal{A}}$$

and $\mathcal{A}$ is a normed algebra. It is complete since the trace-class operators are complete in the Schatten 1-norm. A simple estimate shows that the equality (6.3.16) holds not only on $\mathbb{C}G$ but on $\mathcal{A}$.

Next we show that the inclusion $\mathcal{A} \hookrightarrow A$ satisfies the hypotheses of the Karoubi Density Theorem (Exercise 1.6.16). Since $\mathbb{C} \subseteq \mathcal{A}$, $\mathcal{A}$ is certainly

a dense subalgebra of A. We need to show that if $a \in M_n(\mathcal{A})$ and a is invertible in $M_n(A)$, then it is invertible in $M_n(\mathcal{A})$. With slight abuse of notation (u should really be replaced by $u \otimes 1$, where $1 \in M_n(\mathbb{C})$), this amounts to showing that $a^{-1} - u^{-1}a^{-1}u$ is of trace class if $a - u^{-1}au$ is of trace class. But since the trace-class operators are an ideal, the latter implies $1 - a^{-1}u^{-1}au = a^{-1}(a - u^{-1}au)$ is of trace class, and then

$$a^{-1} - u^{-1}a^{-1}u = -(1 - a^{-1}u^{-1}au)(u^{-1}a^{-1}u)$$

is also of trace class. Thus the hypotheses of Exercise 1.6.16 are satisfied, and the inclusion $\mathcal{A} \hookrightarrow A$ induces an isomorphism on K_0.

Let $\mathcal{D}$ be the double of $\mathcal{A} + \mathcal{L}^1(\mathcal{H}_0)$ along the ideal $\mathcal{L}^1(\mathcal{H}_0)$, so that we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}^1(\mathcal{H}_0) & \longrightarrow & \mathcal{D} & \overset{\longleftarrow}{\longrightarrow} & \mathcal{A} & \longrightarrow & 0 \\
\| & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \| \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & D & \overset{\longleftarrow}{\longrightarrow} & A & \longrightarrow & 0,
\end{array}$$

with the arrows to the left given by $a \mapsto (a, u^{-1}au)$. Now just as before, because of Lemma 6.3.14, we have a map

$$K_0(\mathcal{A}) \to K_0(\mathcal{D}) \cong K_0(\mathcal{A}) \oplus K_0(\mathcal{L}^1) \cong K_0(\mathcal{A}) \oplus \mathbb{Z},$$

whose second component is a map $K_0(\mathcal{A}) \to \mathbb{Z}$. However, the map

$$K_0(\mathcal{D}) \to \mathbb{Z}$$

is induced by $(a, a') \mapsto \mathrm{Tr}(a - a')$. (Recall that the map must vanish on the diagonal, and it is correct on elements of the form $(0, e)$ since by Lemma 6.3.14, the usual operator trace induces the isomorphism $K_0(\mathcal{L}^1) \to \mathbb{Z}$.) Thus the map $K_0(\mathcal{A}) \to \mathbb{Z}$ is induced by $a \mapsto \mathrm{Tr}(a - u^{-1}au)$, which we have seen coincides with φ. Since, by Exercise 1.6.16, the inclusion $\mathcal{A} \hookrightarrow A$ induces an isomorphism on K_0, φ can only take integral values on $K_0(A)$, and the conclusion of the Theorem follows from Corollary 6.3.13. □

6.3.17. Remark. If $n = 1$ in Theorem 6.3.15, then $G = \mathbb{Z}$ and $\ell^2(G)$ can by Fourier analysis be identified with $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in the complex plane. Under this identification (given by the Fourier transform), $C_r^*(G)$ goes over the algebra $C(\mathbb{T})$ of continuous functions on the circle, acting on L^2 by multiplication. So the fact that $C_r^*(G)$ contains no idempotents other than 0 and 1 is due merely to the fact that the circle is connected. However, if $n \geq 2$, it is known that $C_r^*(G)$ is simple [Powers] (and of course highly non-commutative). For a while it was an open problem as to whether any simple unital C*-algebra must contain a projection other than 0 and 1, and Theorem 6.3.15, first proved by Mihai Pimsner and Dan Voiculescu by a different method, provided one of the first examples where the answer is "no." That this would be such an example was first conjectured by Richard Kadison. The proof of Connes is based on earlier ideas of Joachim Cuntz.

Group Rings and Assembly Maps. The last application of cyclic homology which we will discuss concerns the (higher) K-theory of group rings. We will see that an analogue of the construction in Definition 6.2.11 gives an "assembly map" into the K-theory of a group ring, and that in some cases the injectivity of this map may be deduced from comparing it with the corresponding assembly map in cyclic homology, using the Chern character. The assembly map tends to give many interesting classes in the K-theory of group rings, and under optimal circumstances it actually gives a complete calculation of the K-theory.

Unfortunately, the assembly map for K-theory is not as easy to define as the one for cyclic homology, in part because the definition of the higher K-groups is already rather intangible, and depends on homotopy theory. Thus a homotopy definition for the assembly map in higher K-theory is more or less inevitable.

The assembly map for K_0 and K_1 is more concrete but less interesting, so we begin with these cases first. If R is a ring (at the moment not required to be commutative) and G is a group, and if RG denotes the group ring of G over R (this is just $R \otimes_{\mathbb{Z}} \mathbb{Z}G$, with the obvious tensor product multiplication), then the assembly map for RG will roughly speaking be a map from the homology of G with coefficients in the K-theory of R into the K-theory of RG. In lowest degrees it is clear how to proceed: $H_0(G, K_n(R)) \cong K_n(R)$ maps into $K_n(RG)$ via the map of K-groups induced by the inclusion $\iota : R \hookrightarrow RG$. Similarly, $H_1(G, K_n(R)) \cong G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_n(R)$ maps to $K_{n+1}(RG)$ by sending the class of $g \otimes x$, where $g \in G$ and $x \in K_n(RG)$, to the Loday product (see Theorem 5.3.1) $[g] \cdot x$, where $[g]$ is the class of g viewed as an element of $(RG)^{\times} \cong GL(1, RG)$ in $K_1(RG)$. Putting the two of these together, we have a map

$$\text{(6.3.18)} \qquad \text{assembly} : H_0(G, K_n(R)) \oplus H_1(G, K_{n-1}(R)) \to K_n(RG).$$

While one might be inclined to dismiss this as a little formal nonsense, a few special cases make it clear that this is an interesting map:

(a) If $G = \mathbb{Z}$, then $RG = R[t, t^{-1}]$, where t is a generator of t. The assembly map then becomes a map $K_n(R) \oplus K_{n-1}(R) \to K_n(R[t, t^{-1}])$, which one recognizes as the map of the Bass-Heller-Swan Theorem (Theorems 3.2.22 and 3.3.3) and of its generalization to higher K-theory by Quillen (Theorem 5.3.30). These theorems, among the most important in K-theory, assert that the assembly map in this case is always split injective, and that it is an isomorphism if R is left regular.

(b) If $R = \mathbb{Z}$ and G is arbitrary, then $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. The assembly map

$$H_0(G, K_1(\mathbb{Z})) \oplus H_1(G, K_0(\mathbb{Z})) \to K_1(\mathbb{Z}G)$$

is just the obvious map of $G_{\mathrm{ab}} \times \{\pm 1\}$ into $K_1(\mathbb{Z}G)$, and the cokernel of this map is $\mathrm{Wh}(G)$, which measures the failure of the

assembly map to be surjective. The assembly map into $K_1(\mathbb{Z}G)$ is always injective, however. To see this, note that the assembly map is functorial in the group G, so that the map $G \twoheadrightarrow G_{ab}$ induces a commutative diagram

$$\begin{array}{ccc} G_{ab} \times \{\pm 1\} & \xrightarrow{\text{assembly for } G} & K_1(\mathbb{Z}G) \\ \| & & \downarrow \\ G_{ab} \times \{\pm 1\} & \xrightarrow{\text{assembly for } G_{ab}} & K_1(\mathbb{Z}G_{ab}). \end{array}$$

But $\mathbb{Z}G_{ab}$ is a commutative ring and thus

$$K_1(\mathbb{Z}G_{ab}) \supseteq (\mathbb{Z}G_{ab})^{\times} \supseteq G_{ab} \times \{\pm 1\},$$

so the assembly map is an injection for G_{ab} and thus for G.

(c) If $R = \mathbb{F}_p$ and $G = \mathbb{Z}/p$ is cyclic of order p (this case is in some sense the opposite extreme of the "characteristic 0" situation), then $RG = \mathbb{F}_p[t]/(t^p - 1) = \mathbb{F}_p[t]/(t-1)^p$ is a local ring so $K_1(RG) \cong (RG)^{\times} \cong \mathbb{F}_p^{\times} \times (1 + I)$, where I is the radical, the ideal generated by $t-1$. As an abelian group this is isomorphic to $\mathbb{F}_p^{\times} \times \mathbb{F}_p^{p-1}$. The assembly map is split injective with the cokernel a vector space over $\mathbb{F}_p$ of dimension $p-2$.

When G is a free group, then there is a model for BG which is a one-dimensional CW-complex, and all higher homology of G vanishes, so (6.3.18) gives a perfectly reasonable notion of an assembly map in K-theory for RG. Reasoning by analogy from the case $G = \mathbb{Z}$ suggests that this assembly map should always be split injective, and should be an isomorphism for nice enough R. This is in fact the case [Wald]. However, for general groups G, it should be necessary to take the higher homology of G into account in the assembly map. Now we run into a problem, since higher K-theory is defined in terms of homotopy groups rather than homology groups, and thus there is no way to map homology of G directly to K-theory of RG.

A full explanation of the way out of this difficulty would require two complicated bits of machinery: "spectra" in algebraic topology and "spectral sequences" in homological algebra. To avoid these we'll be a bit vague about the general theory, though we will mention some concrete results that can be understood directly.

Recall first that to any ring R we attached in Chapter 5 a topological space $\mathbf{K}(R)$ (chosen to be a CW-complex, and defined up to homotopy equivalence), with the property that the K-groups of R are the homotopy groups of $\mathbf{K}(R)$. However, the homotopy type of the space $\mathbf{K}(R)$ actually carries more information than just the K-groups alone. This extra information is encoded in the functor

$$X \rightsquigarrow [X, \mathbf{K}(R)] \tag{6.3.19}$$

(the notation on the right indicates the homotopy classes of maps $X \to \mathbf{K}(R)$), which was given a more concrete interpretation in Theorem 5.3.32. In analogy with Proposition 1.6.5, one can show that this functor satisfies an excision property, and can be identified with the 0-th term $H^0(X, \mathbf{K}(R))$ of a "generalized cohomology theory" $H^\bullet(X, \mathbf{K}(R))$ on the category of CW-complexes. This theory will satisfy all of the usual Eilenberg-Steenrod axioms for a cohomology theory (cf. [Spanier, Ch. 5, §4] or [Whitehead, Ch. XII, §7]) except for the "dimension axiom" which gives its values on a point. (The theory is not completely determined without imposing some extra natural conditions, but one can arrange for it to have the property that $H^\bullet(pt, \mathbf{K}(R)) \cong K_{-\bullet}(R)$ for all $\bullet \in \mathbb{Z}$.)

A general principle of algebraic topology (discussed in [Whitehead, Ch. XII, §7]) is that each generalized cohomology theory is dual to a unique generalized homology theory. We will need the homology theory $H_\bullet(X, \mathbf{K}(R))$ dual to the cohomology theory just described, which satisfies the property that $H_\bullet(pt, \mathbf{K}(R)) \cong K_\bullet(R)$ for all $\bullet \in \mathbb{Z}$. The definition of the homology theory unfortunately requires the notion of a spectrum, but roughly speaking,

$$H_\bullet(X, \mathbf{K}(R)) \approx \pi_\bullet(X_+ \wedge \mathbf{K}(R)). \tag{6.3.20}$$

Here X_+ denotes X with a disjoint basepoint added so that we get an "unreduced" homology theory instead of a "reduced" one. (This isn't quite right since one needs to "stabilize" the homotopy groups to ensure that excision holds, but it conveys the correct idea. For actually computing $H_\bullet(X, \mathbf{K}(R))$, one rarely needs the definition anyway since usually one can get by with the fact that $\tilde{H}_n(S^i, \mathbf{K}(R)) \cong H_{n-i}(pt, \mathbf{K}(R)) \cong K_{n-i}(R)$ together with various functorial properties.)

Now we can finally define the assembly map. The trick is that we have to replace the homology of G with coefficients in the K-theory of R, which is canonically isomorphic to $H_\bullet(BG; K_\bullet(R))$, by a generalized homology group. (An aside intended only for the reader who knows about spectral sequences: in fact, $H_\bullet(BG; K_\bullet(R))$ will be a good approximation to $H_\bullet(BG, \mathbf{K}(R))$, in the sense that there is an "Atiyah-Hirzebruch-type" spectral sequence converging to it with $H_\bullet(BG; K_\bullet(R))$ as its E^2-term.)

6.3.21. Definition. Let R be a ring and let G be a group. The assembly map

$$H_\bullet(BG, \mathbf{K}(R)) \to K_\bullet(RG)$$

is the map on (stable) homotopy groups induced by the composite

$$BG_+ \wedge \mathbf{K}(R) \xrightarrow{B\iota^+ \wedge id} BGL(RG)^+ \wedge \mathbf{K}(R) \xrightarrow{\mu} \mathbf{K}(RG), \tag{6.3.22}$$

where $B\iota^+ : BG_+ \to BGL(RG)^+$ is the map on classifying spaces induced by the inclusion

$$\iota : G \hookrightarrow GL(1, RG) \hookrightarrow GL(RG),$$

followed by the canonical map $BGL(RG) \to BGL(RG)^+$, and where μ is the Loday product (see Theorem 5.3.1).

If there is a model for BG which is a one-dimensional CW-complex, then one can show that

$$H_n(BG, \mathbf{K}(R)) \cong H_0(G, K_n(R)) \oplus H_1(G, K_{n-1}(R))$$

and that the above definition coincides with (6.3.18). In general, things are not so simple, but part of the relationship between $H_\bullet(BG, \mathbf{K}(R))$ and ordinary homology is reflected in the following.

6.3.23. Lemma. *Let R be a PID (e.g., $\mathbb{Z}$ or a field). Then there is a natural transformation of homology theories*

$$H_\bullet(*, \mathbf{K}(R)) \rightsquigarrow H_\bullet(*; \mathbb{Z})$$

which when $*$ is a point is an isomorphism $H_0(pt, \mathbf{K}(R)) \to \mathbb{Z}$ and is 0 on $H_n(pt, \mathbf{K}(R))$, $n \neq 0$, and which for general spaces is at least a surjection after tensoring with $\mathbb{Q}$.

Sketch of proof. Since homology theories are in natural one-to-one correspondence with cohomology theories, it is really only necessary to define a natural transformation of cohomology theories

$$H^\bullet(*, \mathbf{K}(R)) \rightsquigarrow H^\bullet(*; \mathbb{Z}).$$

On H^0, this will simply be the map

$$H^0(*, \mathbf{K}(R)) \cong [X, \mathbf{K}(R)] \to [X, \mathbb{Z}] \cong H^0(*; \mathbb{Z})$$

induced by the obvious map

$$\mathbf{K}(R) = K_0(R) \times BGL(R)^+ \to \mathbb{Z}$$

which collapses $BGL(R)^+$ to a point and sends each element of $K_0(R)$ to its "rank" in $\mathbb{Z}$. One can now show this extends to a map of cohomology theories in a natural way. In proving this, one uses the fact that since R is a regular ring, the negative K-groups of R vanish, from which it follows by an induction on dimension that $H^n(X, \mathbf{K}(R)) = 0$ for $n > \dim X$. As for the last statement, recall by (6.3.20) that $H_\bullet(X, \mathbf{K}(R))$ is the stable homotopy of $X_+ \wedge \mathbf{K}(R)$. However, it is known that the Hurewicz map induces a rational isomorphism from stable homotopy to ordinary homology, so that

$$H_\bullet(X, \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \tilde{H}_\bullet(X_+ \wedge \mathbf{K}(R); \mathbb{Q}).$$

The right side now splits up by the Künneth Theorem, and our natural transformation is just projection onto the summand $H_\bullet(X; \mathbb{Q})$. □

Up till now, none of what we have said in this subsection has anything to do with cyclic homology, but at the same time we haven't yet proven anything about K-theory of group rings. Now we can use cyclic homology to prove something about non-triviality of the assembly map. In what follows, one could replace HC^- by HH and Ch by Tr without affecting the conclusion.

6.3.24. Theorem. *Let k be either $\mathbb{Z}$ or a field of characteristic 0, and let G be a group. Then the following diagram rationally commmutes*

$$\begin{array}{ccc} H_\bullet(BG,\, \mathbf{K}(k)) & \xrightarrow{\text{assembly}} & K_\bullet(kG) \\ \downarrow & & \downarrow \text{Ch} \\ H_\bullet(G,\, k) & \xrightarrow{\text{assembly}} & HC_\bullet^-(kG). \end{array}$$

Here the map on the left is the one given by applying Lemma 6.3.23 to BG and using Theorem 5.1.27 to identify the homology of BG with that of G. The assembly map on the bottom is given by Definition 6.2.11. Since (when $k = \mathbb{Z}$) these two maps are, respectively, a rational surjection and a split injection, it follows that $K_n(\mathbb{Z}G) \otimes_{\mathbf{Z}} \mathbb{Q}$ contains a copy of $H_n(G,\, \mathbb{Q})$.

Sketch of proof. After tensoring with $\mathbb{Q}$, the assembly map just becomes the map in rational homology induced by (6.3.22), and the map on the left becomes projection onto the summand $H_n(G,\, \mathbb{Q})$. Thus the composite of the natural rational splitting of the map on the left with the map on top is just the map on rational homology induced by $B\iota^+$. The definition of the Chern character in (6.2.14) now identifies this with the assembly map and so the diagram commutes. (The characteristic 0 assumption is needed to make sure that $\mathrm{Tr} : K_0(R) \to R$ is rationally injective.) The assembly map on cyclic homology is a split injection by (6.2.11). □

While Theorem 6.3.24 begins to show the importance of the assembly map for the study of K-theory of group rings, it is by no means the final word on the subject. A much more impressive recent result, with a proof that would take at least another whole volume to explain in detail, is the following.

6.3.25. Theorem [BöHsMad]. *Let G be a group such that $H_i(G,\, \mathbb{Z})$ is finitely generated for each i. Then the assembly map $H_\bullet(BG,\, \mathbf{K}(\mathbb{Z})) \to K_\bullet(\mathbb{Z}G)$ is an injection after tensoring each side with $\mathbb{Q}$, and thus*

$$K_n(\mathbb{Z}G) \otimes_{\mathbf{Z}} \mathbb{Q}$$

contains not only a copy of $H_n(G,\, \mathbb{Q})$ (as promised by Theorem 6.3.24) but also a copy of

$$\bigoplus_{1 \le i \le \left[\frac{n-1}{4}\right]} H_{n-1-4i}(G,\, \mathbb{Q}).$$

Here, to get from the first assertion to the last one needs to use Borel's calculation of $K_\bullet(\mathbb{Z}) \otimes_{\mathbf{Z}} \mathbb{Q}$ (Theorem 5.3.13). The method of proof of Bökstedt, Hsiang, and Madsden is vaguely related to that of Theorem 6.3.24, but requires first replacing K-theory of rings by "K-theory of spaces" and cyclic homology by a more complicated topological analogue. It would be nice also to be able to say something about surjectivity of the assembly map, but it's unlikely anything in this direction could be proved using analogues of the Chern character. Nevertheless, the fact that one can say anything at all about the size of K-groups of group rings already proves the usefulness of cyclic homology.

6.3.26. Exercise. Let R be any ring and take $k = \mathbb{Z}$. By modifying the construction in Definition 6.2.14, construct a Dennis trace map $K_i(R;\, \mathbb{Z}/n) \to HH_i(R;\, \mathbb{Z}/n)$ and a Chern character map $K_i(R;\, \mathbb{Z}/n) \to HC_i^-(R;\, \mathbb{Z}/n)$, $i \geq 2$. (The restriction $i \geq 2$ makes it possible to use Theorem 5.3.7 here.) Show how these could be used (at least in principle) to detect torsion in $K_1(R)$, even when $HH_1(R)$ and $HC_1^-(R)$ are torsion-free.

6.3.27. Exercise. Try to extend to the relative case the idea presented here for detecting torsion elements of class groups with the trace map. Here is a sketch of how to proceed. If R is a Dedekind domain with field of fractions F and if $\mathfrak{a}$ is an ideal in R, one can describe $K_0(R,\, \mathfrak{a})$ in terms of a relative class group constructed from ideals I of R which are relatively prime to $\mathfrak{a}$, modulo those of the form (a), where $a \equiv 1 \mod \mathfrak{a}$. The ideal I gives n-torsion in this group if $I^n = (a)$ for such an a, and one is led to consider

$$\frac{\{a \in R \smallsetminus \{0\} : a \equiv 1 \mod \mathfrak{a},\ \exists I \trianglelefteq R,\ I^n = (a)\}}{\{a \in R \smallsetminus \{0\} : a \equiv 1 \mod \mathfrak{a}\}^n},$$

and to look for a kind of "logarithmic derivative" detecting elements in this group. See if you can find an example where your invariant is non-zero.

6.3.28. Exercise. This exercise illustrates the use of the Chern character to prove non-triviality of a class in higher K-theory of a field. While for K_2 one could give another proof using the Matsumoto relations, similar arguments also apply to K_n for $n > 2$ where such a technique is not available.

Let k be a field of characteristic 0, and let F be a field extension of k of transcendence degree at least 2. Let $t,\, s \in F$ be algebraically independent over k. Viewing F as a k-algebra, show that the differential form $t^{-1}s^{-1}dt\, ds \in \Omega R$ represents a non-zero class in $H_2^{\text{de R}}(F)$. (You may need to use facts about k-linear derivations of a field extension F, as found for example in [Jacobson, II, Proposition 8.17].) Show that this class is basically the image of the Steinberg symbol $\{t,\, s\}$ under the Chern character, and thus deduce that $\{t,\, s\}$ must be an element of $K_2(F)$ of infinite order.

6.3.29. Exercise (Cyclic homology and the Helton-Howe invariant). As in Exercise 4.4.30, let $\mathfrak{A}$ be the "Toeplitz algebra" generated by the Toeplitz operators T_f, $f \in C^\infty(S^1)$, together with $\mathcal{L}^1(\mathcal{H})$, where $\mathcal{H}$ is the Hardy Hilbert space on S^1. Let Tr be the usual operator trace. Show that the map

$$(f,\, g) \mapsto \operatorname{Tr}(T_f T_g - T_g T_f)$$

studied in [HeltonHowe] defines a map $HC_1(C^\infty(S^1)) \to \mathbb{C}$, which may be identified with the composite of $\operatorname{Tr} : HC_0(\mathfrak{A},\, \mathcal{L}^1(\mathcal{H})) \to \mathbb{C}$ with the boundary map in the long exact sequence of the ideal $\mathcal{L}^1(\mathcal{H})$ in $\mathfrak{A}$. Composing with the Chern character $\operatorname{Ch} : K_1(C^\infty(S^1)) \to HC_1(C^\infty(S^1))$, we obtain a map $K_1(C^\infty(S^1)) \to \mathbb{C}$. (Recall that by Theorem 4.4.19, $K_1(C^\infty(S^1))$ is just $(C^\infty(S^1))^\times = C^\infty(S^1,\, \mathbb{C}^\times)$.) Analyze this map and compare it with the calculations in Exercise 4.4.30.

6.3.30. Exercise (More on the image of the trace on idempotents). Suppose Γ is a subgroup of finite index m in a group G. Show that in this case, there is a natural inclusion map $\iota : C_r^*(\Gamma) \hookrightarrow C_r^*(G)$ making $C_r^*(G)$ into a free module of rank m over $C_r^*(\Gamma)$. Thus we have an injection $t : C_r^*(G) \hookrightarrow M_m(C_r^*(\Gamma))$ and a transfer map $t_* : K_0(C_r^*(G)) \to K_0(C_r^*(\Gamma))$ such that $t_* \circ \iota_*$ is multiplication by m. Show that if φ_Γ and φ_G are the traces on $C_r^*(\Gamma)$ and $C_r^*(G)$ as defined in Definition 6.3.11, extended in the obvious way to matrices, then $\varphi_G \circ \iota = \varphi_\Gamma$ and $\varphi_\Gamma \circ t = m\varphi_G$. Conclude that if the image of $(\varphi_\Gamma)_*$ on $K_0(C_r^*(\Gamma))$ is contained in $\mathbb{Z}$, then the image of φ_G on $K_0(C_r^*(G))$ is contained in the rational numbers with denominators dividing m. Explain what this means when Γ is the trivial group and G is finite. Then use Theorem 6.3.15 to get a conclusion about groups with a free subgroup of finite index.

6.3.31. Exercise. Let G be a perfect group, so that one can apply the $+$-construction to BG. Note that the inclusion $\iota : G \hookrightarrow GL(\mathbb{Z}G)$ induces a map $B\iota^+ : BG^+ \to BGL(\mathbb{Z}G)^+$, closely related to the assembly map. Show that the diagram

$$\begin{array}{ccc} \pi_n(BG^+) & \xrightarrow{(B\iota^+)_*} & \pi_n(BGL(\mathbb{Z}G)^+) = K_n(\mathbb{Z}G) \\ {\scriptstyle \text{Hurewicz}}\downarrow & & \downarrow{\scriptstyle \text{Ch}} \\ H_n(G, \mathbb{Z}) & \xrightarrow{\text{assembly}} & HC_n^-(\mathbb{Z}G) \end{array}$$

commutes, and show how this can be used to prove non-triviality of some of the K-groups of $\mathbb{Z}G$. For example, if $G = SL(2, \mathbb{F}_5)$ (the non-trivial central extension of A_5 by $\mathbb{Z}/2$), use this result and Exercise 5.2.16 to show that $K_3(\mathbb{Z}G)$ contains an element of order 120.

References

Books and Monographs on Related Areas of Algebra, Analysis, Number Theory, and Topology

Remark. The works listed in this part of the bibliography fall into several categories. Some are basic textbooks on areas needed for studying certain aspects of algebraic K-theory. Some are textbooks on areas where algebraic K-theory may be applied. A few are more advanced monographs dealing with related areas.

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